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COMBINATORIAL RESULTS FOR SEMIGROUPS OF ORDER-PRESERVING FULL TRANSFORMATIONS WITH PATTERN OF RANGE

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Abstract: Let $\mathcal{O}[n]$ and $\mathcal{C}[n]$ be the semigroup of all order-preserving and all order-preserving and regressive transformations on the set $[n] = \{1, \ldots, n\}$, respectively. In this paper, we give a pattern on $[n]$. Combinatorial properties on subsemigroups of $\mathcal{O}[n]$ with restricted range to the pattern are investigated.

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1. Introduction

For a set $[n] = \{1, \ldots, n\}$. $\mathcal{T}[n]$ denotes the full transformation semigroup on $[n]$. We shall call a transformation $\alpha : [n] \to [n]$ *order-preserving* if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in [n]$, and *regressive* (or *order-decreasing*) if $x\alpha \leq x$ for all $x \in [n]$. We denote by $\mathcal{O}[n]$ the semigroup of all full order-preserving transformations on $[n]$ and denote by $\mathcal{R}[n]$ the semigroup of all full regressive transformations on $[n]$. Let $\mathcal{C}[n] = \mathcal{O}[n] \cap \mathcal{R}[n]$.

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On these semigroups, combinatorial results are well understood by now. For example, Howie in [4] showed that $|\mathcal{O}[n]| = \binom{2n-1}{n-1}$ and $|E\mathcal{O}[n]| = F_{2n}$, where $E\mathcal{O}[n]$ is the set of all idempotents of $\mathcal{O}[n]$ and F_{2n} is the alternate Fibonacci number given by $F_1 = F_2 = 1$. Higgins in [2] and later in (alternative version) [6, 7] Laradji and Umar proved that $|\mathcal{C}[n]| = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number C_n.

In 1975, Symons [11] introduced the subsemigroup of full transformation with restricted range and later Sanwong in [10] studied its regularity. Recently in [5, 12] some classifications on subsemigroups of full order-preserving (regressive) transformations with restricted range were investigated. These papers motivated the study of combinatorial properties of semigroups of transformations with restricted range.

For $t \in \mathbb{N}$ and $n_1, \ldots, n_t, m_1, \ldots, m_t \in \mathbb{N}_0$, a *pattern* $[{m_1}_{n_1}{m_2}_{n_2}\cdots {m_t}_{n_t}]$ is defined to be

$$\{1, 2, \ldots, \sum_{i=1}^{t}(m_i + n_i)\}$$

having the structure of two disjoint subsets

$$\{1, \ldots, m_1\} \cup \bigcup_{k=1}^{t}\left(\sum_{i=1}^{k}(m_i + n_i) + \{1, \ldots, m_{k+1}\}\right),$$

denoted by $\lceil {m_1}_{n_1}{m_2}_{n_2}\cdots {m_t}_{n_t}\rceil$, and

$$(m_1 + \{1, \ldots, n_1\}) \cup \bigcup_{k=1}^{t-1}\left(m_1 + \sum_{i=1}^{k}(n_i + m_{i+1}) + \{1, \ldots, n_{k+1}\}\right),$$

denoted by $\lfloor {m_1}_{n_1}{m_2}_{n_2}\cdots {m_t}_{n_t}\rfloor$, respectively.

For example, $[1_2 3_1] = \{1, 2, 3, 4, 5, 6, 7\}$ with $\lceil 1_2 3_1\rceil = \{1, 4, 5, 6\}$ and $\lfloor 1_2 3_1\rfloor = \{2, 3, 7\}$.

Note that for $i = 1, \ldots, t$, if $m_i = 0$, then

$$[{m_1}_{n_1}\cdots {m_i}_{n_i}\cdots {m_t}_{n_t}] = [{m_1}_{n_1}\cdots {m_{i-1}}_{n_{i-1}+n_i}\cdots {m_t}_{n_t}],$$

and if $n_i = 0$, then

$$[{m_1}_{n_1}\cdots {m_i}_{n_i}\cdots {m_t}_{n_t}] = [{m_1}_{n_1}\cdots (m_i + m_{i+1})_{n_{i+1}}\cdots {m_t}_{n_t}].$$

For brevity, we shall omit to write 0 in any positions of the pattern if it is 0. For example, $[{}_2 5] := [0_2 5_0]$, $[{}_2] := [0_2]$ and $[5] := [5_0]$. When all positions are 0, we will write $[0]$ instead.

Let $\mathcal{S}$ be the set containing all these elements. We define a binary operation on $\mathcal{S}$ in natural way of the concatenation, for example, $[1_2 1][1_3 2_1] = [1_2 2_3 2_1]$,

$[3][1_13] = [4_13]$ and $[{}_1][{}_13_3] = [{}_23_3]$. Then $\mathcal{S}$ is a free monoid with $[0]$ as the identity element.

