SIMPLE PROOFS DETERMINING ALL NONISOMORPHIC SEMIGROUPS OF ORDER 3 WITH TWO IDEMPOTENTS

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Abstract: In this paper we use only elementary ideas to show that there are 9 nonisomorphic semigroups of order 3 with two idempotents.

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1. Introduction

Let \((S, \ast)\) be a semigroup. An element \(a\) of \(S\) is called an idempotent element if \(a^2 = a\). An element \(e\) of \(S\) is called an identity element if \(s \ast e = s = e \ast s\) for all \(s\) in \(S\). A monoid is a semigroup which has an identity element. It is well known that there are 5 nonisomorphic semigroups of order 2. In this paper we show that there are 9 nonisomorphic semigroups of order 3 with two idempotents.

2. Main Results

The following proposition gives some property of semigroups of order 3 with two idempotents.
Theorem 1. Every semigroup of order 3 with two idempotents has an identity element or has a left zero element or has a right zero element.

Proof. Assume that \( S = \{a, b, c\} \) is a semigroup with idempotents \( a, b \) and that \( c \) is not an idempotent element. Thus \( c \neq c \).

We consider two cases of the product \( c \cdot c \). That is, case A: \( c \cdot c = a \) and case B: \( c \cdot c = b \).

Case A. Assume that \( c \cdot c = a \). We have \( a \cdot c = (c \cdot c) \cdot c = c \cdot (c \cdot c) = c \cdot a \).

Consider the product \( a \cdot c = c \cdot a \). If \( a \cdot c = c \cdot a = b \), then \( b = b \cdot b = (a \cdot c) \cdot c = a \cdot (c \cdot c) = a \cdot c = a \cdot a \cdot a = a \). This contradicts \( b \neq a \). Therefore \( a \cdot c = c \cdot a \neq b \). If \( a \cdot c = c \cdot a = c \), then \( a = a \cdot c = a \cdot (a \cdot c) = (a \cdot a) \cdot c = a \cdot c = c \).

Therefore \( a \cdot c = c \cdot a \neq c \). Thus \( a \cdot c = c \cdot a = a \). Next we consider the product \( a \cdot b \). Since \( a \cdot (a \cdot b) = (a \cdot a) \cdot b = a \cdot b \) and \( a \cdot c = a \), \( a \cdot b \neq c \). Therefore \( a \cdot b = a \) or \( a \cdot b = c \). In case \( a \cdot b = a \), we have \( a \) is a left zero element. In case \( a \cdot b = b \), we have \( c \cdot b = c \cdot (a \cdot b) = (c \cdot a) \cdot b = a \cdot b = b \), i.e., \( b \) is a right zero element. So case A is complete.

Case B. Assume that \( c \cdot c = b \). We have \( b \cdot c = (c \cdot c) \cdot c = c \cdot (c \cdot c) = c \cdot b \).

So we consider the product \( b \cdot c = c \cdot b \). If \( b \cdot c = c \cdot b = a \), then \( a = c \cdot b = c \cdot (b \cdot b) = (c \cdot b) \cdot b = a \cdot b = a \cdot (b \cdot b) = a \cdot (b \cdot (c \cdot c)) = a \cdot (b \cdot c) = a \cdot c = (b \cdot c) \cdot c = b \cdot (c \cdot c) = b \cdot c = b \). Therefore \( b \cdot c = c \cdot b \neq a \).

We consider two subcases of case B according to the product \( b \cdot c = c \cdot b \).

Subcase B-1. Assume that \( b \cdot c = c \cdot b = b \). We consider the product \( a \cdot b \).

If \( a \cdot b = c \), then \( c = a \cdot b = a \cdot (b \cdot b) = (a \cdot b) \cdot b = c \cdot b = b \). Therefore \( a \cdot b \neq c \).

In case \( a \cdot b = b \), \( b \) is a right zero element. In case \( a \cdot b = a \), we have \( a \cdot c = (a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot b = a \), i.e., \( a \) is a left zero element.

