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IDEMPOTENT AND REGULAR GENERALIZED COHYPERSUBSTITUTIONS OF TYPE $\tau = (2)$

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Abstract: A mapping σ which assigns to every n_i-ary cooperation symbol f_i a coterm of type $\tau = (n_i)_{i\in I}$ is said to be a generalized cohypersubstitution of type τ. Every generalized cohypersubstition σ of type τ induces a mapping $\hat{\sigma}$ on the set of all coterms of type τ. The set of all generalized cohypersubstitutions of type τ under the binary operation $\circ_{CG}$ which is defined by $\sigma_1 \circ_{CG} \sigma_2 := \hat{\sigma_1} \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$ forms a monoid which is called the monoid of cohypersubstitution of type τ. In this research, we characterize all idempotent and regular elements of $Cohyp(\tau)$ where $\tau = (2)$.

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1. Introduction

Let A be a non-empty set and n be a positive integer. The n-th copower $A^{\sqcup n}$ of A is the union of n disjoint copies of A; formally, we define $A^{\sqcup n}$ as the cartesian product $A^{\sqcup n} := \underline{n} \times A$, where $\underline{n} := \{1, \dots, n\}$. An element (i, a) in this copower corresponds to the element a in the i-th copy of A, for $1 \leq i \leq n$.

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A co-operation on A is a mapping $f^A : A \to A^{\sqcup n}$ for some $n \geq 1$; the natural number n is called the arity of the co-operation f^A. We also need to recall that any n-ary co-operation f^A on set A can be uniquely expressed as a pair (f_1^A, f_2^A) of mappings, $f_1^A : A \to \underline{n}$ and $f_2^A : A \to A$; the first mapping gives the labelling used by f^A in mapping elements to copies of A, and the second mapping tells us what element of A any element is mapped to. We shall denote by $cO_A^{(n)} = \{f^A : A \to A^{\sqcup n}\}$ the set of all $n - ary$ co-operations defined on A, and by $cO_A := \cup_{n\geq 1} cO_A^{(n)}$ the set of all finitary co-operations defined on A. An indexed coalgebra is a pair $(A; (f_i^A)_{i\in I})$, where f_i^A is an n_i-ary cooperation defined on A, and $\tau = (n_i)_{i\in I}$ for $n_i \geq 1$ is called the type of the coalgebra. Coalgebras were studied by Drbohlav [4]. In [1], the following superposition of cooperations was introduced. If $f^A \in cO_A^{(n)}$ and $g_0^A, \ldots, g_{n-1}^A \in cO_A^{(k)}$, then the k-ary co-operation $f^A[g_0^A, \ldots, g_{n-1}^A] : A \to A^{\sqcup k}$ is defined by

$$a \mapsto ((g^A_{f_1^A(a)})_1(f_2^A(a)), (g^A_{f_1^A(a)})_2(f_2^A(a)))$$

for all $a \in A$. The co-operation $f^A[g_0^A, \ldots, g_{n-1}^A]$ is called the *superposition* of f^A and $g_0^A, \ldots, g_{n-1}^A$. It will also be denoted by $comp_k^n(f^A, g_0^A, \ldots, g_{n-1}^A)$.

The *injection co-operations* $\iota_i^{n,A} : A \to A^{\sqcup n}$ are special cooperations which are defined for each $0 \leq i \leq n-1$ by $\iota_i^{n,A} : A \to A^{\sqcup n}$ with $a \mapsto (i, a)$ for all $a \in A$. Then we get a multi-based algebra

$$((cO_A^{(n)})_{n\geq 1}, (comp_k^n)_{k,n\geq 1}, (\iota_i^{n,A})_{0\leq i\leq n-1}),$$

called the *clone of co-operations* on A. In [1] it is mentioned that this algebra is a clone, i.e. it satisfies the three clone axioms $(C1), (C2), (C3)$. In [2], K.Denecke and K.Saengsura gave a full proof of this fact. In [2], the following coterms of type $\tau = (n_i)_{i\in I}$ were introduced. Let $(f_i)_{i\in I}$ be an indexed set of co-operation symbols such that for each $i \in I$, f_i has arity n_i. Let $\bigcup\{e_j^n \mid n \geq 1,\ n \in \mathbb{N},\ 0 \leq j \leq n-1\}$ be a set of symbols which is disjoint from the set $\{f_i \mid i \in I\}$ such that for each $0 \leq j \leq n-1$, e_j^n has arity n. Then coterms of type τ are defined as follows:

(i) For every $i \in I$ the co-operation symbol f_i is an n_i-ary coterm of type τ.

