COMBINATORIAL RESULTS FOR SEMIGROUPS OF
ORDER-PRESERVING FULL TRANSFORMATIONS
WITH PATTERN OF RANGE

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Abstract: Let $O[n]$ and $C[n]$ be the semigroup of all order-preserving and all order-preserving and regressive transformations on the set $[n] = \{1, \ldots, n\}$, respectively. In this paper, we give a pattern on $[n]$. Combinatorial properties on subsemigroups of $O[n]$ with restricted range to the pattern are investigated.

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Key Words: order-preserving transformation semigroups, regressive transformation semigroups, Fibonacci numbers, Catalan numbers

1. Introduction

For a set $[n] = \{1, \ldots, n\}$, $T[n]$ denotes the full transformation semigroup on $[n]$. We shall call a transformation $\alpha : [n] \to [n]$ order-preserving if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in [n]$, and regressive (or order-decreasing) if $x\alpha \leq x$ for all $x \in [n]$. We denote by $O[n]$ the semigroup of all full order-preserving transformations on $[n]$ and denote by $R[n]$ the semigroup of all full regressive transformations on $[n]$. Let $C[n] = O[n] \cap R[n]$.
On these semigroups, combinatorial results are well understood by now. For example, Howie in [4] showed that $|O[n]| = \binom{2n-1}{n}$ and $|EO[n]| = F_{2n}$, where $EO[n]$ is the set of all idempotents of $O[n]$ and $F_{2n}$ is the alternate Fibonacci number given by $F_1 = F_2 = 1$. Higgins in [2] and later in (alternative version) [6, 7] Laradji and Umar proved that $|C[n]| = \frac{1}{n+1}(\binom{2n}{n})$, the $n$th Catalan number $C_n$.

In 1975, Symons [11] introduced the subsemigroup of full transformation with restricted range and later Sanwong in [10] studied its regularity. Recently in [5, 12] some classifications on subsemigroups of full order-preserving (regressive) transformations with restricted range were investigated. These papers motivated the study of combinatorial properties of semigroups of transformations with restricted range.

For $t \in \mathbb{N}$ and $n_1, \ldots, n_t, m_1, \ldots, m_t \in \mathbb{N}_0$, a pattern $[m_1 n_1 m_2 n_2 \cdots m_t n_t]$ is defined to be

$$\{1, 2, \ldots, \sum_{i=1}^{t} (m_i + n_i)\}$$

having the structure of two disjoint subsets

$$\{1, \ldots, m_1\} \cup \bigcup_{k=1}^{t} \left( \sum_{i=1}^{k} (m_i + n_i) + \{1, \ldots, m_{k+1}\} \right),$$

denoted by $[m_1 n_1 m_2 n_2 \cdots m_t n_t]$, and

$$(m_1 + \{1, \ldots, n_1\}) \cup \bigcup_{k=1}^{t-1} \left( m_1 + \sum_{i=1}^{k} (n_i + m_{i+1}) + \{1, \ldots, n_{k+1}\} \right),$$

denoted by $[m_1 n_1 m_2 n_2 \cdots m_t n_t]$, respectively.

For example, $[1231] = \{1, 2, 3, 4, 5, 6, 7\}$ with $[1231] = \{1, 4, 5, 6\}$ and $[12\ 31] = \{2, 3, 7\}$.

Note that for $i = 1, \ldots, t$, if $m_i = 0$, then

$$[m_1 n_1 \cdots m_i n_i \cdots m_t n_t] = [m_1 n_1 \cdots m_{i-1} n_{i-1} n_i \cdots m_t n_t],$$

and if $n_i = 0$, then

$$[m_1 n_1 \cdots m_i n_i \cdots m_t n_t] = [m_1 n_1 \cdots (m_i + m_{i+1}) n_{i+1} \cdots m_t n_t].$$

For brevity, we shall omit to write 0 in any positions of the pattern if it is 0. For example, $[1250] := [0250]$, $[2] := [02]$ and $[5] := [50]$. When all positions are 0, we will write $[0]$ instead.

Let $S$ be the set containing all these elements. We define a binary operation on $S$ in natural way of the concatenation, for example, $[121][1321] = [122321]$. 


$[3][1,3] = [4,1,3]$ and $[1][1,3] = [2,3]$. Then $S$ is a free monoid with $[0]$ as the identity element.

For $[x] \in S$, by considering $[x]$ ($|[x]|$) as the skeleton of $[x]$ and grouping elements in $[x]$ ($|[x]|$) into classes, we have that each class contains all elements in $[x]$ ($|[x]|$) which has no elements in $[x]$ ($|[x]|$) lies between them and it is said to be a lower (upper)-class of $[x]$. A lower-class containing $a$ and an upper-class containing $b$ of $[x]$ are denoted by $\underline{a}$ and $\overline{b}$, respectively.

