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GENERALIZED COHYPERSUBSTITUTIONS OF TYPE $\tau = (n_i)_{i\in I}$

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Abstract: Generalized cohypersubstitutions of type $\tau = (n_i)_{i\in I}$ are mappings which send the n_i-ary operation symbols to coterms of type τ. Coidentities which are closed under these generalized cohypersubstitutions are called generalized cohyperidentities and a covariety is said to be generalized solid if every coidentity in it is a generalized cohyperidentity. These concepts were introduced in this study and the results show that the lattice of all generalized solid classes form a complete sublattice of the lattice of all coalgebras of type τ.

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1. Introduction

Let A be a non-empty set and n be a positive integer. The n-th copower $A^{\sqcup n}$ of A is the union of n disjoint copies of A; formally, we define $A^{\sqcup n}$ as the cartesian product $A^{\sqcup n} := \underline{n} \times A$, where $\underline{n} := \{1, \ldots, n\}$. An element (i, a) in this copower corresponds to the element a in the i-th copy of A, for $1 \leq i \leq n$. A co-operation on A is a mapping $f^A : A \to A^{\sqcup n}$ for some $n \geq 1$; the natural

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number n is called the arity of the co-operation f^A. We also need to recall that any n-ary co-operation f^A on set A can be uniquely expressed as a pair (f_1^A, f_2^A) of mappings, $f_1^A : A \to \underline{n}$ and $f_2^A : A \to A$; the first mapping gives the labelling used by f^A in mapping elements to copies of A, and the second mapping tells us what element of A any element is mapped to. We shall denote by $cO_A^{(n)} = \{f^A : A \to A^{\sqcup n}\}$ the set of all $n - ary$ co-operations defined on A, and by $cO_A := \cup_{n\geq 1} cO_A^{(n)}$ the set of all finitary co-operations defined on A. An indexed coalgebra is a pair $(A; (f_i^A)_{i\in I})$, where f_i^A is an n_i-ary cooperation defined on A, and $\tau = (n_i)_{i\in I}$ for $n_i \geq 1$ is called the type of the coalgebra. Coalgebras were studied by Drbohlav [4]. In [1], the following superposition of cooperations was introduced. If $f^A \in cO_A^{(n)}$ and $g_0^A, \ldots, g_{n-1}^A \in cO_A^{(k)}$, then the k-ary co-operation $f^A[g_0^A, \ldots, g_{n-1}^A] : A \to A^{\sqcup k}$ is defined by

$$a \mapsto ((g^A_{f_1^A(a)})_1(f_2^A(a)), (g^A_{f_1^A(a)})_2(f_2^A(a)))$$

for all $a \in A$. The co-operation $f^A[g_0^A, \ldots, g_{n-1}^A]$ is called the *superposition* of f^A and $g_0^A, \ldots, g_{n-1}^A$. It will also be denoted by $comp_k^n(f^A, g_0^A, \ldots, g_{n-1}^A)$.

The *injection co-operations* $\iota_i^{n,A} : A \to A^{\sqcup n}$ are special cooperations which are defined for each $0 \leq i \leq n-1$ by $\iota_i^{n,A} : A \to A^{\sqcup n}$ with $a \mapsto (i, a)$ for all $a \in A$. Then we get a multi-based algebra

$$((cO_A^{(n)})_{n\geq 1}, (comp_k^n)_{k,n\geq 1}, (\iota_i^{n,A})_{0\leq i\leq n-1}),$$

called the *clone of co-operations* on A. In [1] it is mentioned that this algebra is a clone, i.e. it satisfies the three clone axioms $(C1), (C2), (C3)$. In [2], K. Denecke and K. Saengsura gave a full proof of this fact. From their study, the following coterms of type $\tau = (n_i)_{i\in I}$ were introduced. They let $(f_i)_{i\in I}$ be an indexed set of co-operation symbols such that for each $i \in I$, f_i has arity n_i and also let $\bigcup\{e_j^n \mid n \geq 1,\ n \in \mathbb{N},\ 0 \leq j \leq n-1\}$ be a set of symbols which is disjoint from the set $\{f_i \mid i \in I\}$ such that for each $0 \leq j \leq n-1$, e_j^n has arity n. Then coterms of type τ are defined as follows:

(i) For every $i \in I$ the co-operation symbol f_i is an n_i-ary coterm of type τ.