(ii) For every $n \geq 1$ and $0 \leq j \leq n-1$ the symbol e_j^n is an n-ary coterm of type τ.

(iii) If $t_1, \ldots, t_{n_i}$ are n-ary coterms of type τ, then $f_i[t_1, \ldots, t_{n_i}]$ is an n-ary coterm of type τ for every $i \in I$, and if $t_0, \ldots, t_{n-1}$ are m-ary coterms of type τ, then $e_j^n[t_0, \ldots, t_{n-1}]$ is an m-ary coterm of type τ for every $n \geq 1$ and $0 \leq j \leq n-1$.

Let $cT_\tau^{(n)}$ be the set of all n-ary coterms of type τ and let $cT_\tau := \bigcup_{n\geq 1} cT_\tau^{(n)}$ be the set of all (finitary) coterms of type τ.

The superposition of coterms was introduced in [3] as follows: The operation $S_m^n : cT_\tau^{(n)} \times (cT_\tau^{(m)})^n \to cT_\tau^{(m)}$ is defined by induction on the complexity of coterm definition, as follows:

(i) $S_m^n(e_i^n, t_0, \ldots, t_{n-1}) := t_i$ for $0 \leq i \leq n-1$.

(ii) $S_{n_i}^{n_i}(f_i, e_0^{n_i}, \ldots, e_{n_i-1}^{n_i}) := f_i$ for an n_i-ary co-operation symbol f_i .

(iii) $S_m^{n_j}(g_j, t_1, \ldots, t_{n_j}) := g_j[t_1, \ldots, t_{n_j}]$ if g_j is an n_j-ary co-operation symbol.

(iv) $S_m^n(f_i[s_1, \ldots, s_{n_i}], t_1, \ldots, t_n) := f_i[S_m^n(s_1, t_1, \ldots, t_n), \ldots, S_m^n(s_{n_i}, t_1, \ldots, t_n)]$ where f_i is an n_i-ary co-operation symbol, $s_1, \ldots, s_{n_i}$ are n-ary coterms of type τ and $t_1, \ldots, t_n$ are m-ary coterms of type τ.

These operations give us a heterogeneous algebra

$$c\mathcal{T}_\tau := ((cT_\tau^{(n)})_{n\geq 1}, (S_m^n)_{m,n\geq 1}, (e_j^n)_{1\leq j\leq n}).$$

i.e., that is satisfies the axioms (C1),(C2),(C3).

Theorem 1.1 ([3]). *The heterogeneous algebra $c\mathcal{T}_\tau$ satisfies the following identities:*

(C1) $\hat{S}_m^p(z, \hat{S}_m^n(y_1, x_1, \ldots, x_n), \ldots, \hat{S}_m^n(y_p, x_1, \ldots, x_n)) \approx \hat{S}_m^n(\hat{S}_n^p(z, y_1, \ldots, y_p), x_1, \ldots, x_n), \quad (m, n, p \in \mathbb{N}^+),$

(C2) $\hat{S}_m^n(e_i^n, x_1, \ldots, x_n) \approx x_i \quad (m \in \mathbb{N}^+,\ 1 \leq i \leq n),$

(C3) $\hat{S}_n^n(y, e_1^n, \ldots, e_n^n) \approx y, \quad (n \in \mathbb{N}^+).$

(Here $\hat{S}_m^n, e_i^n$ are operation symbols corresponding to the clone type.)

Let $m \in \mathbb{N}^*$, a *generalized superposition* of coterms was introduced in [6] as a mapping $S^m : CT_\tau^{m+1} \to CT_\tau$ defined inductively by the following steps:

(i) If $t = e_i^n$ and $0 \leq i \leq m-1$, then $e_i^n[t_0, \ldots, t_{m-1}] = t_i$, where $t_1, \ldots, t_{m-1} \in CT_\tau$.