Subcase B-2. Assume that \( b \cdot c = c \cdot b = c \). If \( a \cdot b = c \), then \( c = a \cdot b = (a \cdot a) \cdot b = a \cdot (a \cdot b) = a \cdot c = a \cdot (b \cdot c) = (a \cdot b) \cdot c = c \cdot c = c \).

So \( a \cdot b \neq c \).

We consider two subcases of subcase B-2 according to the product \( a \cdot b \).

Subcase B-2.1. Assume that \( a \cdot b = a \). We have \( (a \cdot c) \cdot c = a \cdot (c \cdot c) = a \cdot b = a \). If follows that \( a \cdot c \) can not be equal to \( b \) or \( c \), i.e., \( a \cdot c = a \).

Therefore \( a \) is a left zero element.

Subcase B-2.2. Assume that \( a \cdot b = b \). So \( a \cdot c = a \cdot (b \cdot c) = (a \cdot b) \cdot c = b \cdot c = c \).

This shows that \( a \) is a left identity element. Next we show that \( a \) is a right identity element. Since \( (c \cdot a) \cdot c = c \cdot (a \cdot c) = c \cdot c = b \), \( c \cdot a \) can not be equal to \( a \) or \( b \), i.e., \( c \cdot a = c \).

Consequently \( b \cdot a = (c \cdot c) \cdot a = c \cdot (c \cdot a) = c \cdot c = b \). Thus \( a \) is a right identity element, i.e., \( S \) has an identity element. This completes the proof.

From [1] we have all nonisomorphic monoids of order 3. So it is easy to check that there are 3 nonisomorphic monoids of order 3 with two idempotents.
Next we consider the case of a semigroup $S$ of order 3 with two idempotents and $S$ does not have an identity element.

**Theorem 2.** There are only 6 nonisomorphic semigroups of order 3 with two idempotents and without identity.

**Proof.** Let $S = \{\{a,b,c\},\ast\}$ be a semigroup of order 3 with idempotents $a, b$ and $S$ is without an identity element. By proposition, $S$ has at least one left zero element or $S$ has at least one right zero element. Every semigroup which has a unique left zero element or has a unique right zero element, has a zero element. We know that every left zero element or right zero element is an idempotent. So $c$ is not a left zero element and $c$ is not a right zero element. So we consider 3 cases:

- case A: $S$ has a zero element;
- case B: $S$ has two right zero elements;
- case C: $S$ has two left zero elements.

First we consider case A. Assume that $S$ has a zero element $a$. So we consider two subcases of case A according to the product $c \ast c$.

Subcase A-1. Assume that $c \ast c = a$.

Since $(b \ast c) \ast c = b \ast (c \ast c) = b \ast a = a$, $b \ast c \neq b$. So we consider two subcases of subcase A-1 according to the product $b \ast c$.

Subcase A-1.1. Assume that $b \ast c = c$. If $c \ast b = c$, then $b$ is an identity element. So we need not consider this case. If $c \ast b = b$, then $a = c \ast c = c \ast (b \ast c) = (c \ast b) \ast c = b \ast c = c$. So $c \ast b \neq b$. Therefore $c \ast b = a$. So we have table 4 (see the table below).

Subcase A-1.2. Assume that $b \ast c = a$. If $c \ast b = b$, then we have $b = b \ast b = b \ast (c \ast b) = (b \ast c) \ast b = a \ast b = a$. This is impossible. So $c \ast b \neq b$. Therefore $c \ast b = a$ or $c \ast b = c$. So we have table 5 and table 6 (see tables below).
Subcase A-2. Assume that \(c \star c = b\). We have \(b \star c = (c \star c) \star c = c \star (c \star c) = c \star b\). If \(b \star c = c \star b = c\), then \(b\) is an identity. So we need not consider this case. Since \(c \star (c \star b) = (c \star c) \star b = b \star b = b\), \(c \star b\) cannot equal \(a\). Therefore \(b \star c = c \star b = b\). So we have table 7 (see the table below). Now we have all tables of case A.

\[
\begin{array}{ccc}
* & a & b & c \\
\hline
a & a & a & a \\
b & a & b & c \\
c & a & a & a
\end{array}
\]

