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REGULAR WEAKLY CLOSED SETS IN IDEAL TOPOLOGICAL SPACES

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Abstract: In this paper, we define I_{rw}-closed sets and I_{rw}-open sets, characterize these sets and discuss their properties. Then, we define $\vee_{rs}$-sets and $\wedge_{rs}$-sets and discuss the relation between them. Also, we give characterizations of I_{rw}-closed sets and rw-closed sets.

A separation axiom stronger than rsT_I-space is defined and various characterizations are given.

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Key Words: I_{rw}-closed, I_{rw}-open, $\mathcal{N}_I$-rw-set, $\wp_I$-rw-set, $\vee_{rs}$-set, $\wedge_{rs}$-set, $I.\vee_{rs}$-set, $I.\wedge_{rs}$-set, rsT_I-spaces

1. Introduction

An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in I$ and $B \subseteq A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space (X, τ) with an ideal I on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^* : \wp(X) \to \wp(X)$, called a local function [6] of A with respect to τ and I is defined as follows: for $A \subseteq X$, $A^*(I, \tau) = \{x \in X | U \cap A \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau | x \in U\}$. We will make use of the basic facts about the local

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function [[5], Theorem 2.3] without mentioning it explicitly. In particular, the local function is monotonic. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(I,\tau)$, called the $*$- topology, finer than τ is defined by $cl^*(A) = A \cup A^*(I,\tau)$ [16]. Clearly, if $I = \{\phi\}$, then $cl^*(A) = cl(A)$ for every subset A of X. When there is no chance for confusion, we will simply write A^* for $A^*(I,\tau)$ and τ^* for $\tau^*(I,\tau)$. If I is an ideal on X, then (X,τ,I) is called an ideal space. A subset A of an ideal space (X,τ,I) is $*$-closed (τ^*-closed) [5] (resp. $*$-dense in itself [4]) if $A^* \subseteq A$ (resp. $A \subseteq A^*$). A subset A of an ideal space (X,τ,I) is I_g-closed [3] if $A^* \subseteq U$ whenever U is open and $A \subseteq U$. By Example 2.1(b) of [11], every $*$-closed and hence every closed set is I_g-closed. A subset A of an ideal space (X,τ,I) is said to be I_g-open if $X - A$ is I_g-closed. An ideal space (X,τ,I) is said to be a T_I-space [3] if every I_g-closed set is $*$-closed. An ideal I is said to be codense [5] or a boundary ideal [16] if $\tau \cap I = \{\phi\}$. $\mathcal{N}$ denotes the ideal of all nowhere dense subset in (X,τ). In this paper, we define I_{rw}-closed and I_{rw}-open sets in ideal spaces, discuss their properties and give various characterizations. Again, for the collection of these sets, we define $\vee_{rs}$-sets and $\wedge_{rs}$-sets and discuss their properties.

By a space (X,τ), we always mean a topological space (X,τ) with no separation properties assumed. If $A \subseteq X$, $cl(A)$ and $int(A)$ will, respectively, denote the closure and interior of A in (X,τ) and $int^*(A)$ will denote the interior of A in (X,τ^*). A subset A of a space (X,τ) is said to be regular open [15] if $A = int(cl(A))$ and A is said to be regular closed [15] if $A = cl(int(A))$. A subset A of a space (X,τ) is said to be semi open [7] (resp. α-open [12]) if $A \subseteq cl(int(A))$ (resp. $A \subseteq int(cl(int(A)))$). The family of all α-open sets in a space (X,τ), denoted by τ^α, is a topology on X finer than τ. The closure of a subset A in (X,τ^α) is denoted by $cl_\alpha(A)$. A subset A of a space (X,τ) is said to be g-closed [8] (resp. αg-closed [9]) if $cl(A) \subseteq U$ (resp. $cl_\alpha(A) \subseteq U$) whenever $A \subseteq U$ and U is open. A subset A of a space (X,τ) is said to be regular semiopen [2] if there is a regular open set U such that $U \subseteq A \subseteq cl(U)$. A subset A of a space (X,τ) is said to be rw-closed [1] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular semiopen. A is said to be rw-open (resp. g-open) if $X - A$ is rw-closed (resp. g-closed). The following lemmas will be useful in the sequel.