For $[\mathbf{x}] \in \mathcal{S}$, by considering $\lceil \mathbf{x} \rceil$ ($\lfloor \mathbf{x} \rfloor$) as the skeleton of $[\mathbf{x}]$ and grouping elements in $\lfloor \mathbf{x} \rfloor$ ($\lceil \mathbf{x} \rceil$) into classes, we have that each class contains all elements in $\lfloor \mathbf{x} \rfloor$ ($\lceil \mathbf{x} \rceil$) which has no elements in $\lceil \mathbf{x} \rceil$ ($\lfloor \mathbf{x} \rfloor$) lies between them and it is said to be a *lower*(*upper*)*-class* of $[\mathbf{x}]$. A lower-class containing a and an upper-class containing b of $[\mathbf{x}]$ are denoted by $\underline{a}$ and $\overline{b}$, respectively.

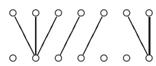
Let $\mathcal{T}[\mathbf{x}]$ be the full transformation semigroup under composition of all mapping of $[\mathbf{x}]$ into $\lceil \mathbf{x} \rceil$. For $\mathcal{O}[\mathbf{x}]$, $\mathcal{R}[\mathbf{x}]$ and $\mathcal{C}[\mathbf{x}]$ are defined analogously.

We already know that the number of integral solutions of the equation

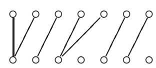
$$x_1 + x_2 + \cdots + x_{n+1} = n - 1$$

that satisfy $x_i \geq 0$ is equal to the cardinality of $\mathcal{O}[n]$, $\binom{2n-1}{n-1}$. For example, when $n = 7$

- $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (1, 0, 0, 1, 1, 3, 0, 0)$ is the representation of

 .

- $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (0, 0, 1, 1, 0, 2, 1, 1)$ is the representation of

 .

It is clear that the cardinality of the set containing all order-preserving from $\{1, \ldots, p\}$ to $\{1, \ldots, q\}$ is $\binom{p+q-1}{p}$. Consequently, the following result is obtained.

Theorem 1.1. *The cardinality of* $\mathcal{O}[m_{1\,k_1} m_{2\,k_2} \cdots {}_{k_{t-1}} m_t]$ *is* $\binom{2m+k-1}{m-1}$ *where* $m = m_1 + \cdots + m_t$ *and* $k = k_1 + \cdots + k_{t-1}$.

The paper is organized as follows: In Section 2 and 3, we deal with cardinalities of $E\mathcal{O}[m_{1\,k_1} m_{2\,k_2} \cdots {}_{k_{t-1}} m_t]$, $\mathcal{C}[m_k]$ and $\mathcal{C}[m_k n]$. In Section 4, cardinalities of some equivalence classes are investigated.

2. The Number of Idempotents

To find the cardinality of $E\mathcal{O}[m_{1\,k_1} m_{2\,k_2} \cdots {}_{k_{t-1}} m_t]$, our strategy will be to look at the sets having some types as follows:

For $\alpha \in E\mathcal{O}[m_{1k_1}m_{2k_2}\cdots{}_{k_{t-1}}m_t]$, if there are exactly distinct n lower-classes, $\underline{g_1}, \ldots, \underline{g_n}$ of $[m_{1k_1}m_{2k_2}\cdots{}_{k_{t-1}}m_t]$ such that for $i \in \{1, \ldots, n\}$,

$$|(\underline{g_i} \cup \{\min \underline{g_i} - 1, \max \underline{g_i} + 1\})\alpha| = 1,$$

then α is called a type of n *closed lower-classes* (or $t-1-n$ *open lower-classes*).

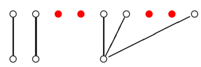
For $s \in \mathbb{N}_0$, let $U_s[m_{1k_1}m_{2k_2}\cdots{}_{k_{t-1}}m_t]$ and $V_s[m_{1k_1}m_{2k_2}\cdots{}_{k_{t-1}}m_t]$ stand for the set of all idempotents in $\mathcal{O}[m_{1k_1}m_{2k_2}\cdots{}_{k_{t-1}}m_t]$ of the type s open lower-classes and s closed lower-classes, respectively.

Example. In $E\mathcal{O}[2_22_21]$,

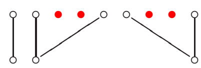
$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 2 & 5 & 5 & 5 & 5 & 5 & 5 \end{pmatrix} \in V_1[2_22_21] = U_1[2_22_21],$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 2 & 2 & 2 & 9 & 9 & 9 & 9 \end{pmatrix} \in V_2[2_22_21] = U_0[2_22_21].$$

Note that in the structure of α, there is the shape of a right triangle, , from $(\{5,6\} \cup \underline{7} \cup \{9\})\alpha = \{5\}$,

 ,

whereas in the structure of β, there is the shape of two right triangles,  , from $(\{2\} \cup \underline{3} \cup \{5\})\beta = \{2\}$ and $(\{6\} \cup \underline{7} \cup \{9\})\beta = \{9\}$,

 .

