EXISTENCE AND CHARACTERIZATION OF FARTHEST POINTS IN NORMED LINEAR SPACES

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Abstract: This paper delineates existence and characterization of a set of farthest points in normed linear spaces.

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1. Introduction

The problem of farthest points in normed linear spaces has been studied by Franchetti and Singer [4]. They obtained some results on characterization and existence of farthest points in normed linear spaces in terms of bounded linear functionals. Asplund [2] and Edelstein [3] have considered the farthest points of sets in uniformly convex Banach spaces. Several results related with farthest

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points in the context of normed linear space and metric space were obtained by Panda, et.al. [7] and Ahuja, et.al. [1] respectively. Section 2 provides the existence of farthest points in normed linear spaces. Some characterizations of farthest points with respect to uniform norm are established in Section 3.

2. Existence of Farthest Points

Definition 2.1. Let G be a bounded subset of a real normed linear space X and $x \in X$. An element $g_0 \in G$ is called a farthest point to $x \in X$ from G if for every $g \in G$, $\|x - g_0\| \geq \|x - g\|$, i.e.

$$\|x - g_0\| = \sup_{g \in G} \|x - g\|.$$

The set of all farthest points to x from G is denoted by $F_G(x)$.

Example 2.2. Let $X = \mathbb{R} \times \mathbb{R}$, $x = (a, b)$ in X we define a l_∞ - norm on X as follows:

$$\|(a, b)\|_\infty = \max\{|a|, |b|\}, a, b \in \mathbb{R}.$$

Let $G = \{(g_1, g_2) : -3 \leq g_1 \leq 3 \text{ and } g_2 = 0\}$ and $x = (5, 0)$. Then we can see that $g_0 = (-3, 0) \in F_G(x)$.

Let $G = \{(g_1, g_2) : g_1 = 0 \text{ and } -5 \leq g_2 \leq 5\}$ and $x = (0, 1)$.

Then clearly $g_0 = (0, -5) \in F_G(x)$.

Definition 2.3. Let G be a bounded subset of a real normed linear space X and $x \in X$. An element $g_0 \in G$ is called a strongly unique farthest point to X from G , if there exists a constant $K_x > 0$ such that for every $g \in G$,

$$\|x - g_0\| \geq \|x - g\| + K_x \|g - g_0\|.$$

Theorem 2.4. Let G be a compact subset in a normed linear space X . Then for every $x \in X$, there exists a farthest point from G .

Proof. Let $d = \sup_{g \in G} \|x - g\|$. Thus by definition of supremum there exists a sequence $\{g_n\}$ in G such that

$$\|x - g_n\| \rightarrow d \text{ as } n \rightarrow \infty.$$

Since G is compact, there exists a sub sequence $\{g_{n_k}\}$ such that $g_{n_k} \rightarrow g_0$, for some $g_0 \in G$.

Now we will show that g_0 is a farthest point to x from G .

Now

$$\begin{aligned} \|x - g_{n_k}\| &\leq \|x - g_0\| + \|g_0 - g_{n_k}\| \\ \Rightarrow \|x - g_0\| &\geq \|x - g_{n_k}\| - \|g_0 - g_{n_k}\| \\ &\rightarrow d \text{ as } k \rightarrow \infty. \end{aligned}$$

Hence

$$\|x - g_0\| \geq d,$$

since $g_0 \in G$, $\|x - g_0\| \leq d$.

Therefore $\|x - g_0\| = d$.

The proof of the following corollaries is obvious.

Corollary 2.5. *Let G be a closed and bounded finite dimensional set in a normed linear space X . Then for every $x \in X$, there exists a farthest point in G .*

Corollary 2.6. *Let G be a complete and totally bounded subset of a normed linear space X . Then for every $x \in X$, there exists a farthest point in G .*

Corollary 2.7. *Let G be a closed subset of a compact normed linear space X . Then for every $x \in X$, there exists a farthest point in G .*

Corollary 2.8. *Let f be a continuous mapping from a normed linear space X into another normed linear space Y . Let G be a compact set in X . Then for every $y \in Y$, there exists a farthest point from $f(X)$.*

Corollary 2.9. *Let f be a continuous mapping from a normed linear space X into another finite dimensional normed linear space Y . Let G be a compact set in X . Then for every $y \in Y$, there exists a farthest point from convex hull of $f(X)$.*

Corollary 2.10. *Let G be a totally bounded subset of a Banach space X . Then for every $x \in X$, there exists a farthest point in G .*

Theorem 2.11. *Every strongly unique farthest point is a unique farthest point.*

Proof. Let X be a normed linear space, G be a bounded subset of X , $x \in X$ and $g_0 \in G$ be a strongly unique farthest point to x from G .