(ii) For every $n \geq 1$ and $0 \leq j \leq n-1$ the symbol e_j^n is an n-ary coterm of type τ.

(iii) If $t_1, \ldots, t_{n_i}$ are n-ary coterms of type τ, then $f_i[t_1, \ldots, t_{n_i}]$ is an n-ary coterm of type τ for every $i \in I$, and if $t_0, \ldots, t_{n-1}$ are m-ary coterms of type τ, then $e_j^n[t_0, \ldots, t_{n-1}]$ is an m-ary coterm of type τ for every $n \geq 1$ and $0 \leq j \leq n-1$.

Let $cT_\tau^{(n)}$ be the set of all n-ary coterms of type τ and let $cT_\tau := \bigcup_{n\geq 1} cT_\tau^{(n)}$ be the set of all (finitary) coterms of type τ.

The superposition of coterms was introduced in [3] as follows: The operation $S_m^n : cT_\tau^{(n)} \times (cT_\tau^{(m)})^n \to cT_\tau^{(m)}$ is defined by induction on the complexity of coterm definition as follows:

(i) $S_m^n(e_i^n, t_0, \ldots, t_{n-1}) := t_i$ for $0 \leq i \leq n-1$.

(ii) $S_{n_i}^{n_i}(f_i, e_0^{n_i}, \ldots, e_{n_i-1}^{n_i}) := f_i$ for an n_i-ary co-operation symbol f_i .

(iii) $S_m^{n_j}(g_j, t_1, \ldots, t_{n_j}) := g_j[t_1, \ldots, t_{n_j}]$ if g_j is an n_j-ary co-operation symbol.

(iv) $S_m^n(f_i[s_1, \ldots, s_{n_i}], t_1, \ldots, t_n) := f_i[S_m^n(s_1, t_1, \ldots, t_n), \ldots, S_m^n(s_{n_i}, t_1, \ldots, t_n)]$ where f_i is an n_i-ary co-operation symbol, $s_1, \ldots, s_{n_i}$ are n-ary coterms of type τ and $t_1, \ldots, t_n$ are m-ary coterms of type τ.

These operations give us a heterogeneous algebra

$$c\mathcal{T}_\tau := ((cT_\tau^{(n)})_{n\geq 1}, (S_m^n)_{m,n\geq 1}, (e_j^n)_{1\leq j\leq n}).$$

i.e., it satisfies the axioms (C1),(C2)and (C3) below.

Theorem 1.1. (see [3]) *The heterogeneous algebra $c\mathcal{T}_\tau$ satisfies the following identities:*

(C1) $\hat{S}_m^p(z, \hat{S}_m^n(y_1, x_1, \ldots, x_n), \ldots, \hat{S}_m^n(y_p, x_1, \ldots, x_n)) \approx \hat{S}_m^n(\hat{S}_n^p(z, y_1, \ldots, y_p), x_1, \ldots, x_n), \quad (m, n, p \in \mathbb{N}^+),$

(C2) $\hat{S}_m^n(e_i^n, x_1, \ldots, x_n) \approx x_i \quad (m \in \mathbb{N}^+,\ 1 \leq i \leq n),$

(C3) $\hat{S}_n^n(y, e_1^n, \ldots, e_n^n) \approx y, \quad (n \in \mathbb{N}^+).$

(Here $\hat{S}_m^n$ and e_i^n are operation symbols corresponding to the clone type.)

Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a coalgebra of type τ. Each coterm t of type τ induces a co-operation t^A on $\mathcal{A}$ by the following inductive definitions:

(i) If f_i is an n_i-ary co-operation symbol, then f_i^A is the induced n_i-ary co-operation on A.