Table 4

\[
\begin{array}{ccc}
* & a & b & c \\
\hline
a & a & a & a \\
b & a & b & a \\
c & a & a & a
\end{array}
\]

Table 5

\[
\begin{array}{ccc}
* & a & b & c \\
\hline
a & a & a & a \\
b & a & b & a \\
c & a & c & a
\end{array}
\]

Table 6

\[
\begin{array}{ccc}
* & a & b & c \\
\hline
a & a & a & a \\
b & a & b & b \\
c & a & b & b
\end{array}
\]

Table 7

Since all tables above have a zero element, it is easy to check that all tables are associative and all tables are nonisomorphic. This completes case A.

Case B. Assume that \(S\) has two right zero elements. Therefore \(a\) and \(b\) are right zero elements. So we consider two subcases according to the product \(c \star c\).

Subcase B-1. Assume that \(c \star c = a\). We have \(a \star c = (c \star c) \star c = c \star (c \star c) = c \star a = a\). Since \((b \star c) \star c = b \star (c \star c) = b \star a = a\), \(b \star c \neq b\). Since \(a \star (b \star c) = (a \star b) \star c = b \star c\), \(b \star c \neq c\). Therefore \(b \star c = a\). So we have table 8 (see the table in next page).

Subcase B-2. Assume that \(c \star c = b\). We have \(b \star c = (c \star c) \star c = c \star (c \star c) = c \star b = b\). Since \((a \star c) \star c = a \star (c \star c) = a \star b = b\), \(a \star c \neq a\). Since \(b \star (a \star c) = (b \star a) \star c = a \star c\) and \(b \star c = b\), \(a \star c \neq c\). Therefore \(a \star c = b\). So we have table 9 (see the table in next page). Now we have all tables of case B.

\[
\begin{array}{ccc}
* & a & b & c \\
\hline
a & a & b & a \\
b & a & b & a \\
c & a & b & a
\end{array}
\]

Table 8

\[
\begin{array}{ccc}
* & a & b & c \\
\hline
a & a & b & b \\
b & a & b & b \\
c & a & b & b
\end{array}
\]

Table 9

It is easy to check that table 8 and table 9 are associative and they are isomorphic. This completes case B.
Case C. Assume that $S$ has two left zero elements. Therefore $a$ and $b$ are left zero elements. So we consider two subcases according to the product $c \ast c$.

Subcase C-1. Assume that $c \ast c = a$. We have $c \ast a = c \ast (c \ast c) = (c \ast c) \ast c = a \ast c = a$. Since $c \ast (c \ast b) = (c \ast c) \ast b = a \ast b = a$, $c \ast b \neq b$. Since $(c \ast c) \ast a = c \ast (b \ast a) = c \ast b$ and $c \ast a = a$, $c \ast b \neq c$. Therefore $c \ast b = a$. So we have table 10 (see the table below).

Subcase C-2. Assume that $c \ast c = b$. We have $c \ast b = c \ast (c \ast c) = (c \ast c) \ast c = c \ast b = b$. Since $c \ast (c \ast a) = (c \ast c) \ast a = b \ast a = b$, $c \ast a \neq a$. Since $(c \ast a) \ast c = c \ast (a \ast c) = c \ast a$, $c \ast a \neq c$. Therefore $c \ast a = b$. So we have table 11 (see the table below). Now we have all tables of case C.

\[
\begin{array}{ccc}
* & a & a & c \\
 a & a & a & a \\
b & b & b & b \\
c & a & a & a \\
\end{array}
\quad
\begin{array}{ccc}
* & a & b & c \\
 a & a & a & a \\
b & b & b & b \\
c & b & b & b \\
\end{array}
\]

Table 10  
Table 11

It is easy to check that table 10 and table 11 are associative and they are isomorphic. This completes case C.

Hence there are 6 nonisomorphic semigroups of order 3 with two idempotents which do not have an identity element.

Now we can summarize the above work in the following theorem.

**Theorem 3.** There are only 9 nonisomorphic semigroups of order 3 with two idempotents.

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References