We denote

$$V_1[m_{1k_1}\cdots{}_{k_{i-1}}m_{i\triangleright}\cdots{}_{k_{t-1}}m_t] \text{ and } V_1[m_{1k_1}\cdots{}_{k_{i-1}}m_{i\triangleleft}\cdots{}_{k_{t-1}}m_t]$$

as two subsets of $V_1[m_{1k_1}\cdots{}_{k_{i-1}}m_{ik_i}\cdots{}_{k_{t-1}}m_t]$, with the closed lower-class at $\underline{g_i} = (m_i + \sum_{j=1}^{i-1}(m_j + k_j)) + \{1, \ldots, k_i\}$ in the structure of  and , respectively. For example,

$$V_1[2_22_21] = V_1[2_\triangleleft 2_21] \,\dot\cup\, V_1[2_\triangleright 2_21] \,\dot\cup\, V_1[2_22_\triangleleft 1] \,\dot\cup\, V_1[2_22_\triangleright 1].$$

For $V_n[m_{1k_1}m_{2k_2}\cdots{}_{k_{t-1}}m_t]$, it can be written as a disjoint union of $2^n\binom{t-1}{n}$ sets defined in analogous way. For example,

$$V_2[2_22_21] = V_2[2_\triangleright 2_\triangleleft 1] \,\dot\cup\, V_2[2_\triangleright 2_\triangleright 1] \,\dot\cup\, V_2[2_\triangleleft 2_\triangleleft 1] \,\dot\cup\, V_2[2_\triangleleft 2_\triangleright 1].$$

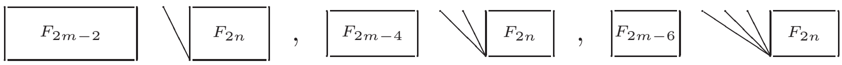
The following lemma is obtained immediately.

Lemma 2.1. $\left|EO[m_{1\,k_1} m_{2\,k_2} \cdots {}_{k_{t-1}} m_t]\right| = \sum_{i=0}^{t-1} \left|V_i[m_{1\,k_1} m_{2\,k_2} \cdots {}_{k_{t-1}} m_t]\right|.$

Lemma 2.2. *For $V_1[m_k n]$, we have that*

$$|V_1[m_{\lhd} n]| = F_{2m-1}F_{2n} \quad \text{and} \quad |V_1[m_{\rhd} n]| = F_{2m}F_{2n-1}.$$

Proof. To illustrate the cardinality of $V_1[m_{\lhd} n]$, we consider the following cases:



Then $|V_1[m_{\lhd} n]| = \left(\sum_{i=1}^{m} F_{2(m-i)} + 1\right) F_{2n}$. By using the Fibonacci identity: $\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$, it follows that $|V_1[m_{\lhd} n]| = F_{2m-1}F_{2n}$ as wanted.
For the rest, it can be proved in the same fashion. □

From Lemma 2.1 and 2.2, the following proposition is obtained.

Proposition 2.3. *The cardinality of $EO[m_k n]$ is*

$$(k+1)F_{2m}F_{2n} + F_{2m-1}F_{2n} + F_{2m}F_{2n-1}.$$

As a consequence of Proposition 2.3 and Howie's result, by taking $k = 0$, we have the following conclusion.

Corollary 2.4. *The cardinality of $EO[m_k n]$ is $kF_{2m}F_{2n} + F_{2(m+n)}$.*

	1	2	3	4	5	6
1	$k+3$	$3k+8$	$8k+21$	$21k+55$	$55k+144$	$144k+377$
2	$3k+8$	$9k+21$	$24k+55$	$63k+144$	$165k+377$	$432k+987$
3	$8k+21$	$24k+55$	$64k+144$	$168k+377$	$440k+987$	$1152k+2584$
4	$21k+55$	$63k+144$	$168k+377$	$441k+987$	$1155k+2584$	$3024k+6765$
5	$55k+144$	$165k+377$	$440k+987$	$1155k+2584$	$3025k+6765$	$7920k+17711$
6	$144k+377$	$432k+987$	$1152k+2584$	$3024k+6765$	$7920k+17711$	$20736k+46368$

Table 1: The cardinality of $EO[m_k n]$

Proposition 2.5. *The cardinality of $EO[m_{1\,k_1} m_{2\,k_2} m_3]$ is*

$$k_1k_2F_{2m_1}F_{2m_2}F_{2m_3} + k_1F_{2m_1}F_{2(m_2+m_3)} + k_2F_{2(m_1+m_2)}F_{2m_3} + F_{2(m_1+m_2+m_3)}.$$

Proof. It is clear that $|V_0[m_{1\,k_1} m_{2\,k_2} m_3]| = (k_1+1)(k_2+1)F_{2m_1}F_{2m_2}F_{2m_3}$. Since $V_1[m_{1\,k_1} m_{2\,k_2} m_3]$ is

$$V_1[m_{1\triangleright} m_{2\,k_2} m_3] \,\dot\cup\, V_1[m_{1\triangleleft} m_{2\,k_2} m_3] \,\dot\cup\, V_1[m_{1\,k_1} m_{2\triangleright} m_3] \,\dot\cup\, V_1[m_{1\,k_1} m_{2\triangleleft} m_3]$$

and $V_2[m_{1\,k_1} m_{2\,k_2} m_3]$ is

$$V_2[m_{1\triangleright} m_{2\triangleright} m_3] \,\dot\cup\, V_2[m_{1\triangleleft} m_{2\triangleleft} m_3] \,\dot\cup\, V_2[m_{1\triangleleft} m_{2\triangleright} m_3] \,\dot\cup\, V_2[m_{1\triangleright} m_{2\triangleleft} m_3],$$