(ii) $(e_j^n)^{\mathcal{A}} := \iota_j^{n,A}$ for every $n \geq 1$ and $j \leq n$, where $\iota_j^{n,A}$ is an n-ary injection.

(iii) If $f_i[g_1, \ldots, g_{n_i}]$ is a coterm and we inductively assume that the induced co-operations $g_1^A, \ldots, g_{n_i}^A$ are defined, then $(f_i[g_1, \ldots, g_{n_i}])^A = f_i^A[g_1^A, \ldots, g_{n_i}^A]$.

(iv) If $g_1^A, \dots, g_n^A$ are assumed to be known, then we define $(e_j^n[g_1, \dots, g_n])^A = g_j^A$ for $1 \le j \le n$.

The concepts of cohypersubstitutions and cohyperidentities of coalgebra of type $\tau = (n_i)_{i \in I}$ were introduced by K. Denecke and K. Saengsura which are analogue of hypersubstitutions and hyperidentities of algebra of type $\tau = (n_i)_{i \in I}$ (see [3]).

2. Generalized Cohypersubstitutions

In this section, we have to define equations of coterms which are satisfied for all replacements of occurring cooperation symbols by cooperations. Such replacements can be defined more precisely by the concept of a generalized cohypersubstitution. First of all, we define the concept of generalized superposition which is inductively defined as follows:

Definition 2.1. Let $m \in \mathbb{N}^*$, a generalized superposition of coterms: $S^m : CT_\tau^{m+1} \to CT_\tau$ defined inductively by the following steps:

(i) If $t = e_i^n$ and $0 \le i \le m-1$, then
$S^m(e_i^n, t_0, \dots, t_{m-1}) = t_i$, where $t_1, \dots, t_{m-1} \in CT_\tau$.

(ii) If $t = e_i^n$ and $0 < m \le i \le n-1$, then
$S^m(e_i^n, t_0, \dots, t_{m-1}) = e_i^n$ where $t_0, \dots, t_{m-1} \in CT_\tau$.

(iii) If $t = f_i[s_1, \dots, s_{n_i}]$, then
$S^m(t, t_1, \dots, t_m) = f_i(S^m(s_1, t_1, \dots, t_m), \dots, S^m(s_{n_i}, t_1, \dots, t_m))$
where $S^m(s_1, t_1, \dots, t_m), \dots, S^m(s_{n_i}, t_1, \dots, t_m) \in CT_\tau$.

The above definition can be written in the following forms:

(i) If $t = e_i^n$ and $0 \le i \le m-1$, then
$e_i^n[t_0, \dots, t_{m-1}] = t_i$, where $t_1, \dots, t_{m-1} \in CT_\tau$.

(ii) If $t = e_i^n$ and $0 < m \le i \le n-1$, then
$e_i^n[t_0, \dots, t_{m-1}] = e_i^n$ where $t_0, \dots, t_{m-1} \in CT_\tau$.

(iii) If $t = f_i[s_1, \dots, s_{n_i}]$, then
$(f_i[s_1, \dots, s_{n_i}])[t_1, \dots, t_m] = f_i[s_1[t_1, \dots, t_m], \dots, s_{n_i}[t_1, \dots, t_m]]$
where $s_1[t_1, \dots, t_m], \dots, s_{n_i}[t_1, \dots, t_m] \in CT_\tau$.

It is easy to see that the image of the superposition of a coterm $t \in CT_\tau$ and coterms $t_1, \dots, t_m \in CT_\tau$ is depend on the arities of injection symbols which occur in a coterm t, i.e., if $e_i^n \in E(t)$ and $i \ge m$, the position of e_i^n have no change and if $i \le m-1$, the position of e_i^n have to change to a coterm t_i. For instance, $(f[e_0^3, e_3^4])[t_1, t_2] = f[t_1, e_3^4]$.