Let $T[x]$ be the full transformation semigroup under composition of all mapping of $[x]$ into $[x]$. For $O[x]$, $R[x]$ and $C[x]$ are defined analogously.

We already know that the number of integral solutions of the equation

$$x_1 + x_2 + \cdots + x_{n+1} = n - 1$$

that satisfy $x_i \geq 0$ is equal to the cardinality of $O[n]$, $\binom{2n-1}{n-1}$. For example, when $n = 7$

- $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (1, 0, 0, 1, 1, 3, 0, 0)$ is the representation of
  \[
  \begin{array}{ccccccc}
  0 & 0 & 0 & 1 & 1 & 3 & 0 & 0
  \end{array}
  \]

- $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (0, 0, 1, 1, 0, 2, 1, 1)$ is the representation of
  \[
  \begin{array}{ccccccc}
  0 & 0 & 1 & 1 & 0 & 2 & 1 & 1
  \end{array}
  \]

It is clear that the cardinality of the set containing all order-preserving from $\{1, \ldots, p\}$ to $\{1, \ldots, q\}$ is $\binom{p+q-1}{p}$. Consequently, the following result is obtained.

**Theorem 1.1.** The cardinality of $O[m_1, k_1, m_2, k_2, \cdots, k_{l-1}, m_t]$ is $\binom{2m+k-1}{m-1}$ where $m = m_1 + \cdots + m_t$ and $k = k_1 + \cdots + k_{l-1}$.

The paper is organized as follows: In Section 2 and 3, we deal with cardinalities of $EO[m_1, k_1, m_2, k_2, \cdots, k_{l-1}, m_t]$, $C[k]$, and $C[m_k n]$. In Section 4, cardinalities of some equivalence classes are investigated.

### 2. The Number of Idempotents

To find the cardinality of $EO[m_1, k_1, m_2, k_2, \cdots, k_{l-1}, m_t]$, our strategy will be to look at the sets having some types as follows:
For \( \alpha \in E\mathcal{O}[m_{1k_1}m_{2k_2}\cdots k_{t-1}m_t] \), if there are exactly distinct \( n \) lower-classes, \( g_1,\ldots,g_n \) of \([m_{1k_1}m_{2k_2}\cdots k_{t-1}m_t]\) such that for \( i \in \{1,\ldots,n\} \),
\[
|\{(g_i \cup \{\min g_i - 1, \max g_i + 1\})\alpha| = 1,
\]
then \( \alpha \) is called a type of \( n \) closed lower-classes (or \( t - 1 - n \) open lower-classes).

For \( s \in \mathbb{N}_0 \), let \( U_s[m_{1k_1}m_{2k_2}\cdots k_{t-1}m_t] \) and \( V_s[m_{1k_1}m_{2k_2}\cdots k_{t-1}m_t] \) stand for the set of all idempotents in \( \mathcal{O}[m_{1k_1}m_{2k_2}\cdots k_{t-1}m_t] \) of the type \( s \) open lower-classes and \( s \) closed lower-classes, respectively.

**Example.** In \( EO[2_22_21] \),
\[
\alpha = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{pmatrix} \in V_1[2_22_21] = U_1[2_22_21],
\]
\[
\beta = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{pmatrix} \in V_2[2_22_21] = U_0[2_22_21].
\]
Note that in the structure of \( \alpha \), there is the shape of a right triangle, \( \nabla \), from \((\{5,6\} \cup \emptyset \cup \{9\})\alpha = \{5\},
\[
\begin{array}{c}
\hline
\hline
\hline
\end{array}
\]
whereas in the structure of \( \beta \), there is the shape of two right triangles, \( \nabla \nabla \), from \((\{2\} \cup \emptyset \cup \{5\})\beta = \{2\} \) and \((\{6\} \cup \emptyset \cup \{9\})\beta = \{9\},
\[
\begin{array}{c}
\hline
\hline
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\begin{array}{c}
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\end{array}
\]
We denote
\[
V_1[m_{1k_1}\cdots k_{t-1}m_t] \text{ and } V_1[m_{1k_1}\cdots k_{t-1}m_t]
\]
as two subsets of \( V_1[m_{1k_1}\cdots k_{t-1}m_t] \), with the closed lower-class at \( g_i = (m_i + \sum_{j=1}^{i-1}(m_j + k_j)) + \{1,\ldots,k_i\} \) in the structure of \( \nabla \) and \( \nabla \), respectively. For example,
\[
V_1[2_22_21] = V_1[2_02_21] \cup V_1[2_22_01] \cup V_1[2_22_21] \cup V_1[2_22_21].
\]
For \( V_n[m_{1k_1}m_{2k_2}\cdots k_{t-1}m_t] \), it can be written as a disjoint union of \( 2^n(t-1) \) sets defined in analogous way. For example,
\[
V_2[2_22_21] = V_2[2_02_21] \cup V_2[2_22_01] \cup V_2[2_22_21] \cup V_2[2_22_21].
\]
The following lemma is obtained immediately.
Lemma 2.1. \(|EO[m_{k_1}m_{2k_2} \cdots m_{k_{t-1}}]| = \sum_{i=0}^{t-1} |V_i[m_{k_1}m_{2k_2} \cdots m_{k_{t-1}}]|.\)