Lemma 1.1.. (see [14], Lemma 1.2) *Let (X,τ,I) be an ideal space and $A \subseteq X$. If $A \subseteq A^*$, then $A^* = cl(A^*) = cl(A) = cl^*(A)$.*

Lemma 1.2.. (see [14], Lemma 1.1) *Let (X,τ,I) be an ideal space. Then I is codense if and only if $A \subseteq A^*$ for every semiopen set A in X.*

2. I_{rw}-Closed and I_{rw}-Open Sets

A subset A of an ideal space (X, τ, I) is said to be a *regular weakly closed set with respect to the ideal I (I_{rw}-closed)* if $A^* \subseteq U$ whenever $A \subseteq U$ and U is regular semiopen. A is called a regular weakly open set (I_{rw}-open) if $X - A$ is an I_{rw}-closed set. Clearly, every I_g-closed set is I_{rw}-closed and every rw-closed set is an I_{rw}-closed set. The following Example 2.1(a) shows that an I_{rw}-closed set is not an I_g-closed set and Example 2.1(b) shows that an I_{rw}-closed set is not an rw-closed set. Theorem 2.2. shows that I_{rw}-closed sets is finitely additive and Example 2.3 shows that the finite intersection of I_{rw}-closed sets is not an I_{rw}-closed set. If $I = \{\phi\}$, then $cl(A) = cl^*(A) = A^*$ for every subset A of X and so I_{rw}-closed sets coincide with rw-closed sets.

Example 2.1. (a) Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $I = \{\phi, \{c\}\}$. Then $A = \{a, c\}$ is not regular semiopen. So the only regular semiopen set containing A is X and hence A is I_{rw}-closed. Since A is open and $A^* = \{a, b\} \nsubseteq A$, A is not an I_g-closed set.

(b) Let X and τ be as in Example 2.1 (a). Let $I = \wp(X)$. Since every subset of X is $*$-closed, every subset of X is I_{rw}-closed. Since $\{a\}$ is regular semiopen and not closed, $\{a\}$ is not rw-closed.

Theorem 2.1.. *Every rw-closed set is I_{rw}-closed.*

Proof. Let A be any rw-closed set in (X, τ, I). Then $cl(A) \subseteq U$ whenever U is regular semiopen and $A \subseteq U$. Since $A^* \subseteq cl(A) \subseteq U$, we have $A^* \subseteq U$ and hence A is I_{rw}-closed. □

Example 2.2. Let $X = \{a, b, c, d, e\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}$ and $I = \{\phi, \{b\}, \{d\}, \{b, d\}\}$. Then the set $\{b\}$ is I_{rw}-closed but not rw-closed.

Remark 2.1.. For a subsets of an ideal space, the following implications hold.

$*$-closed $\Rightarrow$ I_g-closed $\Rightarrow$ I_{rw}-closed $\Leftarrow$ rw-closed
when none of the implications is reversible as shown in [[13], Example 3.7]

Theorem 2.2.. *In an ideal space (X, τ, I), the union of two I_{rw}-closed sets is an I_{rw}-closed set.*

Proof. Let A and B be I_{rw}-closed sets in (X, τ, I). Suppose $A \cup B \subseteq U$ and U is regular semiopen. Then $A \subseteq U$ and $B \subseteq U$. By hypothesis, $cl^*(A) \subseteq U$ and $cl^*(B) \subseteq U$. Therefore, $cl^*(A \cup B) = cl^*(A) \cup cl^*(B) \subseteq U$ which implies that $A \cup B$ is I_{rw}-closed. □

Example 2.3. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $I = \{\phi, \{c\}\}$. Let $A = \{a, c\}$ and $B = \{a, b\}$. Since X is the only regular semiopen set containing A and B, A and B are I_{rw}-closed. Now $A \cap B = \{a\}$ is regular semiopen and $(A \cap B)^* = \{a, b\} \nsubseteq \{a\}$. Therefore, $A \cap B$ is not an I_{rw}-closed set.

The following Theorem 2.3. gives a property of I_{rw}-closed sets and Example 2.4 shows that the converse of Theorem 2.3. is not true.