Proposition 2.1. *Let $t, t_1, \ldots, t_m \in CT_\tau$. If $j \geq m$ for all $0 \leq j \leq n-1$ and $e^n_j \in E(t)$, then $t[t_1, \ldots, t_m] = t$.*

Proof. We give a proof by induction on the complexity of a coterm t. If $t = e^n_j$, by Definition 2.1 we get that $e^n_j[t_1, \ldots, t_m] = e^n_j$. Assume that $t = f[s_1, \ldots, s_{n_i}]$ and that $s_j[t_1, \ldots, t_m] = s_j$ for all $j = 1, \ldots, n_i$.

Then we have that

$$\begin{aligned} t[t_1, \ldots, t_m] &= (f[s_1, \ldots, s_{n_i}])[t_1, \ldots, t_m] \\ &= f[s_1[t_1, \ldots, t_m], \ldots, s_{n_i}[t_1, \ldots, t_m]] \\ &= f[s_1, \ldots, s_{n_i}] \\ &= t. \end{aligned}$$ □

Definition 2.2. A generalized cohypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \to CT_\tau$. The extension of σ is a mapping $\hat{\sigma} : CT_\tau \to CT_\tau$ which is inductively defined by the following steps:

(i) $\hat{\sigma}(e^n_j) := e^n_j$ for every $n \geq 1$ and $0 \leq j \leq n-1$,

(ii) $\hat{\sigma}(f_i) := \sigma(f_i)$ for every $i \in I$,

(iii) $\hat{\sigma}(f_i[t_1, \ldots, t_{n_i}] := \sigma(f_i)[\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ for $t_1, \ldots, t_{n_i} \in CT^{(n)}_\tau$.

Let $Cohyp_G(\tau)$ be the set of all generalized cohypersubstitutions of type τ.

Let t be a coterm of type τ and $\{f_i \mid i \in J\}$ be a set of cooperation symbols occurring in a coterm t. It is easily shown that the extension of a generalized cohypersubstitution σ maps a coterm t to a coterm of type τ by substitute $\sigma(f_i)$ for f_i for all $i \in J$. For example, consider the type $\tau = (3)$ and the generalized hypersubstitution $\sigma : CT_{(3)} \to CT_\tau$ which maps $f \mapsto f(f(e^3_0, e^3_2, e^3_1), e^3_1, f(e^3_2, e^3_1, e^3_0))$. For a coterm $t = f(e^3_2, f(e^3_1, e^3_0, e^3_2), e^3_1)$, we get that

$$\begin{aligned} \hat{\sigma}(f(e^3_2, f(e^3_1, e^3_0, e^3_2), e^3_1)) &= \sigma(f)(e^3_2, \sigma(f)(e^3_1, e^3_0, e^3_2), e^3_1) \\ &= f(f(e^3_2, e^3_1, f(f(e^3_1, e^3_2, e^3_0), e^3_0, f(e^3_2, e^3_0, e^3_1))), \\ &\quad f(f(e^3_1, e^3_2, e^3_0), e^3_0, f(e^3_2, e^3_0, e^3_1)), \\ &\quad f(e^3_1, f(f(e^3_1, e^3_2, e^3_0), e^3_0, f(e^3_2, e^3_0, e^3_1)), e^3_2) \end{aligned}$$

Proposition 2.2. *If $t, t_1, \ldots, t_n \in CT_\tau$ and $\sigma \in Cohyp_G(\tau)$, then $\hat{\sigma}(t[t_1, \ldots, t_n]) = \hat{\sigma}(t)[\hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_n)]$.*

Proof. We will give a proof by induction on the complexity of the coterm t. If $t = e^n_i$ for $0 \leq i \leq n-1$, then

$$\begin{aligned} \hat{\sigma}(e^n_i[t_1, \ldots, t_n]) &= \hat{\sigma}(t_i) \\ &= e^n_i[\hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_n)] \\ &= \hat{\sigma}(e^n_i)[\hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_n)]. \end{aligned}$$

If $t = e_j^m$ for $j \geq n$, then

$$\begin{aligned}\hat{\sigma}(e_j^m[t_1,\ldots,t_n]) &= e_j^m\\ &= e_j^m[\hat{\sigma}(t_1),\ldots,\hat{\sigma}(t_n)]\\ &= \hat{\sigma}(e_j^m)[\hat{\sigma}(t_1),\ldots,\hat{\sigma}(t_n)].\end{aligned}$$

