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SIMPLE PROOFS DETERMINING ALL NONISOMORPHIC SEMIGROUPS OF ORDER 3 WITH TWO IDEMPOTENTS

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Abstract: In this paper we use only elementary ideas to show that there are 9 nonisomorphic semigroups of order 3 with two idempotents.

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1. Introduction

Let $(S, *)$ be a semigroup. An element a of S is called an idempotent element if $a^2 = a$. An element e of S is called an identity element if $s*e = s = e*s$ for all s in S. A monoid is a semigroup which has an identity element. It is well known that there are 5 nonisomorphic semigroups of order 2. In this paper we show that there are 9 nonisomorphic semigroups of order 3 with two idempotents.

2. Main Results

The following proposition gives some property of semigroups of order 3 with two idempotents.

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Theorem 1. *Every semigroup of order 3 with two idempotents has an identity element or has a left zero element or has a right zero element.*

Proof. Assume that $S = (\{a, b, c\}, *)$ is a semigroup with idempotents a, b and that c is not an idempotent element. Thus $c * c \neq c$.

We consider two cases of the product $c * c$. That is, case A: $c * c = a$ and case B: $c * c = b$.

Case A. Assume that $c*c = a$. We have $a*c = (c*c)*c = c*(c*c) = c*a$. Consider the product $a * c = c * a$. If $a * c = c * a = b$, then $b = b * b = (a * c) * (c * a) = a * (c * c) * a = a * a * a = a$. This contradicts $b \neq a$. Therefore $a*c = c*a \neq b$. If $a*c = c*a = c$, then $a = a*c = a*(a*c) = (a*a)*c = a*c = c$. Therefore $a * c = c * a \neq c$. Thus $a * c = c * a = a$. Next we consider the product $a * b$. Since $a * (a * b) = (a * a) * b = a * b$ and $a * c = a$, $a * b \neq c$. Therefore $a * b = a$ or $a * b = c$. In case $a * b = a$, we have a is a left zero element. In case $a * b = b$, we have $c * b = c * (a * b) = (c * a) * b = a * b = b$, i.e., b is a right zero element. So case A is complete.

Case B. Assume that $c * c = b$. We have $b * c = (c * c) * c = c * (c * c) = c * b$. So we consider the product $b * c = c * b$. If $b * c = c * b = a$, then $a = c * b = c * (b * b) = (c * b) * b = a * b = a * (b * b) = a * (b * (c * c)) = a * (b * c) * c = a * a * c = a * c = (b * c) * c = b * (c * c) = b * c = b$. Therefore $b * c = c * b \neq a$. We consider two subcases of case B according to the product $b * c = c * b$.

Subcase B-1. Assume that $b * c = c * b = b$. We consider the product $a * b$. If $a * b = c$, then $c = a * b = a * (b * b) = (a * b) * b = c * b = b$. Therefore $a * b \neq c$. In case $a * b = b$, b is a right zero element. In case $a * b = a$, we have $a * c = (a * b) * c = a * (b * c) = a * b = a$, i.e., a is a left zero element.

Subcase B-2. Assume that $b * c = c * b = c$. If $a * b = c$, then $c = a * b = (a * a) * b = a * (a * b) = a * c = a * (b * c) = (a * b) * c = c * c = b$. So $a * b \neq c$. We consider two subcases of subcase B-2 according to the product $a * b$.

Subcase B-2.1. Assume that $a * b = a$. We have $(a * c) * c = a * (c * c) = a * b = a$. If follows that $a * c$ can not be equal to b or c, i.e., $a * c = a$. Therefore a is a left zero element.

Subcase B-2.2. Assume that $a * b = b$. So $a * c = a * (b * c) = (a * b) * c = b * c = c$. This shows that a is a left identity element. Next we show that a is a right identity element. Since $(c*a)*c = c*(a*c) = c*c = b$, $c*a$ can not be equal to a or b, i.e., $c*a = c$. Consequently $b*a = (c*c)*a = c*(c*a) = c*c = b$. Thus a is a right identity element, i.e., S has an identity element. This completes the proof. □

From [1] we have all nonisomorphic monoids of order 3. So it is easy to check that there are 3 nonisomorphic monoids of order 3 with two idempotents.

See tables below.

$*$	a	b	c
a	a	b	c
b	b	b	b
c	c	b	b

Table 1

$*$	a	b	c
a	a	b	c
b	b	b	c
c	c	c	b

Table 2

$*$	a	b	c
a	a	b	c
b	b	b	b
c	c	b	a

Table 3

Next we consider the case of a semigroup S of order 3 with two idempotents and S does not have an identity element.