Theorem 2.3.. *Let (X, τ, I) be an ideal space and $A \subseteq X$. If A is I_{rw}-closed, then $cl^*(A) - A$ contains no nonempty regular semiclosed set.*

Proof. Let A be an I_{rw}-closed set of (X, τ, I). Suppose a regular semiclosed set F is contained in $cl^*(A) - A = cl^*(A) \cap (X - A)$. Since $F \subseteq X - A$, we have $A \subseteq X - F$ and $X - F$ is regular semiopen. Therefore, $cl^*(A) \subseteq X - F$ and so $F \subseteq X - cl^*(A)$. Already, we have $F \subseteq cl^*(A)$ and so $F = \phi$. Hence $cl^*(A) - A$ contains no nonempty regular semiclosed sets. □

Example 2.4. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$ and $I = \{\phi\}$. Let $A = \{a\}$. Then $cl^*(A) - A = cl(A) - A = \{a, c, d\} - \{a\} = \{c, d\}$ which does not contain any nonempty regular semiclosed set. But A is not I_{rw}-closed, since A is regular semiopen and $cl^*(A) = cl(A) = \{a, c, d\} \nsubseteq A$.

Theorem 2.4.. *If A is an I_{rw}-closed subset of an ideal space (X, τ, I) and $A \subseteq B \subseteq cl^*(A)$, then B is also I_{rw}-closed. In particular, the $*$-closure of every I_{rw}-closed set is an I_{rw}-closed set.*

Proof. Suppose $B \subseteq U$ and U is regular semiopen. Since A is I_{rw}-closed and $A \subseteq U$, $cl^*(A) \subseteq U$. Since $B \subseteq cl^*(A)$, $cl^*(B) \subseteq cl^*(A)$. Therefore, $cl^*(B) \subseteq U$ and so B is I_{rw}-closed. □

Theorem 2.5.. *If (X, τ, I) is an ideal space and A is a $*$-dense in itself, I_{rw}-closed subset of X, then A is a rw-closed set.*

Proof. If A is a $*$-dense in itself, I_{rw}-closed subset of X and U is any regular semiopen set containing A, then $cl^*(A) \subseteq U$. Since A is $*$-dense in itself, by Lemma 1.1., $cl(A) \subseteq U$ and so A is rw-closed. □

Corollary 2.1.. *Let A and B be subsets of an ideal space (X, τ, I) such that $A \subseteq B \subseteq A^*$. If A is I_{rw}-closed, then A and B are rw-closed.*

Proof. Since $A \subseteq B \subseteq A^*$, $A \subseteq B \subseteq cl^*(A)$. Since A is I_{rw}-closed, by Theorem 2.4., B is I_{rw}-closed. Since $A \subseteq B \subseteq A^*$, $B^* = A^*$. Therefore, A and B are $*$-dense in itself. By Theorem 2.5., A and B are rw-closed. □

Corollary 2.2.. *Let (X, τ, I) be an ideal space where I is codense. If A*

is a semiopen, I_{rw}-closed subset of X, then A is rw-closed.

Proof. Since A is a semiopen, by Lemma 1.2., $A \subseteq A^*$. So, A is $*$-dense in itself and I_{rw}-closed. Therefore, A is rw-closed, by Theorem 2.5. □

The following Theorem 2.6. gives another property of I_{rw}-closed sets.

Theorem 2.6.. *Let (X, τ, I) be an ideal space and A be an I_{rw}-closed set. Then the following are equivalent.*

(a) A is a $*$-closed set.

(b) $cl^*(A) - A$ is a regular semiclosed set.

(c) $A^* - A$ is a regular semiclosed set.

Proof. $(a) \Rightarrow (b)$ If A is $*$-closed, then $cl^*(A) - A = \phi$ and so $cl^*(A) - A$ is regular semiclosed.

$(b) \Rightarrow (a)$ Suppose $cl^*(A) - A$ is regular semiclosed. Since A is I_{rw}-closed, by Theorem 2.3., $cl^*(A) - A = \phi$ and so A is $*$-closed.