by applying Lemma 2.2, we have

$$\begin{aligned} |V_1[m_{1\,k_1} m_{2\,k_2} m_3]| &= (k_1+1)[F_{2m_1}F_{2m_2}F_{2m_3-1} + F_{2m_1}F_{2m_2-1}F_{2m_3}] \\ &\quad + (k_2+1)[F_{2m_1}F_{2m_2-1}F_{2m_3} + F_{2m_1-1}F_{2m_2}F_{2m_3}], \\ |V_2[m_{1\,k_1} m_{2\,k_2} m_3]| &= F_{2m_1}F_{2m_2-1}F_{2m_3-1} + F_{2m_1-1}F_{2m_2-1}F_{2m_3} \\ &\quad + F_{2m_1-1}F_{2m_2}F_{2m_3-1} + F_{2m_1}F_{2m_2-2}F_{2m_3}. \end{aligned}$$

Using the Fibonacci identity: $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$ and Howie's result when $k_1 = k_2 = 0$, our proof is finished. □

From Proposition 2.5, we have an identity of 3-term Fibonacci numbers (see also in [3] and [9]): $F_{2(m_1+m_2+m_3)}$ is

$$2F_{2m_1}F_{2m_2}F_{2m_3} + F_{2m_1}F_{2m_2+1}F_{2m_3-1} + F_{2m_1+1}F_{2m_2-1}F_{2m_3} + F_{2m_1-1}F_{2m_2}F_{2m_3+1}.$$

As a polynomial in the variables $k_1, \ldots, k_{t-1}$ with coefficient in $\mathbb{Z}$, the leading coefficient of $k_{i_1}\cdots k_{i_d}$ in $\left|E\mathcal{O}[m_{1\,k_1} m_{2\,k_2} \cdots {}_{k_{t-1}} m_t]\right|$ is denoted by $c(k_{i_1}\cdots k_{i_d})$.

Lemma 2.6. *In* $\left|E\mathcal{O}[m_{1\,k_1} m_{2\,k_2} \cdots {}_{k_{t-1}} m_t]\right|$, $c(k_i k_{i+n_1} \cdots k_{i+n_1+\cdots+n_r})$ *is*

$$F_{2(m_1+\cdots+m_i)}F_{2(m_{i+1}+\cdots+m_{i+n_1})}\cdots F_{2(m_{i+n_1+\cdots+n_r+1}+\cdots+m_t)}.$$

Proof. By Corollary 2.4 and Proposition 2.5, the result holds for $|E\mathcal{O}[m_{1\,k_1} m_2]|$ and $|E\mathcal{O}[m_{1\,k_1} m_{2\,k_2} m_3]|$. We prove the result by induction on the number of lower-classes. Suppose the result holds for $\left|E\mathcal{O}[m_{1\,k_1} m_{2\,k_2} \cdots {}_{k_{p-1}} m_p]\right|$ where $p = 2, \ldots, t-1$.

To find $c(k_i k_{i+n_1} \cdots k_{i+n_1+\cdots+n_r})$, let $q \in \{i, i+n_1, \ldots, i+n_1+\cdots+n_r\}$. By considering all terms having k_q as a factor in $\left|E\mathcal{O}[m_{1\,k_1} m_{2\,k_2} \cdots {}_{k_{t-1}} m_t]\right|$, it suffices to consider the set of all idempotent such that $\underline{g_q} = (m_i + \sum_{j=1}^{q-1}(m_j + k_j)) + \{1, \ldots, k_q\}$ is an open lower-class, namely A_q. Then

$$|A_q| = (k_q+1)\left|E\mathcal{O}[m_{1\,k_1} \cdots {}_{k_{q-1}} m_q]\right| \cdot \left|E\mathcal{O}[m_{q+1\,k_{q+1}} \cdots {}_{k_{t-1}} m_t]\right|.$$

WLOG, we let $q = i + (n_1 + \cdots + n_d)$. Then

$$c(k_i k_{i+n_1} \cdots k_{i+n_1+\cdots+n_{d-1}} k_{i+n_1+\cdots+n_d} k_{i+n_1+\cdots+n_{d+1}} \cdots k_{i+n_1+\cdots+n_r})$$

is the product of

$$c(k_i k_{i+n_1} \cdots k_{i+n_1+\cdots+n_{d-1}}) \text{ and } c(k_{i+n_1+\cdots+n_{d+1}} \cdots k_{i+n_1+\cdots+n_r})$$

in $\left|\mathcal{EO}[m_{1\,k_1} \cdots {}_{k_{q-1}} m_q]\right|$ and $\left|\mathcal{EO}[m_{q+1\,k_{q+1}} \cdots {}_{k_{t-1}} m_t]\right|$, respectively. By induction, the proof is finished. □

Theorem 2.7. *The cardinality of* $\mathcal{EO}[m_{1\,k_1} m_{2\,k_2} \cdots {}_{k_{t-1}} m_t]$ *is*

$$F_{2(m_1+m_2+\cdots m_t)} + \sum_{A\in\mathcal{P}(\{1,\ldots,t-1\})\{\emptyset\}} \left[c(\textstyle\prod_{i\in A} k_i) \cdot \prod_{i\in A} k_i\right]$$

Proof. From Lemma 2.6 and by using Howie's result when $k_1 = k_2 = \cdots = k_{t-1} = 0$, our proof is finished. □