(ii) If $t = e_i^n$ and $0 < m \leq i \leq n-1$, then $e_i^n[t_0, \ldots, t_{m-1}] = e_i^n$ where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(iii) If $t = f_i[s_1, \ldots, s_{n_i}]$, then $(f_i[s_1, \ldots, s_{n_i}])[t_1, \ldots, t_m] = f_i[s_1[t_1, \ldots, t_m], \ldots, s_{n_i}[t_1, \ldots, t_m]]$ where $s_1[t_1, \ldots, t_m], \ldots, s_{n_i}[t_1, \ldots, t_m] \in CT_\tau$.

And they were also introduced a generalized cohypersubstitution of type τ as a mapping $\sigma : \{f_i \mid i \in I\} \to CT_\tau$ from the set of all cooperation symbols to the set of all coterms which need not to be preserves the arities. The extension of σ is a mapping $\hat{\sigma} : CT_\tau \to CT_\tau$ which is inductively defined by the following steps:

(i) $\hat{\sigma}(e_j^n) := e_j^n$ for every $n \geq 1$ and $0 \leq j \leq n-1$,

(ii) $\hat{\sigma}(f_i) := \sigma(f_i)$ for every $i \in I$,

(iii) $\hat{\sigma}(f_i[t_1, \ldots, t_{n_i}] := \sigma(f_i)[\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ for $t_1, \ldots, t_{n_i} \in CT_\tau^{(n)}$.

They let $Cohyp_G(\tau)$ be the set of all generalized cohypersubstitutions of type τ and obtained the following results:

Proposition 1.1. [6] *If $t, t_1, \ldots, t_n \in CT_\tau$ and $\sigma \in Cohyp_G(\tau)$, then*

$$\hat{\sigma}(t[t_1, \ldots, t_n]) = \hat{\sigma}(t)[\hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_n)].$$

Proposition 1.2. [6] *The set $Cohyp_G(\tau)$ of all generalized cohypersubstitutions of type τ is associates with a binary operation $\circ_{CG}$ and the generalized cohypersubstitution σ_{id} is an identity of $Cohyp_G(\tau)$.*

Proposition 1.1 means that the mapping $\hat{\sigma}$ is compatible with the operations S^n for all $n \in \mathbb{N}^*$ and the binary operation $\circ_{CG} : Cohyp_G(\tau) \times Cohyp_G(\tau) \to Cohyp_G(\tau)$ in Proposition 1.2 was defined by $\sigma_1 \circ_{CG} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$ where $\circ$ is the usual composition of mappings and the generalized cohypersubstitution σ_{id} was defined by $\sigma_{id}(f_i) := f_i$ for all $i \in I$. Then they forms a monoid $(Cohyp_G; \circ_{CG}, \sigma_{id})$ and called a monoid of generalized cohypersubstitutions.

2. Idempotent and Regular of Generalized Cohypersubstitutions of Type $\tau = (2)$

In this section, we characterized all idempotent and regular of generalized cohypersubstitutions of type $\tau = (2)$. Recalling that an element a of a semigroup S is called idempotent if $aa = a$, and called regular if there exists $x \in S$ such that $axa = a$. We denoted $E(S)$ and $R(S)$ for the set of all idempotent elements and the set of all regular elements of S, respectively (see [5]) . For any $\sigma \in Cohyp_G(\tau)$ and $\tau = (2)$, if $\sigma(f) = t$, we denoted σ by σ_t. For any positive integer n we called the symbol e_j^n the injection symbol for all $0 \leq j \leq n-1$ and for each coterm t, let $E(t)$ be the set of all injection symbols which occur in t.

Lemma 2.1. *Let $t, s, r \in CT_\tau$, where $\tau = (2)$. Then:*

(1) If $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$, then $t[e_0^2, s] = t$.

(2) If $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$, then $t[s, e_1^2] = t$.

(3) If $E(t) \cap \{e_0^2, e_1^2\} = \emptyset$, then $t[s, r] = t$.

Proof. We give a proof by induction on the complexity of the coterm t.
(1) If $t = e_0^2$, then $e_0^2[e_0^2, s] = e_0^2$.
If $t = e_j^m$ for $j \geq 2$, then $e_j^m[e_0^2, s] = e_j^m$.
Now we assume that $t = f[t_1, t_2]$ and that $t_1[e_0^2, s] = t_1$ and $t_2[e_0^2, s] = t_2$.
Then we obtain that

$$\begin{aligned} t[e_0^2, s] &= (f[t_1, t_2])[e_0^2, s] \\ &= f[t_1[e_0^2, s], t_2[e_0^2, s]] \\ &= f[t_1, t_2] \\ &= t. \end{aligned}$$

The proofs of (2) and (3) are similar. □

The next result is a condition for an element of $Cohyp_G(2)$ to be idempotent.