$(b) \Leftrightarrow (c)$. The proof follows from the fact that $cl^*(A) - A = A^* - A$. □

Theorem 2.7.. *Let (X, τ, I) be an ideal space and A be an I_{rw}-closed set. Then $A \cup (X - A^*)$ is also an I_{rw}-closed set.*

Proof. Suppose that A is an I_{rw}-closed set. If U is any regular semiopen set such that $A \cup (X - A^*) \subseteq U$, then $X - U \subseteq X - (A \cup (X - A^*)) = (X - A) \cap A^* = A^* - A$. Since $X - U$ is regular semiclosed and A is I_{rw}-closed, by Theorem 2.3., it follows that $X - U = \phi$ and so $X = U$. Hence X is the only regular semiopen set containing $A \cup (X - A^*)$ and so $A \cup (X - A^*)$ is I_{rw}-closed. □

The following Example 2.5 shows that the converse of the above Theorem 2.7. is not true.

Example 2.5. Let (X, τ) and I be as in Example 2.4. Let $A = \{a\}$. Then A is not I_{rw}-closed. Now, $A \cup (X - A^*) = \{a\} \cup (X - \{a, c, d\}) = \{a\} \cup \{b\} = \{a, b\}$. Since the only regular semiopen set containing $A \cup (X - A^*)$ is X, $A \cup (X - A^*)$ is I_{rw}-closed.

Theorem 2.8.. *Let (X, τ, I) be an ideal space. Then the following are equivalent.*

(a) Every subset of X is I_{rw}-closed.

(b) Every regular semiopen set is $*$-closed.

Proof. $(a) \Rightarrow (b)$ Suppose every subset of X is I_{rw}-closed. If U is regular semiopen, then by hypothesis, U is I_{rw}-closed and so $U^* \subseteq U$. Hence U is $*$-closed.

$(b) \Rightarrow (a)$ Suppose every regular semiopen set is $*$-closed. Let A be a subset of X. If U is a regular semiopen set such that $A \subseteq U$, then $A^* \subseteq U^* \subseteq U$ and so A is I_{rw}-closed. □

The following Theorem 2.9. gives a characterization of I_{rw}-open sets.

Theorem 2.9.. *Let (X, τ, I) be an ideal space. A subset $A \subseteq X$ is I_{rw}-open if and only if $F \subseteq int^*(A)$ whenever F is regular semiclosed and $F \subseteq A$.*

Proof. Suppose A is I_{rw}-open. Let F be a regular semiclosed set contained in A. Then $cl^*(X - A) \subseteq X - F$ and so $F \subseteq X - cl^*(X - A) = int^*(A)$. Conversely, suppose $X - A \subseteq U$ and U is regular semiopen. By hypothesis, $X - U \subseteq int^*(A)$ which implies that $cl^*(X - A) \subseteq U$. Therefore, $X - A$ is I_{rw}-closed and so A is I_{rw}-open. □

Theorem 2.10.. *In an ideal space (X, τ, I), if A is an I_{rw}-open set, then $G = X$ whenever G is regular semiopen and $int^*(A) \cup (X - A) \subseteq G$.*

Proof. Let A be an I_{rw}-open set. Suppose G is a regular semiopen set such that $int^*(A) \cup (X - A) \subseteq G$. Then $X - G \subseteq (X - int^*(A)) \cap A = (X - int^*(A)) - (X - A) = cl^*(X - A) - (X - A)$. Since $X - A$ is I_{rw}-closed, by Theorem 2.3., $X - G = \phi$ and so $G = X$. □

The following Example 2.6 shows that the converse of the above Theorem 2.10. is not true.

Example 2.6. Let (X, τ) and I be as in Example 2.4. The nonempty regular semiopen sets are $\{a\}, \{b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, c, d\}, \{b, c, d\}$ and X. Let $A = \{b, c, d\}$. Then $int^*(A) \cup (X - A) = \{b\} \cup \{a\} = \{a, b\}$ and X is the only regular semiopen set containing $int^*(A) \cup (X - A)$. Since A is regular semiclosed and $A \nsubseteq int^*(A) = int(A) = \{b\}$, A is not I_{rw}-open.

Theorem 2.11.. *If A is an I_{rw}-closed set in an ideal space (X, τ, I), then $cl^*(A) - A$ is I_{rw}-open.*

Proof. Since A is I_{rw}-closed, by Theorem 2.3., ϕ is the only regular semiclosed set contained in $cl^*(A) - A$ and so by Theorem 2.9., $cl^*(A) - A$ is I_{rw}-open. □

The following Example 2.7 shows that the converse of the above Theorem 2.11. is not true.