Claim. g_0 is a unique farthest point. If possible assume that $g_1 \in G$ is a farthest point to x from G .

Then $\|x - g_1\| \geq \|x - g\|$ for every $g \in G$

$$\Rightarrow \|x - g_1\| \geq \|x - g_0\|. \quad (2.1)$$

Since g_0 is a strongly unique farthest point to x from G , we have

$$\begin{aligned} \|x - g_0\| &\geq \|x - g\| + t\|g - g_0\| \text{ for every } g \in G \text{ and for some } t > 0 \\ \Rightarrow \|x - g_0\| &\geq \|x - g_1\| + t\|g_1 - g_0\|. \end{aligned} \quad (2.2)$$

From (2.1) and (2.2) it follows that

$$\begin{aligned} \|x - g_1\| &\geq \|x - g_1\| + t\|g_1 - g_0\|. \\ \Rightarrow t\|g_1 - g_0\| &= 0 \\ \Rightarrow g_1 &= g_0, \end{aligned}$$

since $t > 0$.

3. Characterization of Farthest Points

The characterization of farthest points are obtained in this section.

Definition 3.1. Let $f \in C[a, b]$. Then uniform norm or L_∞ -norm f is defined by

$$\|f\|_\infty = \sup_{x \in [a, b]} |f(x)|.$$

The following notation is used in the subsequent result. The set of extreme points of a function $f \in C[a, b]$ is defined by

$$E(f) = \{x \in [a, b] : |f(x)| = \|f\|_\infty\}.$$

The following results provide a characterization of farthest points with respect to uniform norm corresponding to Kolmogorov [5].

Theorem 3.2. Let G be a compact subset of $C[a, b]$, containing $0, f \in C[a, b]$ and $g_0 \in G$. The following statements are equivalent:

- (i) the function g_0 is a farthest point to f from G
- (ii) for every function $g \in G$,

$$\min_{t \in E(f-g)} (f(t) - g(t))(g_0(t) - g(t)) \leq 0.$$

Proof. (ii) $\Rightarrow$ (i). Suppose that (ii) holds and let $g \in G$ be given. By (ii) there exists a point $t \in E(f - g_0)$ such that

$$(f(t) - g(t))(g_0(t) - g(t)) \leq 0.$$

Then we have

$$\begin{aligned} \|f - g_0\|_\infty &\geq |f(t) - g_0(t)| \\ &= |f(t) - g(t) - (g_0(t) - g(t))| \\ &= |f(t) - g(t)| + |g_0(t) - g(t)| \\ &\geq \|f - g\|_\infty. \end{aligned}$$

Which shows that (i) holds.

(i) $\Rightarrow$ (ii). Suppose that (ii) fails. Then there exists a function $g_1 \in G$ such that for all $t \in E(f - g_1)$,

$$(f(t) - g_1(t))(g_0(t) - g_1(t)) > 0.$$

Claim. $E(f - g_1)$ is compact.

Since $E(f - g_1)$ is a bounded subset of $\mathbb{R}$, it is sufficient to show that $E(f - g_1)$ is closed.

Let $\{t_n\} \in E(f - g_1)$ such that $t_n \rightarrow t$. Then

$$\begin{aligned} |f(t) - g_1(t)| &= |f(\lim t_n) - g_1(\lim t_n)| \\ &= \lim |f(t_n) - g_1(t_n)| \\ &= \lim |f - g_1|_\infty \\ &= \|f - g_1\|_\infty \end{aligned}$$

Thus $t \in E(f - g_1)$. Hence the claim is true.

Since $E(f - g_1)$ is compact, there exists a real number $c > 0$ such that for all $t \in E(f - g_1)$,

$$(f(t) - g_1(t))(g_0(t) - g_1(t)) > C.$$

Moreover, there exists a neighborhood U of $E(f - g_1)$ such that for all $t \in U$,

$$(f(t) - g_1(t))(g_0(t) - g_1(t)) > \frac{c}{2} \quad (3.1)$$

and

$$|f(t) - g_1(t)| \geq \frac{1}{2} \|f - g_1\|_\infty. \quad (3.2)$$

Since $[a, b] \setminus U$ is compact, $\max_{t \in [a, b] \setminus U} |f(t) - g_1(t)|$ exists. It is clear that for all $t \in [a, b] \setminus U$,

$$\begin{aligned} |f(t) - g_1(t)| &\leq \max_{t \in [a, b] \setminus U} |f(t) - g_1(t)| \\ &= \|f - g_1\|_\infty - d, \end{aligned}$$

where $d = \|f - g_1\|_\infty - \max_{t \in [a, b] \setminus U} |f(t) - g_1(t)|$.