Lemma 2.2. For \(V_1[m_{k}n]\), we have that
\[|V_1[m_{k}n]| = F_{2m-1}F_{2n} \quad \text{and} \quad |V_1[m_{k}n]| = F_{2m}F_{2n-1}.\]

Proof. To illustrate the cardinality of \(V_1[m_{k}n]\), we consider the following cases:

\[
\begin{align*}
F_{2m-2} & \quad F_{2n} & \quad F_{2m-4} & \quad F_{2n} & \quad F_{2m-6} & \quad F_{2n} & \quad \ldots.
\end{align*}
\]

Then \(|V_1[m_{k}n]| = (\sum_{i=1}^{m} F_{2(m-i)} + 1) F_{2n}$. By using the Fibonacci identity: \(\sum_{i=1}^{m} F_{2i} = F_{2n+1} - 1\), it follows that \(|V_1[m_{k}n]| = F_{2m-1}F_{2n}\) as wanted. For the rest, it can be proved in the same fashion.

From Lemma 2.1 and 2.2, the following proposition is obtained.

Proposition 2.3. The cardinality of \(EO[m_{k}n]\) is
\[(k + 1)F_{2m}F_{2n} + F_{2m-1}F_{2n} + F_{2m}F_{2n-1}.\]

As a consequence of Proposition 2.3 and Howie’s result, by taking \(k = 0\), we have the following conclusion.

Corollary 2.4. The cardinality of \(EO[m_{k}n]\) is \(kF_{2m}F_{2n} + F_{2(m+n)}\).

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<td>(8k + 21)</td>
<td>(21k + 55)</td>
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<td>(63k + 144)</td>
<td>(168k + 377)</td>
<td>(441k + 987)</td>
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<td>(20736k + 46368)</td>
</tr>
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Table 1: The cardinality of \(EO[m_{k}n]\)

Proposition 2.5. The cardinality of \(EO[m_{k_1}m_{2k_2}m_3]\) is
\[k_1k_2F_{2m_1}F_{2m_2}F_{2m_3} + k_1F_{2m_1}F_{2(m_2+m_3)} + k_2F_{2(m_1+m_2)}F_{2m_3} + F_{2(m_1+m_2+m_3)}\].
Proof. It is clear that $|V_0[m_{1k_1}m_{2k_2}m_3]| = (k_1 + 1)(k_2 + 1)F_{2m_1}F_{2m_2}F_{2m_3}$. Since $V_1[m_{1k_1}m_{2k_2}m_3]$ is

$$V_1[m_{1k_1}m_{2k_2}m_3] \cup V_1[m_{1k_1}m_{2k_2}m_3] \cup V_1[m_{1k_1}m_{2k_2}m_3] \cup V_1[m_{1k_1}m_{2k_2}m_3]$$

and $V_2[m_{1k_1}m_{2k_2}m_3]$ is

$$V_2[m_{1k_1}m_{2k_2}m_3] \cup V_2[m_{1k_1}m_{2k_2}m_3] \cup V_2[m_{1k_1}m_{2k_2}m_3] \cup V_2[m_{1k_1}m_{2k_2}m_3],$$

by applying Lemma 2.2, we have

$$|V_1[m_{1k_1}m_{2k_2}m_3]| = (k_1 + 1)[F_{2m_1}F_{2m_2}F_{2m_3-1} + F_{2m_1}F_{2m_2-1}F_{2m_3}],$$

$$|V_2[m_{1k_1}m_{2k_2}m_3]| = F_{2m_1}F_{2m_2-1}F_{2m_3-1} + F_{2m_1-1}F_{2m_2-1}F_{2m_3} + F_{2m_1-1}F_{2m_2}F_{2m_3-1} + F_{2m_1}F_{2m_2-2}F_{2m_3}. $$

Using the Fibonacci identity: $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$ and Howie’s result when $k_1 = k_2 = 0$, our proof is finished. \qed