Example 2.7. Consider the ideal space (X, τ, I) of Example 2.4. Let $A = \{a\}$. Then $cl^*(A) - A = cl(A) - A = \{c, d\}$. Since ϕ is the only regular semiclosed set contained in $cl^*(A) - A$, by Theorem 2.9., $cl^*(A) - A$ is I_{rw}-open. But A is not I_{rw}-closed.

Theorem 2.12.. *Let (X, τ, I) be an ideal space and $A \subseteq X$. If A is*

I_{rw}-open and $int^*(A) \subseteq B \subseteq A$, then B is I_{rw}-open.

Proof. Since $int^*(A) \subseteq B \subseteq A$, we have $int^*(A) = int^*(B)$. Suppose F is regular semiclosed and $F \subseteq B$, then $F \subseteq A$. Since A is I_{rw}-open, by Theorem 2.9., $F \subseteq int^*(A) = int^*(B)$. So, again by Theorem 2.9., B is I_{rw}-open. □

Definition 2.1.. A subset A of an ideal topological space (X, τ, I) is said to be:

1. a $\mathcal{N}_I$-rw-set if $A = U \cap V$, where U is a regular semiopen set and V is a $*$-perfect set.

2. a $\wp_I$-rw-set if $A = U \cap V$, where U is a regular semiopen set and V is a $*$-closed set.

Theorem 2.13.. *A subset A of an ideal topological space (X, τ, I) is a $\mathcal{N}_I$-rw-set and a I_{rw}-closed set, then A is a $*$-closed set.*

Proof. Let A be a $\mathcal{N}_I$-rw-set and a I_{rw}-closed set. Since A is a $\mathcal{N}_I$-rw-set, $A = U \cap V$, where U is a regular semiopen set and V is a $*$-perfect set. Now, $A = U \cap V \subseteq U$ and A is a I_{rw}-closed set implies that $A^* \subseteq U$. Also, $A = U \cap V \subseteq V$ and V is $*$-perfect set implies that $A^* \subseteq V$. Thus, $A^* \subseteq U \cap V = A$. Hence, A is a $*$-closed set. □

Theorem 2.14.. *For a subset A of an ideal topological space (X, τ, I), the following are equivalent.*

1. *A is a $*$-closed set.*

2. *A is a $\wp_I$-rw-set and a I_{rw}-closed set.*

Proof. (1) $\Rightarrow$ (2). Let A be a $*$-closed set and $A = X \cap V$, where X is regular semiopen set and V is a $*$-closed set. Hence, A is a $\wp_I$-rw-set. Assume that A be a $*$-closed set and U be a regular semiopen set such that $A \subseteq U$. Then $A^* \subseteq U$ and hence A is a I_{rw}-closed set.

(2) $\Rightarrow$ (1) Let A be a $\wp_I$-rw-set and a I_{rw}-closed set. Since A is a $\wp_I$-rw-set, $A = U \cap V$, where U is a regular semiopen set and V is a $*$-closed set. Now, $A \subseteq U$ and A is a I_{rw}-closed set implies that $A^* \subseteq U$, Also, $A \subseteq V$ and V is a $*$-closed set implies that $A^* \subseteq V$. Thus, $A^* \subseteq U \cap V = A$. Hence, A is a $*$-closed set. □

3. $\bigvee_{rs}$ and $\bigwedge_{rs}$-Sets

Let (X, τ) be a space. If $B \subseteq X$, we define $B^{\vee}_{rs} = \cup\{F : F \subseteq B$ and F is regular semiclosed$\}$ and $B^{\wedge}_{rs} = \cap\{U : B \subseteq U$ and U is regular semiopen$\}$. The following Theorem 3.1. gives properties of these operators, the proof of which is omitted, since the proofs are similar to the corresponding proofs for open sets.

Theorem 3.1.. *Let (X, τ) be a space. If A and B are subsets of X, then the following hold.*

(a) $\phi_{rs}^{\vee} = \phi$ *and* $\phi_{rs}^{\wedge} = \phi$.