Theorem 2.1. *If $\sigma_t \in Cohyp_G(2)$, then σ_t is an idempotent if and only if $\hat{\sigma}_t(t) = t$.*

Proof. Assume that σ_t is an idempotent. Then

$$\hat{\sigma}_t(t) = \hat{\sigma}_t(\sigma_t(f)) = \sigma_t \circ_{CG} \sigma_t(f) = \sigma_t(f) = t.$$

Conversely, assume that $\hat{\sigma}_t(t) = t$. Then

$$\sigma_t \circ_{CG} \sigma_t(f) = \hat{\sigma}_t(\sigma_t(f)) = \hat{\sigma}_t(t) = t = \sigma_t(f).$$

Therefore, σ_t is an idempotent. □

By applying the definition of extension of generalized cohypersubstitutions and Theorem 2.1, we then obtain

Corollary 2.1. *$\sigma_{e_0^2}$, $\sigma_{e_1^2}$ and σ_{id} are idempotent.*

Corollary 2.2. *If $\sigma_t \in Cohyp_G(2)$ and $E(t) \cap \{e_0^2, e_1^2\} = \emptyset$, then σ_t is an idempotent.*

Theorem 2.2. *If $t = f[t_1, t_2]$ and $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$, then σ_t is an idempotent if and only if $t_1 = e_0^2$.*

Proof. Assume that σ_t is idempotent.

$$\begin{array}{rcl} \text{Then} \quad f[t_1,t_2] &=& \sigma_t(f) \\ &=& \sigma_t \circ_{CG} \sigma_t(f) \\ &=& \hat{\sigma_t}(\sigma_t(f)) \\ &=& \hat{\sigma_t}(f[t_1,t_2]) \\ &=& \sigma_t(f)[\hat{\sigma_t}(t_1),\hat{\sigma_t}(t_2)] \\ &=& f[t_1,t_2][\hat{\sigma_t}(t_1),\hat{\sigma_t}(t_2)] \\ &=& f[t_1[\hat{\sigma_t}(t_1),\hat{\sigma_t}(t_2)],t_2[\hat{\sigma_t}(t_1),\hat{\sigma_t}(t_2)]]. \end{array}$$

Therefore, $t_1 = t_1[\hat{\sigma_t}(t_1),\hat{\sigma_t}(t_2)]$.

Suppose that $t_1 \neq e_0^2$. Then we have 2 following cases.

Case (i) If $t_1 = e_j^m$ for $j \geq 2$, then $\hat{\sigma}(t_1) = t_1$. Since $E(t) \cap \{e_0^2, e_1^2\} = e_0^2$, then $e_0^2 \in E(t_2)$, so $t_2[\hat{\sigma_t}(t_1),\hat{\sigma_t}(t_2)] \neq t_2$. This yields a contradiction.

Case (ii). If the number of cooperation symbol f which occur in a coterm t_1 is greater than or equal 1 and hence $\hat{\sigma_t}(t_1) \neq e_0^2$. There follows $t_1 \neq t_1[\hat{\sigma_t}(t_1),\hat{\sigma_t}(t_2)]$. This also yields a contradiction.

Therefore, $t_1 = e_0^2$.

Conversely, let $t_1 = e_0^2$.

$$\begin{array}{rcll} \text{Then} \quad \hat{\sigma_t}(t) &=& \hat{\sigma_t}(f[t_1,t_2]) & \\ &=& \hat{\sigma_t}(f[e_0^2,t_2]) & \\ &=& \sigma_t(f)[\hat{\sigma_t}(e_0^2),\hat{\sigma_t}(t_2)] & \\ &=& (f[e_0^2,t_2])[e_0^2,\hat{\sigma_t}(t_2)] & \\ &=& f[e_0^2,t_2] & \text{(by Lemma 2.1(1))} \\ &=& t. & \end{array}$$

□

Similarly, we will have

Theorem 2.3. *If $t = f[t_1,t_2]$ and $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$, then σ_t is an idempotent if and only if $t_2 = e_1^2$.*

The next result is a condition for an element of $\sigma_t \in Cohyp_G(2)$ to be idempotent, where $\{e_0^2, e_1^2\} \subseteq E(t)$.