Thus for all $t \in [a, b] \setminus U$,

$$|f(t) - g_1(t)| \leq \|f - g_1\|_\infty - d. \quad (3.3)$$

By multiplying $(g_0 - g_1)$ with an appropriate positive factor, we may assume without loss of generality that

$$\|g_0 - g_1\|_\infty \leq \min\{d, \frac{1}{2}\|f - g_1\|_\infty\}. \quad (3.4)$$

Case (i) Let $t \in [a, b] \setminus U$. Then

$$\begin{aligned} |f(t) - g_0(t)| &\leq |f(t) - g_1(t)| + |g_0(t) - g_1(t)| \\ &\leq \|f - g_1\|_\infty - d + \|g_0 - g_1\|_\infty \quad \text{by (3.3)} \\ &< \|f - g_1\|_\infty. \quad \text{by (3.4)} \end{aligned}$$

Case (ii) Let $t \in U$. Then

$$\begin{aligned} |f(t) - g_0(t)| &= |(f(t) - g_1(t)) - (g_0(t) - g_1(t))| \\ &= ||f(t) - g_1(t)| - |g_0(t) - g_1(t)|| \\ &= |f(t) - g_1(t)| - |g_0(t) - g_1(t)| \quad \text{by (3.2) and (3.4)} \\ &< |f(t) - g_1(t)| \\ &= \|f - g_1\|_\infty. \end{aligned}$$

Thus for all $t \in [a, b]$,

$$|f(t) - g_0(t)| < \|f - g_1\|_\infty.$$

This implies that

$$\|f - g_0\|_\infty < \|f - g_1\|_\infty,$$

which shows that g_0 is not a farthest point to f from G , a contradiction.

Theorem 3.3. *Let G be a compact subset of $C[a, b]$ containing $0, f \in C[a, b] \setminus G$ and $g_0 \in G$.*

Consider the following statements.

(i) the function g_0 is a strongly unique farthest point to f from G .

(ii) for $g \in G$,

$$\min_{t \in E(f-g)} (f(t) - g(t))(g_0(t) - g(t)) < 0.$$

(iii) there exists a constant $K_f > 0$ such that for every $g \in G$,

$$\min_{t \in E(f-g)} (f(t) - g(t))(g_0(t) - g(t)) \leq -K_f \|f - g\|_\infty \|g_0 - g\|_\infty.$$

Then (i) $\Leftrightarrow$ (iii) and (iii) $\Rightarrow$ (ii). If $G \setminus \{g_0\}$ is compact, then (ii) $\Rightarrow$ (iii).

Proof. (iii) $\Rightarrow$ (i). Suppose that (iii) holds and let $g \in G$ be given.

By (iii) there exists a point $t \in E(f - g)$ such that

$$(f(t) - g(t))(g_0(t) - g(t)) \leq -K_f \|f - g\|_\infty \|g - g_0\|_\infty.$$

Then it follows that

$$\begin{aligned} \|f - g_0\|_\infty &\geq |f(t) - g_0(t)| \\ &= |(f(t) - g(t)) - (g_0(t) - g(t))| \\ &= |f(t) - g(t)| + |g_0(t) - g(t)| \\ &\geq \|f - g\|_\infty + K_f \frac{\|f - g\|_\infty \|g_0 - g\|_\infty}{|f(t) - g(t)|} \\ &= \|f - g\|_\infty + K_f \|g_0 - g\|_\infty. \end{aligned}$$

(i) $\Rightarrow$ (iii). Suppose that (iii) fails, then there exists a function $g_1 \in G$ such that for all $t \in E(f - g_1)$,

$$(f(t) - g_1(t))(g_0(t) - g_1(t)) > -K_f \|f - g_1\|_\infty \|g_0 - g_1\|_\infty.$$

It can be easily proved that $E(f - g_1)$ is compact.

Then there exists a neighborhood U of $E(f - g_1)$ such that for all $t \in U$,

$$(f(t) - g_1(t))(g_0(t) - g_1(t)) > -K_f \|f - g_1\|_\infty \|g_0 - g_1\|_\infty$$

and

$$|f(t) - g_1(t)| \geq \frac{1}{2} \|f - g_1\|_\infty. \quad (3.5)$$

Moreover, the neighborhood U can be chosen sufficiently small such that for all $t \in U$ with

$$(f(t) - g_1(t))(g_0(t) - g_1(t)) < 0,$$

$$|g_0(t) - g_1(t)| < K_f \|g_0 - g_1\|_\infty. \quad (3.6)$$

Since $[a, b] \setminus U$ is compact, there exists a real number $c > 0$ such that for all $t \in [a, b] \setminus U$,

$$|(f(t) - g_1(t))| \leq \|f - g_1\|_\infty - c. \quad (3.7)$$

By multiplying $(g_0 - g_1)$ with an appropriate positive factor, assume with out loss of generality that

$$\|g_0 - g_1\|_\infty \leq \min\{c, \frac{1}{2}\|f - g_1\|_\infty\}. \quad (3.8)$$