From Proposition 2.5, we have an identity of 3-term Fibonacci numbers (see also in [3] and [9]): $F_{2(m_1+m_2+m_3)}$ is

$$2F_{2m_1}F_{2m_2}F_{2m_3} + F_{2m_1}F_{2m_2+1}F_{2m_3-1} + F_{2m_1+1}F_{2m_2-1}F_{2m_3} + F_{2m_1-1}F_{2m_2}F_{2m_3+1}. $$

As a polynomial in the variables $k_1, \ldots, k_{t-1}$ with coefficient in $\mathbb{Z}$, the leading coefficient of $k_1 \cdots k_d$ in $|EO[m_{1k_1}m_{2k_2} \cdots k_{t-1}m_1]|$ is denoted by $c(k_1 \cdots k_d)$. 

Lemma 2.6. In $|EO[m_{1k_1}m_{2k_2} \cdots k_{t-1}m_1]|$, $c(k_1k_{i+n_1} \cdots k_{i+n_1+\cdots+n_r})$ is

$$F_{2(m_1+m_1+m_1)}F_{2(m_1+\cdots+m_1)} \cdots F_{2(m_1+\cdots+m_1+n_r+\cdots+m_1+n_r)}. $$

Proof. By Corollary 2.4 and Proposition 2.5, the result holds for $|EO[m_{1k_1}m_{2k_2}]|$ and $|EO[m_{1k_1}m_{2k_2}m_3]|$. We prove the result by induction on the number of lower-classes. Suppose the result holds for $|EO[m_{1k_1}m_{2k_2} \cdots k_{p-1}m_p]|$ where $p = 2, \ldots, t-1$.

To find $c(k_1k_{i+n_1} \cdots k_{i+n_1+\cdots+n_r})$, let $q \in \{ i, i+n_1, \ldots, i+n_1+\cdots+n_r \}$. By considering all terms having $k_q$ as a factor in $|EO[m_{1k_1}m_{2k_2} \cdots k_{t-1}m_1]|$, it suffices to consider the set of all idempotent such that $g_q = (m_i + \sum_{j=1}^{q-1}(m_j + k_j)) + \{ 1, \ldots, k_q \}$ is an open lower-class, namely $A_q$. Then

$$|A_q| = (k_q + 1)|EO[m_{1k_1} \cdots k_{q-1}m_q]| \cdot |EO[m_{q+1k_{q+1}} \cdots k_{t-1}m_t]|.$$
WLOG, we let \( q = i + (n_1 + \cdots + n_d) \). Then
\[
c(k_i k_{i+n_1} \cdots k_{i+n_1+\cdots+n_{d-1}} k_{i+n_1+\cdots+n_{d-1}+n_d} k_{i+n_1+\cdots+n_{d+1}} \cdots k_{i+n_1+\cdots+n_r})
\]
is the product of
\[
c(k_i k_{i+n_1} \cdots k_{i+n_1+\cdots+n_{d-1}}) \text{ and } c(k_{i+n_1+\cdots+n_{d+1}} \cdots k_{i+n_1+\cdots+n_r})
\]
in \( |EO[m_1 k_1 \cdots k_{q-1} m_q]| \) and \( |EO[m_{q+1} k_{q+1} \cdots k_{t-1} m_t]| \), respectively. By induction, the proof is finished. \( \square \)

**Theorem 2.7.** The cardinality of \( EO[m_1 k_1 m_2 k_2 \cdots k_{t-1} m_t] \) is

\[
F_{2(m_1+m_2+\cdots+m_l)} + \sum_{A \in \mathcal{P}([1,\ldots,t-1])} [c(\prod_{i \in A} k_i) \cdot \prod_{i \in A} k_i]
\]

**Proof.** From Lemma 2.6 and by using Howie’s result when \( k_1 = k_2 = \cdots = k_{t-1} = 0 \), our proof is finished. \( \square \)

From theorem above, the cardinality of \( EO[m_1 k_1 m_2 k_2 m_3 k_3 m_4] \) is

\[
F_{2(m_1+m_2+m_3+m_4)} + k_1 k_2 k_3 F_{2m_1} F_{2m_2} F_{2m_3} F_{2m_4} + k_1 F_{2m_1} F_{2(m_2+m_3+m_4)} + k_2 F_{2(m_1+m_2)} F_{2(m_3+m_4)}
\]

\[
+ k_3 F_{2(m_1+m_2+m_3)} F_{2m_4} + k_1 k_2 F_{2m_1} F_{2m_2} F_{2(m_3+m_4)} + k_1 k_3 F_{2m_1} F_{2(m_2+m_3)} F_{2m_4}
\]

\[
+ k_2 k_3 F_{2(m_1+m_2)} F_{2m_3} F_{2m_4}.
\]