Theorem 2.4. *If $t = f[t_1,t_2]$ and $\{e_0^2, e_1^2\} \subseteq E(t)$, then σ_t is an idempotent if and only if $t_1 = e_0^2$ and $t_2 = e_1^2$.*

Proof. Assume that σ_t is an idempotent. From the proof of Theorem 2.2, we have that $f[t_1,t_2] = f[t_1[\hat{\sigma_t}(t_1),\hat{\sigma_t}(t_2)],t_2[\hat{\sigma_t}(t_1),\hat{\sigma_t}(t_2)]]$.

Suppose that $t_1 \neq e_0^2$ or $t_2 \neq e_1^2$. Then we have three cases for t_1 and t_2.

Case (1) If $t_1 \notin \{e_0^2, e_1^2\}$ and $t_2 \in \{e_0^2, e_1^2\}$, then $\hat{\sigma_t}(t_1) \notin \{e_0^2, e_1^2\}$ and $\hat{\sigma_t}(t_2) \in \{e_0^2, e_1^2\}$. Let us consider, if $t_1 = e_j^m$ for $j \geq 2$, then $\hat{\sigma_t}(t_1) = t_1$. Since $\{e_0^2, e_1^2\} \subseteq E(t)$, then $\{e_0^2, e_1^2\} \subseteq E(t_2)$, so $t_2 \neq t_2[\hat{\sigma_t}(t_1),\hat{\sigma_t}(t_2)]$, which is a contradiction.

If $t_1 = f[s_1, s_2]$ for some $s_1, s_2 \in CT_{(2)}$, then the number of operation symbols which occur in the coterm $\hat{\sigma_t}(t_1)$ is greater than or equal to 1. Therefore, the number of operation symbols which occur in the coterm

$$f[t_1[\hat{\sigma_t}(t_1), \hat{\sigma_t}(t_2)], t_2[\hat{\sigma_t}(t_1), \hat{\sigma_t}(t_2)]]$$

and $f[t_1, t_2]$ are different. This gives

$$f[t_1, t_2] \neq f[t_1[\hat{\sigma_t}(t_1), \hat{\sigma_t}(t_2)], t_2[\hat{\sigma_t}(t_1), \hat{\sigma_t}(t_2)]].$$

Case (2) If $t_1 \in \{e_0^2, e_1^2\}$ and $t_2 \notin \{e_0^2, e_1^2\}$, then $\hat{\sigma_t}(t_1) \in \{e_0^2, e_1^2\}$ and $\hat{\sigma_t}(t_2) \notin \{e_0^2, e_1^2\}$.

If $t_2 = e_j^m$ for $j \geq 2$, then $\hat{\sigma_t}(t_2) = t_2$. Since $\{e_0^2, e_1^2\} \subseteq E(t)$, then $\{e_0^2, e_1^2\} \subseteq E(t_1)$, so $t_1 \neq t_1[\hat{\sigma_t}(t_1), \hat{\sigma_t}(t_2)]$, which is a contradiction. If $t_1 = f[s_1, s_2]$ for some $s_1, s_2 \in CT_{(2)}$, then the number of operation symbols which occur in the coterm $\hat{\sigma_t}(t_2)$ is greater than or equal to 1. Therefore, the number of operation symbols which occur in the coterm $f[t_1[\hat{\sigma_t}(t_1), \hat{\sigma_t}(t_2)], t_2[\hat{\sigma_t}(t_1), \hat{\sigma_t}(t_2)]]$ and $f[t_1, t_2]$ are different.

This gives $f[t_1, t_2] \neq f[t_1[\hat{\sigma_t}(t_1), \hat{\sigma_t}(t_2)], t_2[\hat{\sigma_t}(t_1), \hat{\sigma_t}(t_2)]]$.