Case (i) Let $t \in [a, b] \setminus U$. Then

$$\begin{aligned} |f(t) - g_0(t)| &\leq |f(t) - g_1(t)| + |g_0(t) - g_1(t)| \\ &\leq \|f - g_1\|_\infty - c + \|g_0 - g_1\|_\infty \quad \text{by (3.7)} \\ &< \|f - g_1\|_\infty \quad \text{by (3.8)} \end{aligned}$$

Case (ii) Let $t \in U$. If $(f(t) - g_1(t))(g_0(t) - g_1(t)) < 0$, then it follows that

$$\begin{aligned} |f(t) - g_0(t)| &= |(f(t) - g_1(t)) - (g_0(t) - g_1(t))| \\ &= |f(t) - g_1(t)| + |g_0(t) - g_1(t)| \\ &< |f(t) - g_1(t)| + K_f \|g_0 - g_1\|_\infty \quad \text{by (3.6)} \\ &= \|f - g_1\|_\infty + K_f \|g_0 - g_1\|_\infty. \end{aligned}$$

If $(f(t) - g_1(t))(g_0(t) - g_1(t)) \geq 0$, then it follows that

$$\begin{aligned} |f(t) - g_0(t)| &= |f(t) - g_1(t) - (g_0(t) - g_1(t))| \\ &= ||f(t) - g_1(t)| - |g_0(t) - g_1(t)|| \\ &= |f(t) - g_1(t)| - |g_0(t) - g_1(t)| \quad \text{by (3.5) and (3.8)} \\ &< |f(t) - g_1(t)| \\ &= \|f - g_1\|_\infty. \end{aligned}$$

Thus for all $t \in [a, b]$,

$$|f(t) - g_0(t)| < \|f - g_1\|_\infty + K_f \|g_0 - g_1\|_\infty.$$

This implies that $\|f - g_0\|_\infty < \|f - g_1\|_\infty + K_f \|g_0 - g_1\|_\infty$. Which shows that g_0 is not a strongly unique farthest point to f from G , a contradiction.

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Assume that $G \setminus \{g_0\}$ is compact and (ii) holds.

Define a map $F : G \setminus \{g_0\} \rightarrow \mathbb{R}$ by

$$F(g) = \min_{t \in E(f-g)} \frac{(f(t) - g(t))(g_0(t) - g(t))}{\|f - g\|_\infty \|g_0 - g\|_\infty}.$$

It is easy to check that F is continuous. By (ii), $F(g) < 0$ for all $g \in G \setminus \{g_0\}$. Since F is continuous on the compact set $G \setminus \{g_0\}$, there exists a constant $K_f > 0$ such that for all $g \in G \setminus \{g_0\}$, $F(g) < -K_f$.

Thus $\min_{t \in E(f-g)} (f(t) - g(t))(g_0(t) - g(t)) \leq -K_f \|f - g\|_\infty \|g_0 - g\|_\infty$.

References

- [1] G.C. Ahuja, T.D. Narang, T. Swaran, On farthest points, *Journal of Indian Mathematical Society*, **39** (1975), 293-297.
- [2] E. Asplund, Farthest points in reflexive locally uniformly rotund Banach spaces, *Israel Journal of Mathematics*, **4** (1966), 213-216.
- [3] M. Edelstein, Farthest points of sets in uniformly convex Banach spaces, *Israel Journal of Mathematics*, **4** (1966), 171-176.
- [4] C. Franchetti, I. Singer, Deviation and farthest points in normed linear spaces, *Romanian Journal of Pure and Applied Mathematics*, **24** (1979), 373-381.
- [5] A.N. Kolmogorov, Eine Bemerkung zu den Polynomen Von P.L. Tschebycheff, die Von einer gegebenen Funktion am Wenigsten abweichen, *USP. Math. Nauk*, **3** (1948), 216-221.
- [6] T.D. Narang, A study of farthest point, *New Archive for Mathematics*, **25** (1977), 54-79.
- [7] B.B. Panda, O.P. Kapoor, On farthest points of sets, *Journal of Mathematical Analysis and Applications*, **62** (1978), 345-353.
- [8] R. Saravanan, *Some Contributions to Approximation Theory*, Ph.D. Thesis, University of Madras (1997).
- [9] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer-Verlag, New York (1970).