**Corollary 2.8.** The cardinality of \( EO[m_k m_m \cdots k_m] \), (with \( n - 1 \) lower-classes), is

\[
F_{2nm} + \sum_{p_1+p_2=n} F_{2(p_1m)} F_{2(p_2m)} k + \sum_{p_1+p_2+p_3=n} F_{2(p_1m)} F_{2(p_2m)} F_{2(p_3m)} k^2 + \cdots + \sum_{p_1+\cdots+p_{n-1}=n} F_{2(p_1m)} \cdots F_{2(p_{n-1}m)} k^{n-2} + (F_{2m})^n k^{n-1}.
\]

### 3. The Number of Regressive Maps

We aim now to find the cardinality of \( C[m_k] \) and \( C[m_k n] \), respectively. Let \( l(n, t) \) be the cardinality of the set \( L(n, t) \) containing all maps in \( C[n] \) whose \( n \) is sent to \( n - t \). It is clear that \( l(n, 0) = l(n, 1) = C_{n-1} \).
Table 2: The cardinality of $EO[m_km_k\cdots k_m]$ with $n - 1$ lower-classes

To find $l(n, t)$ when $t \geq 2$, let $M(n, t, k)$ be the set containing all maps $\alpha$ in $L(n, t)$ such that $(n - t - k + 1)\alpha = n - t - k$ for $k \in \{0, 1, \ldots, n - t - 1\}$ and let

$$S(n, t, k) := M(n, t, k) \bigcup_{i=0}^{k-1} S(n, t, i).$$

It is clear that $|S(n, t, 0)| = C_{n-t}$.

**Remark 3.1.** The following results are obtained immediately:

1. $L(n, t) = S(n, t, 0) \cup S(n, t, 1) \cup \cdots \cup S(n, t, n - t - 1)$.
2. $C[n] = L(n, 0) \cup L(n, 1) \cup \cdots \cup L(n, n - 1)$.
3. $C[n_m] = L(n + m, m) \cup L(n + m, m + 1) \cup \cdots \cup L(n + m, n + m - 1)$.

Consider $S(n, t, k)$ for all $k = 0, 1, \ldots, n - t - 1$. Given $\alpha \in S(n, t, k)$. We firstly consider the structure of $\alpha$ as follows:

\[
\begin{array}{cccccccccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(\alpha_{n-t-k+1}) & (\alpha_{n-t-k}) & (\alpha_{n-t-k-1}) & (\alpha_{n-t-k-2}) & (\alpha_{n-t-k-3}) & \cdots & (\alpha_{n-t-1}) & (\alpha_{n-t}) & \cdots & (\alpha_{n-t-k}) & (\alpha_{n-t-k-1}) & (\alpha_{n-t-k-2}) & (\alpha_{n-t-k+1}) & \vdots & \vdots \\
\end{array}
\]

It is clear that

$$\left|\{\alpha|_{(1, \ldots, n-t-k)} : \alpha \in S(n, t, k)\}\right| = C_{n-t-k}.$$  

We denote $s(n, t, k)$ as the cardinality of $\{\alpha|_{(n-t-k+2, \ldots, n-1)} : \alpha \in S(n, t, k)\}$.

**Lemma 3.2.** The following statements hold:

(i) $s(n, 2, k) = C_{k}$ for $k = 0, 1, \ldots, n - 3$.

(ii) $s(n, t, k) = |C[(k + 1)t-3]|$ for $t \geq 3$ and $k = 0, 1, \ldots, n - t - 1$.  

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<td>$21 + 25k + 9k^2 + k^3$</td>
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<tr>
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<td>$14930352 + 13353984k + 2985984k^2$</td>
<td>$14930352 + 13353984k + 2985984k^2 + 429981696k^3$</td>
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Proof. (i) Let $\alpha \in S(n, 2, k)$. Suppose $k > 0$. Then $(n-1)\alpha \leq n-3$. We consider $\alpha|_{\{n-k, \ldots, n-1\}} : \{n-k, \ldots, n-1\} \to \{n-2-k, \ldots, n-3\}$ where $R = \{n-2-k, \ldots, n-3\}$.

Then $\alpha|_{D_2}$ acts as an order-preserving map $\beta$ on $\{1, \ldots, k\}$. It remains to show that $\beta$ is regressive. Suppose there is $x \in \{1, \ldots, k\}$ such that $(x)\beta > x$. Since $\alpha$ is regressive, it implies that either $(x)\beta = x+1$ or $(x)\beta = x+2$. Then $\alpha \in S(n, 2, t)$ for some $t < k$ which is a contradiction. Then $s(n, 2, k) = C_k$.