Theorem 2. *There are only* 6 *nonisomorphic semigroups of order 3 with two idempotents and without identity.*

Proof. Let $S = (\{a, b, c\}, *)$ be a semigroup of order 3 with idempotents a, b and S is without an identity element. By proposition, S has at least one left zero element or S has at least one right zero element. Every semigroup which has a unique left zero element or has a unique right zero element, has a zero element. We know that every left zero element or right zero element is an idempotent. So c is not a left zero element and c is not a right zero element. So we consider 3 cases:

case A: S has a zero element;

case B: S has two right zero elements;

case C: S has two left zero elements.

First we consider case A. Assume that S has a zero element a. So we consider two subcases of case A according to the product $c * c$.

Subcase A-1. Assume that $c * c = a$.

Since $(b*c)*c = b*(c*c) = b*a = a$, $b*c \neq b$. So we consider two subcases of subcase A-1 according to the product $b * c$.

Subcase A-1.1. Assume that $b * c = c$. If $c * b = c$, then b is an identity element. So we need not consider this case. If $c * b = b$, then $a = c * c = c * (b * c) = (c * b) * c = b * c = c$. So $c * b \neq b$. Therefore $c * b = a$. So we have table 4 (see the table below).

Subcase A-1.2. Assume that $b * c = a$. If $c * b = b$, then we have $b = b * b = b * (c * b) = (b * c) * b = a * b = a$. This is impossible. So $c * b \neq b$. Therefore $c * b = a$ or $c * b = c$. So we have table 5 and table 6 (see tables below).

Subcase A-2. Assume that $c*c = b$. We have $b*c = (c*c)*c = c*(c*c) = c*b$. If $b*c = c*b = c$, then b is an identity. So we need not consider this case. Since $c*(c*b) = (c*c)*b = b*b = b$, $c*b$ can not equal a. Therefore $b*c = c*b = b$. So we have table 7 (see the table below). Now we have all tables of case A.

$*$	a	b	c
a	a	a	a
b	a	b	c
c	a	a	a

Table 4

$*$	a	b	c
a	a	a	a
b	a	b	a
c	a	a	a

Table 5

$*$	a	b	c
a	a	a	a
b	a	b	a
c	a	c	a

Table 6

$*$	a	b	c
a	a	a	a
b	a	b	b
c	a	b	b

Table 7

Since all tables above have a zero element, it is easy to check that all tables are associative and all tables are nonisomorphic. This completes case A.

Case B. Assume that S has two right zero elements. Therefore a and b are right zero elements. So we consider two subcases according to the product $c*c$.

Subcase B-1. Assume that $c*c = a$. We have $a*c = (c*c)*c = c*(c*c) = c*a = a$. Since $(b*c)*c = b*(c*c) = b*a = a$, $b*c \neq b$. Since $a*(b*c) = (a*b)*c = b*c$, $b*c \neq c$. Therefore $b*c = a$. So we have table 8 (see the table in next page).

Subcase B-2. Assume that $c*c = b$. We have $b*c = (c*c)*c = c*(c*c) = c*b = b$. Since $(a*c)*c = a*(c*c) = a*b = b$, $a*c \neq a$. Since $b*(a*c) = (b*a)*c = a*c$ and $b*c = b$, $a*c \neq c$. Therefore $a*c = b$. So we have table 9 (see the table in next page). Now we have all tables of case B.

$*$	a	b	c
a	a	b	a
b	a	b	a
c	a	b	a

Table 8

$*$	a	b	c
a	a	b	b
b	a	b	b
c	a	b	b

Table 9

It is easy to check that table 8 and table 9 are associative and they are isomorphic. This completes case B.

Case C. Assume that S has two left zero elements. Therefore a and b are left zero elements. So we consider two subcases according to the product $c * c$.

Subcase C-1. Assume that $c * c = a$. We have $c * a = c * (c * c) = (c * c) * c = a * c = a$. Since $c * (c * b) = (c * c) * b = a * b = a$, $c * b \neq b$. Since $(c * b) * a = c * (b * a) = c * b$ and $c * a = a$, $c * b \neq c$. Therefore $c * b = a$. So we have table 10 (see the table below).

Subcase C-2. Assume that $c * c = b$. We have $c * b = c * (c * c) = (c * c) * c = c * b = b$. Since $c * (c * a) = (c * c) * a = b * a = b$, $c * a \neq a$. Since $(c * a) * c = c * (a * c) = c * a$, $c * a \neq c$. Therefore $c * a = b$. So we have table 11 (see the table below). Now we have all tables of case C.

$*$	a	a	c
a	a	a	a
b	b	b	b
c	a	a	a

Table 10

$*$	a	b	c
a	a	a	a
b	b	b	b
c	b	b	b

Table 11

It is easy to check that table 10 and table 11 are associative and they are isomorphic. This completes case C.

Hence there are 6 nonisomorphic semigroups of order 3 with two idempotents which do not have an identity element. □

Now we can summarize the above work in the following theorem.

Theorem 3. *There are only* 9 *nonisomorphic semigroups of order 3 with two idempotents.*

Acknowledgments

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