From theorem above, the cardinality of $\mathcal{EO}[m_{1\,k_1} m_{2\,k_2} m_{3\,k_3} m_4]$ is

$$\begin{aligned}
&F_{2(m_1+m_2+m_3+m_4)} \\
&+ k_1k_2k_3 F_{2m_1}F_{2m_2}F_{2m_3}F_{2m_4} + k_1 F_{2m_1}F_{2(m_2+m_3+m_4)} + k_2 F_{2(m_1+m_2)}F_{2(m_3+m_4)} \\
&+ k_3 F_{2(m_1+m_2+m_3)}F_{2m_4} + k_1k_2 F_{2m_1}F_{2m_2}F_{2(m_3+m_4)} + k_1k_3 F_{2m_1}F_{2(m_2+m_3)}F_{2m_4} \\
&\qquad + k_2k_3 F_{2(m_1+m_2)}F_{2m_3}F_{2m_4}.
\end{aligned}$$

Corollary 2.8. *The cardinality of* $\mathcal{EO}[m_k m_k \cdots {}_k m]$, *(with* $n-1$ *lower-classes), is*

$$\begin{aligned}
&F_{2nm} + \sum_{p_1+p_2=n} F_{2(p_1m)}F_{2(p_2m)}k + \sum_{p_1+p_2+p_3=n} F_{2(p_1m)}F_{2(p_2m)}F_{2(p_3m)}k^2 \\
&+ \cdots + \sum_{p_1+\cdots+p_{n-1}=n} F_{2(p_1m)} \cdots F_{2(p_{n-1}m)}k^{n-2} + (F_{2m})^n k^{n-1}.
\end{aligned}$$

3. The Number of Regressive Maps

We aim now to find the cardinality of $\mathcal{C}[m_k]$ and $\mathcal{C}[m_k n]$, respectively. Let $l(n,t)$ be the cardinality of the set $L(n,t)$ containing all maps in $\mathcal{C}[n]$ whose n is sent to $n-t$. It is clear that $l(n,0) = l(n,1) = C_{n-1}$.

	2	3	4
1	$3+k$	$8+6k+k^2$	$21+25k+9k^2+k^3$
2	$21+9k$	$144+126k+27k^2$	$987+1305k+567k^2+81k^3$
3	$144+64k$	$2584+2304k+512k^2$	$46368+62080k+27648k^2+4096k^3$
4	$987+441k$	$46368+41454k+9261k^2$	$2178309+2921625k+1305801k^2+194481k^3$
5	$6765+3025k$	$832040+744150k+166375k^2$	$102334155+137289625k+61392375k^2+9150625k^3$
6	$46368+20736k$	$14930352+13353984k+2985984k^2$	$4807526976+6449932800k+2884460544k^2+429981696k^3$

Table 2: The cardinality of $E\mathcal{O}[m_k m_k \cdots {}_k m]$ with $n-1$ lower-classes

To find $l(n,t)$ when $t \geq 2$, let $M(n,t,k)$ be the set containing all maps α in $L(n,t)$ such that $(n-t-k+1)\alpha = n-t-k$ for $k \in \{0,1,\ldots,n-t-1\}$ and let

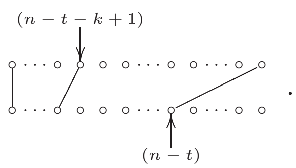
$$S(n,t,k) := M(n,t,k) \bigcup_{i=0}^{k-1} S(n,t,i).$$

It is clear that $|S(n,t,0)| = C_{n-t}$.

Remark 3.1. The following results are obtained immediately:

1. $L(n,t) = S(n,t,0) \,\dot\cup\, S(n,t,1) \,\dot\cup\, \cdots \,\dot\cup\, S(n,t,n-t-1)$.
2. $\mathcal{C}[n] = L(n,0) \,\dot\cup\, L(n,1) \,\dot\cup\, \cdots \,\dot\cup\, L(n,n-1)$.
3. $\mathcal{C}[n_m] = L(n+m,m) \,\dot\cup\, L(n+m,m+1) \,\dot\cup\, \cdots \,\dot\cup\, L(n+m,n+m-1)$.

Consider $S(n,t,k)$ for all $k = 0,1,\ldots,n-t-1$. Given $\alpha \in S(n,t,k)$. We firstly consider the structure of α as follows:



It is clear that

$$\left|\{\alpha|_{\{1,\ldots,n-t-k\}} \;:\; \alpha \in S(n,t,k)\}\right| = C_{n-t-k}.$$

We denote $s(n,t,k)$ as the cardinality of $\{\alpha|_{\{n-t-k+2,\ldots,n-1\}} \;:\; \alpha \in S(n,t,k)\}$.

Lemma 3.2. *The following statements hold:*

(i) $s(n,2,k) = C_k$ *for* $k = 0,1,\ldots,n-3$.

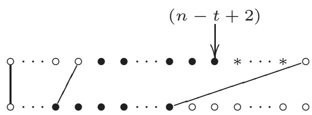
(ii) $s(n,t,k) = |\mathcal{C}[(k+1)_{t-3}]|$ *for* $t \geq 3$ *and* $k = 0,1,\ldots,n-t-1$.