(ii) For any $\alpha \in S(n, t, k)$, we have that $\{n-t-k+2, \ldots, n-1\} \alpha \subseteq \{n-t-k, \ldots, n-t\}$.

To show that $\{\alpha|_{\{n-t-k+2, \ldots, n-1\}} : \alpha \in S(n, t, k)\}$ and the set $B$ containing all order-preserving and regressive maps from $\{1, \ldots, t+k-2\}$ to $\{1, \ldots, k+1\}$ have the same cardinality, we consider the restriction of $\alpha \in S(n, t, k)$ on $\{n-t+3, \ldots, n-1\}$. It is clear that $\{n-t+3, \ldots, n-1\} \alpha \subseteq \{(n-t+2)\alpha, \ldots, n-t\}$. It remains to consider the restriction of $\alpha \in S(n, t, k)$ on $\{n-t-k+2, \ldots, n-t+2\}$. By the same argument as before, our proof is finished. \qed

Remark 3.3. From Remark 3.1 and Lemma 3.2, we have that

1. $s(n, 3, k) = C_{k+1}$ for $k = 0, 1, \ldots, n-4$.
2. For $t \geq 4$ and $k = 0, 1, \ldots, n-t-1$,

$$s(n, t, k) = C_{t+k-2} - \sum_{i=0}^{t-4} l(t+k-2, i).$$

3. For $t \geq 2$,

$$l(n, t) = \sum_{k=0}^{n-t-1} C_{n-t-k}s(n, t, k).$$

We also have an identity for Catalan numbers as follows:

**Proposition 3.4.** For $m \geq n+1$,

$$C_{n-1} = \sum_{i=0}^{n-2} s(m, n-i, i).$$
Proof. Since $C_n + \sum_{i=2}^{n} l(n, i)$

\[
2C_n + \sum_{i=2}^{n} l(n, i) = 2C_n + \sum_{i=2}^{n} \sum_{j=0}^{n-i} |S(n+1, i, j)|
= 2C_n + [C_{n-1}s(n+1, 2, 0) + C_{n-2}s(n+1, 2, 1) + \cdots + C_1s(n+1, 2, n-2)]
+ [C_{n-2}s(n+1, 3, 0) + \cdots + C_1s(n+1, 3, n-3)]
+ \cdots
+ [C_2s(n+1, n-1, 0) + C_1s(n+1, n-1, 1)]
+ C_1s(n+1, n, 0).
\]

The proof is finished, by applying the recursion for Catalan numbers: $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$. 

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Table 3: $s(n, t, k)$

Using the recursion for Catalan numbers $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$, the following results are obtained.

1. $l(n, 0) = l(n, 1) = C_{n-1}$.
2. $l(n, 2) = C_{n-1} - C_{n-2}$.
3. $l(n, 3) = C_{n-1} - 2C_{n-2}$.

From Remark 3.3, for $t \geq 4$, $l(n, t)$ is obtained by following recurrence relation:

\[
l(n, t) = \sum_{k=0}^{n-t-1} C_{n-t-k} \left( C_{t+k-2} - \sum_{i=0}^{t-4} l(t + k - 2, i) \right).
\]  

(3.4.1)

Hence the following result is directly obtained.
Proposition 3.5. For $k, m, n \in \mathbb{N}$, we have that:

1. $|\mathcal{C}[m_k]| = C_{m+k} - \sum_{i=0}^{k-1} l(m+k,i)$.

2. $|\mathcal{C}[m_kn]| = \sum_{i=0}^{n} |\mathcal{C}[m_{k+i}]| \cdot C_{n-i}$.

4. Cardinalities of Some Equivalence Classes

There are several equivalence relations on semigroups, which play an important role in the structure theory. For $\alpha \in \mathcal{T}(X)$, the symbol $\pi_\alpha$ will denote the partition of $X$ induced by $\alpha$, namely,

\[ \pi_\alpha = \{x\alpha^{-1} : x \in \text{ran } \alpha\}. \]

The Green’s relations on $\mathcal{T}(X)$ are described by

- $\alpha \mathcal{L} \beta$ if and only if $\text{ran } \alpha = \text{ran } \beta$
- $\alpha \mathcal{R} \beta$ if and only if $\pi_\alpha = \pi_\beta$.

For a nonempty subset $Y$ of a chain $X$, given a transformation $\alpha : X \to Y$, the skeleton of $\alpha$ consists of the partial map of $\alpha$ by restricted its domain on $Y$ and $\text{ran } \alpha$. Define an equivalence relation $\mathcal{K}$ on $\mathcal{T}(X)$ by

\[ \alpha \mathcal{K} \beta \text{ if and only if } \alpha|_Y = \beta|_Y \text{ and } \text{ran } \alpha = \text{ran } \beta. \]

This relation helps us to classify semigroups of full regressive transformations with restricted range (see [5, 12]).