Case (3) If $t_1, t_2 \in \{e_0^2, e_1^2\}$, then $t_1 = e_1^2$ and $t_2 = e_0^2$, so $\hat{\sigma_t}(t_1) = e_1^2 \neq e_0^2$ and $\hat{\sigma_t}(t_2) = e_0^2 \neq e_1^2$. Therefore, $t_1[\hat{\sigma_t}(t_1), \hat{\sigma_t}(t_2)] \neq t_1$ and $t_2[\hat{\sigma_t}(t_1), \hat{\sigma_t}(t_2)] \neq t_2$. This gives a contradiction.

Therefore, we get that $t_1 = e_0^2$ and $t_2 = e_1^2$.

Conversely, it is easy to see that $\hat{\sigma_t}(t) = t$ if $t = f[e_0^2, e_1^2]$. □

The characterization of idempotent element of $Cohyp_G(2)$ gives us three disjoint sets of all idempotent elements of $Cohyp_G(2)$ as follow:

$E_0 := \{\sigma_{e_0^2}, \sigma_{e_1^2}, \sigma_{id}\}$,

$E_1 := \{\sigma_t \mid t = f[e_0^2, s] \text{ where } E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}, s \in Cohyp(2)\}$,

$E_2 := \{\sigma_t \mid t = f[s, e_1^2] \text{ where } E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}, s \in Cohyp(2)\}$, and

$E_3 := \{\sigma_t \mid E(t) \cap \{e_0^2, e_1^2\} = \emptyset\}$.

Now, we characterize all regular elements of $Cohyp_G(2)$.

Theorem 2.5. *Let $t \in Cohyp_G(2)$ and $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$. Then σ_t is regular if and only if t is of the form*

(1) $t = f[e_0^2, s]$ for some $s \in cT_{(2)}$ or

(2) $t = f[s, e_0^2]$ for some $s \in cT_{(2)}$.

Proof. ($\Leftarrow$) (1)Assume that $t = f[e_0^2, s]$ for some $s \in cT_{(2)}$.
Let $u \in cT_{(2)}$ such that $u = f[e_0^2, v]$ for some $v \in cT_{(2)}$. Then

$$\begin{aligned} \hat{\sigma_u}(t) &= \hat{\sigma_u}(f[e_0^2, v]) \\ &= \sigma_u(f)[\hat{\sigma_u}(e_0^2), \hat{\sigma_u}(v)] \\ &= (f[e_0^2, v])[\hat{\sigma_u}(e_0^2), \hat{\sigma_u}(v)] \\ &= (f[e_0^2, v])[e_0^2, \hat{\sigma_u}(v)] \\ &= f[e_0^2[e_0^2, \hat{\sigma_u}(v)], v[e_0^2, \hat{\sigma_u}(v)]] \\ &= f[e_0^2, v[e_0^2, \hat{\sigma_u}(v)]]. \end{aligned}$$

So, $$\begin{aligned} \hat{\sigma}_t(\hat{\sigma_u}(t)) &= \hat{\sigma}_t(f[e_0^2, v[e_0^2, \hat{\sigma_u}(v)]]) \\ &= \sigma_t(f)[\hat{\sigma}_t(e_0^2), \hat{\sigma}_t(v[e_0^2, \hat{\sigma_u}(v)])] \\ &= (f[e_0^2, s])[e_0^2, \hat{\sigma}_t(v[e_0^2, \hat{\sigma_u}(v)])] \\ &= f[e_0^2, s] && \text{(by Lemma 2.1(1))} \\ &= t. \end{aligned}$$

Therefore, $(\sigma_t \circ_{CG} \sigma_u \circ_{CG} \sigma_t)(f) = t = \sigma_t(f)$. Hence σ_t is regular.
The proof of (2) is similar.

($\Rightarrow$) Assume that σ_t is regular. Suppose that a condition (1) is not true, we have to show that (2) is true.