If $t = f[g_1,\ldots,g_{n_i}]$, for $g_1,\ldots,g_{n_i} \in CT_\tau$ and assume that $\hat{\sigma}(g_k[t_1,\ldots,t_n]) = \hat{\sigma}(g_k)[\hat{\sigma}(t_1),\ldots,\hat{\sigma}(t_n)]$ for all $k = 1,\ldots,n_i$, then

$$\begin{aligned}\hat{\sigma}(t[t_1,\ldots,t_n]) &= \hat{\sigma}((f[g_1,\ldots,g_{n_i}])[t_1,\ldots,t_n])\\ &= \hat{\sigma}((f[g_1[t_1,\ldots,t_n],\ldots,g_{n_i}[t_1,\ldots,t_n]]))\\ &= \sigma(f)[\hat{\sigma}(g_1[t_1,\ldots,t_n]),\ldots,\hat{\sigma}(g_{n_i}[t_1,\ldots,t_n])]\\ &= \sigma(f)[\hat{\sigma}(g_1)[\hat{\sigma}(t_1),\ldots,\hat{\sigma}(t_n)],\ldots,\hat{\sigma}(g_{n_i})[\hat{\sigma}(t_1),\ldots,\hat{\sigma}(t_n)]]\\ &= (\sigma(f)[\hat{\sigma}(g_1),\ldots,\hat{\sigma}(g_{n_i})])[\hat{\sigma}(t_1),\ldots,\hat{\sigma}(t_n)]\\ &= \hat{\sigma}(f[g_1,\ldots,g_{n_i}])[\hat{\sigma}(t_1),\ldots,\hat{\sigma}(t_n)]\\ &= \hat{\sigma}(t)[\hat{\sigma}(t_1),\ldots,\hat{\sigma}(t_n)].\end{aligned}$$

□

Proposition 2.2 means that the mapping $\hat{\sigma}$ is compatible with the operations S^n for all $n \in \mathbb{N}^*$.

On the set $Cohyp_G(\tau)$ of all generalized cohypersubstitutions of type τ we define a function $\circ_{CG} : Cohyp_G(\tau) \times Cohyp_G(\tau) \to Cohyp_G(\tau)$ by $\sigma_1 \circ_{CG} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$ where $\circ$ is the usual composition of mappings. Let σ_{id} be the generalized cohypersubstitution defined by $\sigma_{id}(f_i) := f_i$ for all $i \in I$. Then the following are obtained.

Proposition 2.3. *The set $Cohyp_G(\tau)$ of all generalized cohypersubstitutions of type τ is associates with a binary operation $\circ_{CG}$ and the generalized cohypersubstitution σ_{id} is an identity of $Cohyp_G(\tau)$.*

From Proposition 2.3, we obtain a monoid $(Cohyp_G; \circ_{CG}, \sigma_{id})$. By $Cohyp(\tau)$ we denote the set of all usual cohypersubstitutions of type τ and in [3] it was shown that $(Cohyp(\tau); \circ_{coh}, \sigma_{id})$ is a monoid where a binary operation $\circ_{coh}$ is defined by $\sigma \circ_{coh} \gamma := \hat{\sigma} \circ \gamma$. It is easy to see that $(Cohyp(\tau); \circ_{coh}, \sigma_{id})$ is a submonoid of $(Cohyp_G; \circ_{CG}, \sigma_{id})$.

Definition 2.3. Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a coalgebra of type τ and let $\sigma \in Cohyp_G(\tau)$. The coalgebra derived from $\mathcal{A}$ by the generalized cohypersubstitution σ is defined by $\sigma(\mathcal{A}) := (A; (\sigma(f_i)^{\mathcal{A}})_{i\in I})$ where $\sigma(f_i)^{\mathcal{A}}$ is the cooperation induced by the coterm $\sigma(f_i)$.