We denote $\mathcal{L}$-class, $\mathcal{R}$-class and $\mathcal{K}$-class containing $\alpha$ by $L_\alpha$, $R_\alpha$ and $K_\alpha$, respectively.

Proposition 4.1. Let $\alpha$ be a right identity in $\mathcal{O}[m_{1k_1}m_{2k_2} \cdots k_{t-1}m_t]$. Then $|L_\alpha| = |K_\alpha| = (k_1 + 1) \cdots (k_{t-1} + 1)$ and $|R_\alpha| = 1$.

Proof. Let $\alpha$ be a right identity in $\mathcal{O}[m_{1k_1}m_{2k_2} \cdots k_{t-1}m_t]$. We have that $\text{ran } \alpha = [m_{1k_1}m_{2k_2} \cdots k_{t-1}m_t]$. It follows that $L_\alpha$ and $K_\alpha$ are the same. To build such a map $\beta \in L_\alpha$, by order-preserving property, there are $|a| + 1$ ways in each lower-class $a$ for sending via $\beta$ into its range. Then we have the result. Next, we consider $R_\alpha$. Since $\alpha$ is a right identity, it follows that for each $A \in \pi_\alpha$, $A \cap \text{ran } \alpha = \{n_A\}$ and $A = (n_A)\alpha^{-1}$. As $\alpha$ is order-preserving, we have that $A$ is one of the following convex sets: $\{n_A\}$, $\{n_A, n_A + 1, \ldots, n_A + m\}$ or $\{n_A, n_A - 1, \ldots, n_A - n\}$ for some $m, n \in \mathbb{N}$. These imply that $R_\alpha$ is trivial. \(\square\)
It is known [8] that every regressive subsemigroup $S$ of $\mathcal{T}(X)$ when $X$ is an ordered set, is $\mathcal{R}$-trivial, that is, $|R_\alpha| = 1$ for all $\alpha \in S$. In next propositions, we find the cardinality of $\mathcal{K}$-trivial subsets of $O[m_{1k_1}m_{2k_2}\cdots k_{t-1}m_t]$ and $\mathcal{C}[m_{1k_1}m_{2k_2}\cdots k_{t-1}m_t]$, respectively. It is straightforward to verify the following lemma.

**Lemma 4.2.** For $\alpha \in O[m_{kn}]$, $K_\alpha$ is trivial if and only if one of the following statements hold:

(i) $|(m + \{0, 1, \ldots, k, k + 1\})\alpha| = 1$.

(ii) $k = 1$ and $((m + 1)\alpha)\alpha^{-1} = \{m + 1\}$.

**Theorem 4.3.** Suppose $k_i > 1$ for all $i = 1, \ldots, t-1$ and $m = m_1+\cdots+m_t$. Then we have:

(i) $|\{\alpha \in O[m_{1k_1}m_{2k_2}\cdots k_{t-1}m_t] : K_\alpha \text{ is trivial}\}|$ is $\binom{2m-t}{m-1}$.

(ii) $|\{\alpha \in O[m_{1k_1}m_{2k_2}\cdots k_{t-1}m_t] : K_\alpha \text{ is trivial}\}|$ is

\[
\begin{pmatrix} t-1 \\ 0 \end{pmatrix} \binom{2m-t}{m-1} + \begin{pmatrix} t-1 \\ 1 \end{pmatrix} \binom{2m-t}{m-3} + \cdots + \begin{pmatrix} t-1 \\ t-2 \end{pmatrix} \binom{2m-t}{m-2t+3} + \begin{pmatrix} t-1 \\ t-1 \end{pmatrix} \binom{2m-t}{m-2t+1}.
\]

**Proof.** By applying Lemma 4.2, for each lower-class $\underline{a}$, we group $\underline{a} \cup \{\max \underline{a} + 1, \min \underline{a} - 1\}$ together. Then the cardinalities of $\{\alpha \in O[m_{1k_1}m_{2k_2}\cdots k_{t-1}m_t] : K_\alpha \text{ is trivial}\}$ and $\{\beta : \{1, \ldots, m-t+1\} \to \{1, \ldots, m\} : \beta \text{ is order-preserving}\}$ are the same. Then (i) is proved.