Proof. (i) Let $\alpha \in S(n,2,k)$. Suppose $k > 0$. Then $(n-1)\alpha \leq n-3$. We consider $\alpha|_{\{n-k,\dots,n-1\}} : \{n-k,\dots,n-1\} \to \{n-2-k,\dots,n-3\}$ where $R = \{n-2-k,\dots,n-3\}$,

 .

Then $\alpha|_{D_2}$ acts as an order-preserving map β on $\{1,\dots,k\}$. It remains to show that β is regressive. Suppose there is $x \in \{1,\dots,k\}$ such that $(x)\beta > x$. Since α is regressive, it implies that either $(x)\beta = x+1$ or $(x)\beta = x+2$. Then $\alpha \in S(n,2,t)$ for some $t < k$ which is a contradiction. Then $s(n,2,k) = C_k$.

(ii) For any $\alpha \in S(n,t,k)$, we have that $\{n-t-k+2,\dots,n-1\}\alpha \subseteq \{n-t-k,\dots,n-t\}$,



.

To show that $\{\alpha|_{\{n-t-k+2,\dots,n-1\}} : \alpha \in S(n,t,k)\}$ and the set B containing all order-preserving and regressive maps from $\{1,\dots,t+k-2\}$ to $\{1,\dots,k+1\}$ have the same cardinality, we consider the restriction of $\alpha \in S(n,t,k)$ on $\{n-t+3,\dots,n-1\}$. It is clear that $\{n-t+3,\dots,n-1\}\alpha \subseteq \{(n-t+2)\alpha,\dots,n-t\}$. It remains to consider the restriction of $\alpha \in S(n,t,k)$ on $\{n-t-k+2,\dots,n-t+2\}$. By the same argument as before, our proof is finished. $\square$

Remark 3.3. From Remark 3.1 and Lemma 3.2 , we have that

1. $s(n,3,k) = C_{k+1}$ for $k = 0,1,\dots,n-4$.
2. For $t \geq 4$ and $k = 0,1,\dots,n-t-1$,
$$s(n,t,k) = C_{t+k-2} - \sum_{i=0}^{t-4} l(t+k-2,i).$$
3. For $t \geq 2$,
$$l(n,t) = \sum_{k=0}^{n-t-1} C_{n-t-k}\, s(n,t,k).$$

We also have an identity for Catalan numbers as follows:

Proposition 3.4. *For $m \geq n+1$,*
$$C_{n-1} = \sum_{i=0}^{n-2} s(m,n-i,i).$$

Proof. Since $\mathcal{C}[n+1] = L(n+1,0) \,\dot{\cup}\, L(n+1,1) \,\dot{\cup}\, L(n+1,2) \,\dot{\cup}\, \cdots \,\dot{\cup}\, L(n+1,n)$, it follows that C_{n+1} is

$$\begin{aligned}
&2C_n + \textstyle\sum_{i=2}^{n} l(n+1,i) \\
&= 2C_n + \textstyle\sum_{i=2}^{n}\sum_{j=0}^{n-i} |S(n+1,i,j)| \\
&= 2C_n + [C_{n-1}s(n+1,2,0) + C_{n-2}s(n+1,2,1) + \cdots + C_1 s(n+1,2,n-2)] \\
&\qquad + [C_{n-2}s(n+1,3,0) + \cdots + C_1 s(n+1,3,n-3)] \\
&\qquad \vdots \\
&\qquad + [C_2 s(n+1,n-1,0) + C_1 s(n+1,n-1,1)] \\
&\qquad + \ C_1 s(n+1,n,0).
\end{aligned}$$

The proof is finished, by applying the recursion for Catalan numbers: $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$. □

	0	1	2	3	4	5	6	7	8
2	1	1	2	5	14	42	132	429	1430
3	1	2	5	14	42	132	429	1430	4862
4	1	3	9	28	90	297	1001	3432	11934
5	1	4	14	48	165	572	2002	7072	25194
6	1	5	20	75	275	1001	3640	13260	48450
7	1	6	27	110	429	1638	6188	23256	87210

Table 3: $s(n,t,k)$

Using the recursion for Catalan numbers $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$, the following results are obtained.

1. $l(n,0) = l(n,1) = C_{n-1}$.
2. $l(n,2) = C_{n-1} - C_{n-2}$.
3. $l(n,3) = C_{n-1} - 2C_{n-2}$.

From Remark 3.3, for $t \geq 4$, $l(n,t)$ is obtained by following recurrence relation:

$$l(n,t) = \sum_{k=0}^{n-t-1} C_{n-t-k}\left(C_{t+k-2} - \sum_{i=0}^{t-4} l(t+k-2,i)\right). \tag{3.4.1}$$

Hence the following result is directly obtained.

Proposition 3.5. *For $k, m, n \in \mathbb{N}$, we have that:*

(*i*) $|\mathcal{C}[m_k]| = C_{m+k} - \sum_{i=0}^{k-1} l(m+k, i)$.

(*ii*) $|\mathcal{C}[m_k n]| = \sum_{i=0}^{n} |\mathcal{C}[m_{k+i}]| \cdot C_{n-i}$.