The definition above shows that the derived coalgebra $\sigma(\mathcal{A})$ may have a different type of $\mathcal{A}$. We define the fundamental cooperations of $\sigma(\mathcal{A})$ by $f_i^{\sigma(\mathcal{A})} := \sigma(f_i)^{\mathcal{A}}$. Then the following are obtained.

Lemma 2.1. *Let $\mathcal{A}$ be a coalgebra of type τ, let $t \in cT_\tau$ be a coterm of type τ and let $\sigma \in Cohyp_G(\tau)$ be a generalized cohypersubstitution of type τ. Then $t^{\sigma(\mathcal{A})} = \hat{\sigma}(t)^{\mathcal{A}}$.*

Let K be a class of coalgebras of type τ and Σ be a set of equations of type τ. For any $\sigma \in Cohyp_G(\tau)$, we define $\sigma(K) := \{\sigma(\mathcal{A}) \mid \mathcal{A} \in \mathcal{K}\}$ and $\sigma(\Sigma) := \{\hat{\sigma}(s) \approx \hat{\sigma}(t) \mid s \approx t \in \Sigma\}$. If M is a submonoid of $Cohyp_G(\tau)$, we define $\chi_M^{GA}(K) = \bigcup_{\sigma\in M} \sigma(K)$ and $\chi_M^{GE}(\Sigma) = \bigcup_{\sigma\in M} \sigma(\Sigma)$. We see that the mapping χ_M^{GA} maps subclasses $K \subseteq Coalg(\tau)$ of coalgebras of type τ to subclasses $\bigcup_{\sigma\in M} \sigma(K)$ of coalgebras of type J with $|\tau| = |J|$ and χ_M^{GE} maps subsets $\Sigma \subseteq CT_\tau \times CT_\tau$ of equations of type τ to subsets $\sigma(\Sigma) \subseteq CT_\tau \times CT_\tau$ of equations of type τ.

Corollary 2.1. *Let $\mathcal{A}$ be a coalgebra of type τ and $s \approx t$ be an equation of type τ. If M is a submonoid of $Cohyp_G(\tau)$, then*

$$\chi_M^{GA}(\mathcal{A}) \models_{coid} s \approx t \Leftrightarrow \mathcal{A} \models_{coid} \chi_M^{GE}(s \approx t).$$

Proof. Let $\sigma \in M$. Then we have

$$\begin{aligned} \sigma(\mathcal{A}) \models_{coid} s \approx t \quad &\Leftrightarrow \quad s^{\sigma(\mathcal{A})} = t^{\sigma(\mathcal{A})} \\ &\Leftrightarrow \quad \hat{\sigma}(s)^{\mathcal{A}} = \hat{\sigma}(t)^{\mathcal{A}} \\ &\Leftrightarrow \quad \mathcal{A} \models_{coid} \hat{\sigma}(s \approx t). \end{aligned}$$

Therefore, $\sigma(\mathcal{A}) \models_{coid} s \approx t \Leftrightarrow \mathcal{A} \models_{coid} \hat{\sigma}(s \approx t)$ for all $\sigma \in M$.
Hence, $\chi_M^{GA}(\mathcal{A}) \models_{coid} s \approx t \Leftrightarrow \mathcal{A} \models_{coid} \chi_M^{GE}(s \approx t)$. ☐

In Corollary 2.1, we see that a pair $(\chi_M^{GA}, \chi_M^{GE})$ is a conjugate pair with respect to $\models_{coid}$, but not a conjugate pair of additive closure because χ_M^{GA} is not a closure operator. Next we will show that χ_M^{GE} is an additive closure operator on cT_τ.

Lemma 2.2. *The operator χ_M^{GE} for a submonoid M of $Cohyp_G(\tau)$ is an additive closure operator.*

Proof. Extensivity of the operator χ_M^{GE} follows from $\sigma_{id} \in M$. Additivity implies monotonicity and idempotency follows from the closure of M under $\circ_{CG}$. ☐

Let V be a covariety of type τ and $Coid\ V$ be the set of all coidentities satisfied in the variety V. The definition of M-generalized cohyperidentity is defined as follows.