To show (ii), given a $(m+t)$-tuple $(x_1, x_2, \ldots, x_{m+t})$ as an integral solution of the equation

\[x_1 + x_2 + \cdots + x_{m+t} = m - 1\]

that satisfy the following conditions:

1. $x_i \geq 0$ for all $i \in \{1, \ldots, m+t\}$,

2. for $j \in \{m_1 + 1, m_1 + m_2 + 2, \ldots, m_1 + \cdots + m_{t-1} + t - 1\}$, either $x_j = x_{j+1} = 0$ or $x_j, x_{j+1} \geq 1$.

By the condition 2, it guarantees that there is 1-1 corresponding between $(x_1, x_2, \ldots, x_{m+t})$ and a map in $\{\alpha \in O[m_{1m_1}m_{2m_2}\cdots m_t] : K_\alpha \text{ is trivial}\}$. It is clear that there are $\binom{2m-t}{m-1}$ solutions for the case $x_j = 0$ for all $j \in \{m_1 + 1, m_1 + m_2 + 2, \ldots, m_1 + \cdots + m_{t-1} + t - 1\}$. For other cases, it can be proved directly. Then our proof is finished. \qed
For $\alpha \in \mathcal{O}[n]$, we denote $\text{Fix}(\alpha) = \{x \in [n] : x\alpha = x\}$. For $Y \subseteq [n]$, we define

$$\mathcal{O}_Y[n] = \{\alpha \in \mathcal{O}[n] : \text{Fix}(\alpha) = Y\} \quad \text{and} \quad \mathcal{C}_Y[n] = \{\alpha \in \mathcal{C}[n] : \text{Fix}(\alpha) = Y\}.$$  

The following results were proved by G. Ayik, H. Ayik and M. Koc (see in [1]):

For $Y = \{m_1, m_2, \ldots, m_r\}$ with $m_1 < m_2 < \ldots < m_r$, we have

- $|\mathcal{O}_Y[n]| = \prod_{j=1}^{r+1} C_{k_j}$, where $k_1 = m_1 - 1$, $k_j = m_j - m_{j-1}$ ($2 \leq j \leq r$) and $k_{r+1} = n - m_r$.  

- $|\mathcal{C}_Y[n]| = \prod_{j=1}^{r} C_{k_{j-1}}$, where $k_j = m_{j+1} - m_j$ ($1 \leq j \leq r - 1$) and $k_r = n - m_r + 1$.

**Lemma 4.4.** For $\alpha \in \mathcal{C}[m_{k}n]$, $K_{\alpha}$ is trivial if and only if one of the following statements hold:

(i) $|(m + \{0, 1, \ldots, k, k + 1\})\alpha| = 1$.

(ii) $k = 1$ and $((m + 1)\alpha)^{-1} = \{m + 1\}$.

(iii) $(m)\alpha = m$.

By Lemma 4.4 (iii), it is clear that the set of all maps $\alpha \in \mathcal{C}[m_{k_1}m_{k_2} \cdots k_{l-1}m_{l}]$ such that $\{m_1, m_1 + k_1 + m_2, \ldots, m_1 + k_1 + m_2 + \cdots + k_{l-2} + m_{l-1}\} \subseteq \text{Fix}(\alpha)$, is $K$-trivial. By applying the result in [1], the following proposition is obtained.

**Proposition 4.5.** For $Y = \{1, m_1, m_1 + k_1 + m_2, \ldots, m_1 + k_1 + m_2 + \cdots + k_{l-2} + m_{l-1}\}$, $\mathcal{C}[m_{k_1}m_{k_2} \cdots k_{l-1}m_{l}]$ is $K$-trivial with $C_{m_1-2}C_{m_2-1} \cdots C_{m_{l-1}-1} C_{m_l}$ as its cardinality.

**Proposition 4.6.** Let $\gamma$ be the right identity in $V_{l-1}[m_{1}p, m_{2p}, \ldots, m_{lt}]$. Then $\{\gamma \alpha : \alpha \in \mathcal{C}[m_{k_1}m_{k_2} \cdots k_{l-1}m_{l}]\}$ is $K$-trivial with $C_{m_1+1} \cdots C_{m_l-t+1}$ as its cardinality.

**Proof.** It is clear that for $\alpha \in \mathcal{C}[m_{k_1}m_{k_2} \cdots k_{l-1}m_{l}]$, $\gamma \alpha$ satisfies (i) in Lemma 4.4. Clearly, $\text{ran}(\gamma \beta) \subseteq \text{ran} \gamma$ and $|\text{ran} \gamma| = m_1 + \cdots + m_l - t + 1$. That is, $\{\gamma \alpha : \alpha \in \mathcal{C}[m_{k_1}m_{k_2} \cdots k_{l-1}m_{l}]\}$ is isomorphic to $\mathcal{C}[m_1 + \cdots + m_l - t + 1]$. 

$\square$
References