Let $t = f[t_1, t_2]$ such that $t_1 \neq e_0^2$. Suppose that $t_2 \neq e_0^2$. Then for any $\sigma_u \in Cohyp_G(2)$ and $\sigma_u(f) = f[v_1, v_2]$, we get that

$$\begin{aligned} \hat{\sigma}_u(t) &= \hat{\sigma}_u(f[t_1, t_2]) \\ &= \sigma_u(f)[\hat{\sigma}_u(t_1), \hat{\sigma}_u(t_2)] \\ &= (f[v_1, v_2])[\hat{\sigma}_u(t_1), \hat{\sigma}_u(t_2)] \\ &= f[v_1[\hat{\sigma}_u(t_1), \hat{\sigma}_u(t_2)], v_2[\hat{\sigma}_u(t_1), \hat{\sigma}_u(t_2)]] \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{\sigma}_t \circ \hat{\sigma}_u(t) &= \hat{\sigma}_t(f[v_1[\hat{\sigma}_u(t_1), \hat{\sigma}_u(t_2)], v_2[\hat{\sigma}_u(t_1), \hat{\sigma}_u(t_2)]]) \\ &= \sigma_t(f)[\hat{\sigma}_t(v_1[\hat{\sigma}_u(t_1), \hat{\sigma}_u(t_2)]), \hat{\sigma}_t(v_2[\hat{\sigma}_u(t_1), \hat{\sigma}_u(t_2)])]) \\ &= (f[t_1, t_2])[\hat{\sigma}_t(v_1[\hat{\sigma}_u(t_1), \hat{\sigma}_u(t_2)]), \hat{\sigma}_t(v_2[\hat{\sigma}_u(t_1), \hat{\sigma}_u(t_2)])]) \\ &= f[t_1[\hat{\sigma}_t(v_1[\hat{\sigma}_u(t_1), \hat{\sigma}_u(t_2)]), \hat{\sigma}_t(v_2[\hat{\sigma}_u(t_1), \hat{\sigma}_u(t_2)])], \\ &\quad\; t_2[\hat{\sigma}_t(v_1[\hat{\sigma}_u(t_1), \hat{\sigma}_u(t_2)]), \hat{\sigma}_t(v_2[\hat{\sigma}_u(t_1), \hat{\sigma}_u(t_2)])]]. \end{aligned}$$

Since $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$ and $t_2 \neq e_0^2$, then
$t_2[\hat{\sigma}_t(v_1[\hat{\sigma}_u(t_1), \hat{\sigma}_u(t_2)]), \hat{\sigma}_t(v_2[\hat{\sigma}_u(t_1), \hat{\sigma}_u(t_2)])] \neq t_2$ for all $\sigma_u \in cT_{(2)}$, which contradicts to σ_t is regular. Therefore, we get that $t_2 = e_0^2$. □

Theorem 2.6. *Let $t \in cT_{(2)}$ and $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$. Then σ_t is regular if and only if t is of the form*

(1) $t = f[s, e_1^2]$ for some $s \in cT_{(2)}$ or

(2) $t = f[e_1^2, s]$ for some $s \in cT_{(2)}$.

Proof. The proof is similar to that of Theorem 2.5 and is thus omitted. □

Theorem 2.7. *Let $t = f[t_1, t_2]$ and $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2, e_1^2\}$. Then:*

(1) If $t_1 = e_0^2$ and $t_2 = e_1^2$, then σ_t is regular and

(2) If $t_1 = e_1^2$ and $t_2 = e_0^2$, then σ_t is regular.

Proof. (1) Assume that $t = f[e_0^2, e_1^2]$. By Theorem 2.4, σ_t is an idempotent, so σ_t is regular.

(2) If $t = f[e_1^2, e_0^2]$, then we let $s = f[e_1^2, e_0^2]$.

$$\begin{array}{llll} \text{Thus} & \hat{\sigma_s}(t) & = & \hat{\sigma_s}(f[e_1^2, e_0^2]) \\ & & = & \sigma_s(f)[\hat{\sigma_s}(e_1^2), \hat{\sigma_s}(e_0^2)] \\ & & = & (f[e_1^2, e_0^2])[e_1^2, e_0^2] \\ & & = & f[e_0^2, e_1^2]. \\ \text{So} & \hat{\sigma_t}(\hat{\sigma_s}(t)) & = & \hat{\sigma_t}(f[e_0^2, e_1^2]) \\ & & = & \sigma_t(f)[\hat{\sigma_t}(e_0^2), \hat{\sigma_t}(e_1^2)] \\ & & = & (f[e_1^2, e_0^2])[e_0^2, e_1^2] \\ & & = & f[e_1^2, e_0^2]. \end{array}$$

Therefore $(\sigma_t \circ_{CG} \sigma_s \circ_{CG} \sigma_t)(f) = t = \sigma_t(f)$. Hence, σ_t is regular. □

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