Definition 2.4. Let M be a submonoid of $Cohyp_G(\tau)$. A coidentity $s \approx t$ in V is called an M-generalized cohyperidentity if $\hat{\sigma}(s) \approx \hat{\sigma}(t)$ are coidentities in V for all generalized hypersubstitution $\sigma \in M$. In this case we write

$V \models_{M-GCoh} s \approx t$. For $M = Cohyp_G(\tau)$ we speak of a generalized cohyperidentity. A covariety V is called M-generalized solid if $\chi_M^{GE}(Coid\ V) = Coid\ V$, i.e., if every coidentity in V is an M-generalized cohyperidentity. For $M = Cohyp_G(\tau)$ we speak of a generalized solid.

Clearly, every M-generalized solid variety is M-solid. Every trivial variety is M-generalized solid and the variety $Coalg(\tau)$ consisting of all coalgebras of type τ is generalized solid. From the first section, we know that M-solid classes of coalgebras form complete sublattices of the lattice $\mathcal{L}_{comod}(\tau)$ and the M-cohyperequational theories form complete sublattices of the lattice $\epsilon_{coid}(\tau)$. The following argumentation shows that all M-generalized solid varieties of type τ form a complete lattice. For a class K of coalgebras of type τ, for a set Σ of coequations of type τ and for a monoid M of $Cohyp_G(\tau)$ we define $MGcoid\ K := \{s \approx t \mid \forall \mathcal{A} \in K(\mathcal{A} \models \chi_M^{GE}(s \approx t))\}$ and $MGcomod\Sigma =: \{\mathcal{A} \mid \forall s \approx t \in \Sigma(\mathcal{A} \models \chi_M^{GE}(s \approx t))\}$. It is clear that the pair $(MGcoid, MGcomod)$ forms a Galois connection. The Galois-closed sets are called M-generalized solid classes of coalgebras and M-generalized cohyperequational theories, respectively. By using the general theory of conjugate pairs of additive closure operators in [6], we get that the M-generalized solid classes of coalgebras form complete sublattice of the lattice $\mathcal{L}_{comod}(\tau)$ and the M-generalized cohyperequational theories form complete sublattices of the lattice $\epsilon_{coid}(\tau)$. To verify the above mentions, we then prove theorem 2.1 below.

Theorem 2.1. *Let K be a class of coalgebras of type τ satisfying the equation $K = Comod\ Coid\ K$ and let $M \subseteq Cohyp_G(\tau)$ be a monoid of generalized cohypersubstitutions. Then the following propositions are equivalent:*

(i) $K = MGcomod MGcoid K$,

(ii) $Coid K = MGcoid K$

(iii) $\chi_M^{GE}[Coid K] = Coid K$.

Proof. (i)⇒(ii): Since $K = MGcomod MGcoid K$, then

$$\begin{aligned} CoidK &= Coid(MGcomod MGcoid K) \\ &= Coid\, Comod \chi_M^{GE}(MGcoid K) \\ &= MGcoid MGcomod MGcoid K \\ &= MGcoid K. \end{aligned}$$

(ii) ⇒ (iii): Since $Coid K = MGcoid K$, then

$$\begin{aligned} \chi_M^{GE}(Coid K) &= \chi_M^{GE}(MGcoid K) \\ &= MGcoid K \\ &= Coid K. \end{aligned}$$

(iii) ⇒ (i): Since $\chi_M^{GE}[CoidK] = CoidK$, then

$$\begin{aligned} MGcomodMGcoidMGcomod(CoidK) &= MGcomod(CoidK) \\ &= Comod(\chi_M^{GE}(CoidK)) \\ &= Comod(CoidK) \\ &= K. \end{aligned}$$

Therefore,

$$\begin{aligned} MGcomodMGcoidK &= MGcomod(CoidK) \\ &= K. \end{aligned}$$

□

From the results above, we then obtain.

Corollary 2.2. *All generalized solid covarieties of a given type form a complete lattice which is contained in the complete lattice of all solid covarieties.*

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