4. Cardinalities of Some Equivalence Classes

There are several equivalence relations on semigroups, which play an important role in the structure theory. For $\alpha \in \mathcal{T}(X)$, the symbol π_α will denote the partition of X induced by α, namely,

$$\pi_\alpha = \{x\alpha^{-1} \ : \ x \in \operatorname{ran}\alpha\}.$$

The Green's relations on $\mathcal{T}(X)$ are described by

$$\begin{array}{lll} \alpha\mathcal{L}\beta & \text{if and only if} & \operatorname{ran}\alpha = \operatorname{ran}\beta \\ \alpha\mathcal{R}\beta & \text{if and only if} & \pi_\alpha = \pi_\beta. \end{array}$$

For a nonempty subset Y of a chain X, given a transformation $\alpha : X \to Y$, the *skeleton* of α consists of the partial map of α by restricted its domain on Y and $\operatorname{ran}\alpha$. Define an equivalence relation $\mathcal{K}$ on $\mathcal{T}(X)$ by

$$\alpha\mathcal{K}\beta \quad \text{if and only if} \quad \alpha|_Y = \beta|_Y \ \text{ and } \ \operatorname{ran}\alpha = \operatorname{ran}\beta.$$

This relation helps us to classify semigroups of full regressive transformations with restricted range (see [5, 12]).

We denote $\mathcal{L}$-class, $\mathcal{R}$-class and $\mathcal{K}$-class containing α by L_α, R_α and K_α, respectively.

Proposition 4.1. *Let α be a right identity in $\mathcal{O}[m_{1_{k_1}} m_{2_{k_2}} \cdots {}_{k_{t-1}} m_t]$. Then $|L_\alpha| = |K_\alpha| = (k_1+1)\cdots(k_{t-1}+1)$ and $|R_\alpha| = 1$.*

Proof. Let α be a right identity in $\mathcal{O}[m_{1_{k_1}} m_{2_{k_2}} \cdots {}_{k_{t-1}} m_t]$. We have that $\operatorname{ran}\alpha = \lceil m_{1_{k_1}} m_{2_{k_2}} \cdots {}_{k_{t-1}} m_t \rceil$. It follows that L_α and K_α are the same. To build such a map $\beta \in L_\alpha$, by order-preserving property, there are $|\underline{a}| + 1$ ways in each lower-class $\underline{a}$ for sending via β into its range. Then we have the result. Next, we consider R_α. Since α is a right identity, it follows that for each $A \in \pi_\alpha$, $A \cap \operatorname{ran}\alpha = \{n_A\}$ and $A = (n_A)\alpha^{-1}$. As α is order-preserving, we have that A is one of the following convex sets: $\{n_A\}$, $\{n_A, n_A+1, \ldots, n_A+m\}$ or $\{n_A, n_A-1, \ldots, n_A-n\}$ for some $m, n \in \mathbb{N}$. These imply that R_α is trivial. $\square$

It is known [8] that every regressive subsemigroup S of $\mathcal{T}(X)$ when X is an ordered set, is $\mathcal{R}$-trivial, that is, $|R_\alpha| = 1$ for all $\alpha \in S$. In next propositions, we find the cardinality of $\mathcal{K}$-trivial subsets of $\mathcal{O}[m_{1k_1}m_{2k_2}\cdots{}_{k_{t-1}}m_t]$ and $\mathcal{C}[m_{1k_1}m_{2k_2}\cdots{}_{k_{t-1}}m_t]$, respectively. It is straightforward to verify the following lemma.

Lemma 4.2. *For $\alpha \in \mathcal{O}[m_k n]$, K_α is trivial if and only if one of the following statements hold*:

(*i*) $|(m+\{0,1,\ldots,k,k+1\})\alpha| = 1$.

(*ii*) $k = 1$ *and* $((m+1)\alpha)\alpha^{-1} = \{m+1\}$.

Theorem 4.3. *Suppose $k_i > 1$ for all $i = 1,\ldots,t-1$ and $m = m_1+\cdots+m_t$. Then we have:*

(*i*) $|\{\alpha \in \mathcal{O}[m_{1k_1}m_{2k_2}\cdots{}_{k_{t-1}}m_t] \ : \ K_\alpha$ *is trivial*$\}|$ *is* $\binom{2m-t}{m-1}$.

(*ii*) $|\{\alpha \in \mathcal{O}[m_{11}m_{21}\cdots{}_1m_t] \ : \ K_\alpha$ *is trivial*$\}|$ *is*

$$\binom{t-1}{0}\binom{2m-t}{m-1} + \binom{t-1}{1}\binom{2m-t}{m-3} + \cdots + \binom{t-1}{t-2}\binom{2m-t}{m-2t+3} + \binom{t-1}{t-1}\binom{2m-t}{m-2t+1}.$$

Proof. By applying Lemma 4.2, for each lower-class $\underline{a}$, we group $\underline{a}\cup\{\max\underline{a}+1, \min\underline{a}-1\}$ together. Then the cardinalities of $\{\alpha \in \mathcal{O}[m_{1k_1}m_{2k_2}\cdots{}_{k_{t-1}}m_t] \ : \ K_\alpha$ is trivial$\}$ and $\{\beta : \{1,\ldots,m-t+1\} \to \{1,\ldots,m\} \ : \ \beta$ is order-preserving$\}$ are the same. Then (i) is proved.

To show (ii), given a $(m+t)$-tuple $(x_1, x_2\ldots, x_{m+t})$ as an integral solution of the equation

$$x_1 + x_2 + \cdots + x_{m+t} = m-1$$

that satisfy the following conditions:

1. $x_i \geq 0$ for all $i \in \{1,\ldots,m+t\}$,
2. for $j \in \{m_1+1, m_1+m_2+2, \ldots, m_1+\cdots+m_{t-1}+t-1\}$, either $x_j = x_{j+1} = 0$ or $x_j, x_{j+1} \geq 1$.

By the condition 2, it guarantees that there is 1-1 corresponding between $(x_1, x_2\ldots, x_{m+t})$ and a map in $\{\alpha \in \mathcal{O}[m_{11}m_{21}\cdots{}_1m_t] \ : \ K_\alpha$ is trivial$\}$. It is clear that there are $\binom{2m-t}{m-1}$ solutions for the case $x_j = 0$ for all $j \in \{m_1+1, m_1+m_2+2, \ldots, m_1+\cdots+m_{t-1}+t-1\}$. For other cases, it can be proved directly. Then our proof is finished. ∎

For $\alpha \in \mathcal{O}[n]$, we denote $\mathrm{Fix}(\alpha) = \{x \in [n] \ : \ x\alpha = x\}$. For $Y \subseteq [n]$, we define

$$\mathcal{O}_Y[n] = \{\alpha \in \mathcal{O}[n] \ : \ \mathrm{Fix}(\alpha) = Y\} \ \text{ and } \ \mathcal{C}_Y[n] = \{\alpha \in \mathcal{C}[n] \ : \ \mathrm{Fix}(\alpha) = Y\}.$$

The following results were proved by G. Ayik, H. Ayik and M. Koc (see in [1]):

For $Y = \{m_1, m_2, \ldots, m_r\}$ with $m_1 < m_2 < \ldots < m_r$, we have

- $|\mathcal{O}_Y[n]| = \prod_{j=1}^{r+1} C_{k_j}$, where $k_1 = m_1 - 1$, $k_j = m_j - m_{j-1}$ $(2 \le j \le r)$ and $k_{r+1} = n - m_r$.
- $|\mathcal{C}_Y[n]| = \prod_{j=1}^{r} C_{k_j - 1}$, $(m_1 = 1)$, where $k_j = m_{j+1} - m_j$ $(1 \le j \le r-1)$ and $k_r = n - m_r + 1$.

Lemma 4.4. *For $\alpha \in \mathcal{C}[m_k n]$, K_α is trivial if and only if one of the following statements hold:*

(i) $|(m + \{0, 1, \ldots, k, k+1\})\alpha| = 1$.

(ii) $k = 1$ *and* $((m+1)\alpha)\alpha^{-1} = \{m+1\}$.

(iii) $(m)\alpha = m$.

By Lemma 4.4 (iii), it is clear that the set of all maps $\alpha \in \mathcal{C}[m_{1\,k_1} m_{2\,k_2} \cdots {}_{k_{t-1}} m_t]$ such that $\{m_1, m_1 + k_1 + m_2, \ldots, m_1 + k_1 + m_2 + \cdots + k_{t-2} + m_{t-1}\} \subseteq \mathrm{Fix}(\alpha)$, is $\mathcal{K}$-trivial. By applying the result in [1], the following proposition is obtained.

Proposition 4.5. *For $Y = \{1, m_1, m_1 + k_1 + m_2, \ldots, m_1 + k_1 + m_2 + \cdots + k_{t-2} + m_{t-1}\}$, $\mathcal{C}_Y[m_{1\,k_1} m_{2\,k_2} \cdots {}_{k_{t-1}} m_t]$ is $\mathcal{K}$-trivial with $C_{m_1-2} C_{m_2-1} \cdots C_{m_{t-1}-1} C_{m_t}$ as its cardinality.*

Proposition 4.6. *Let γ be the right identity in $V_{t-1}[m_{1\triangleright} m_{2\triangleright} \cdots {}_{\triangleright} m_t]$. Then $\{\gamma\alpha \ : \ \alpha \in \mathcal{C}[m_{1\,k_1} m_{2\,k_2} \cdots {}_{k_{t-1}} m_t]\}$ is $\mathcal{K}$-trivial with $C_{m_1 + \cdots + m_t - t + 1}$ as its cardinality.*

Proof. It is clear that for $\alpha \in \mathcal{C}[m_{1\,k_1} m_{2\,k_2} \cdots {}_{k_{t-1}} m_t]$, $\gamma\alpha$ satisfies (i) in Lemma 4.4. Clearly, $\mathrm{ran}\,(\gamma\beta) \subseteq \mathrm{ran}\,\gamma$ and $|\mathrm{ran}\,\gamma| = m_1 + \cdots + m_t - t + 1$. That is, $\{\gamma\alpha \ : \ \alpha \in \mathcal{C}[m_{1\,k_1} m_{2\,k_2} \cdots {}_{k_{t-1}} m_t]\}$ is isomorphic to $\mathcal{C}[m_1 + \cdots + m_t - t + 1]$. □

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