(b) $X_{rs}^{\vee} = X$ *and* $X_{rs}^{\wedge} = X$.

(c) $A_{rs}^{\vee} \subseteq A$ *and* $A \subseteq A_{rs}^{\wedge}$.

(d) $(A_{rs}^{\vee})_{rs}^{\vee} = A_{rs}^{\vee}$.

(e) $(A_{rs}^{\wedge})_{rs}^{\wedge} = A_{rs}^{\wedge}$.

(f) $A \subseteq B \Rightarrow A_{rs}^{\vee} \subseteq B_{rs}^{\vee}$.

(g) $A \subseteq B \Rightarrow A_{rs}^{\wedge} \subseteq B_{rs}^{\wedge}$.

(h) $A_{rs}^{\vee} \cup B_{rs}^{\vee} \subseteq (A \cup B)_{rs}^{\vee}$.

(i) $A_{rs}^{\wedge} \cup B_{rs}^{\wedge} \subseteq (A \cup B)_{rs}^{\wedge}$.

(j) $(A \cap B)_{rs}^{\vee} \subseteq A_{rs}^{\vee} \cap B_{rs}^{\vee}$.

(k) $(A \cap B)_{rs}^{\wedge} \subseteq A_{rs}^{\wedge} \cap B_{rs}^{\wedge}$.

(l) $A_{rs}^{\vee} \subseteq A^{\vee}$ *and* $A_{rs}^{\wedge} \supseteq A^{\wedge}$.

A subset B of a space (X, τ) is said to be a $\vee$-set [10] (resp. $\wedge$-set [10]) if $B = B^{\vee}$ (resp. $B = B^{\wedge}$) where $B^{\vee} = \cup\{F | F \subseteq B, X - F \in \tau\}$ and $B^{\wedge} = \cap\{U | B \subseteq U, U \in \tau\}$. A subset B of a space (X, τ) is said to be a $\vee_{rs}$-set if $B = B_{rs}^{\vee}$. A subset B of X is said to be a $\wedge_{rs}$-set if $B = B_{rs}^{\wedge}$. Every regular semiclosed set is a $\vee_{rs}$-set and every regular semiopen set is a $\wedge_{rs}$-set. But a $\vee_{rs}$-set need not be a regular semiclosed set and $\wedge_{rs}$-set need not be a regular semiopen set.

Theorem 3.2.. *Let (X, τ) be a space and A be a subset of X. Then the following hold.*

(a) If A is a $\vee_{rs}$-set, then it is a $\vee$-set.

(b) If A is a $\wedge_{rs}$-set, then it is a $\wedge$-set.

Proof. (a) Always, $A^{\vee} \subseteq A$. Since A is a $\vee_{rs}$-set, $A = A_{rs}^{\vee} \subseteq A^{\vee}$, by Theorem 3.1.(l). Therefore, $A = A^{\vee}$ and so A is a $\vee$-set.

(b) Clearly, $A \subseteq A^{\wedge}$. Since A is a $\wedge_{rs}$-set, $A = A_{rs}^{\wedge} \supseteq A^{\wedge}$ and so A is a $\wedge$-set. □

The following Example 3.1 shows that a $\vee$-set need not be a $\vee_{rs}$-set.

Example 3.1. Consider the space (X, τ) of Example 2.6 Let $A = \{c, d\}$. Since A is closed, it is a $\vee$-set. But $A^{\vee}_{rs} = \phi$, since there is no regular semiclosed set contained in A and so A is not a $\vee_{rs}$-set.

Theorem 3.3.. *Let (X, τ) be a space. Then $(X - B)^{\wedge}_{rs} = X - B^{\vee}_{rs}$ for every subset B of X.*

Proof. The proof follows from the definitions. □

Corollary 3.1.. *Let (X, τ) be a space. Then $(X - B)^{\vee}_{rs} = X - B^{\wedge}_{rs}$ for every subset B of X.*

Corollary 3.2.. *Let (X, τ) be a space. Then, a subset B of X is $\vee_{rs}$-set if and only if $X - B$ is a $\wedge_{rs}$-set.*

Remark 3.1.. Let (X, τ, I) be an ideal space. It is clear that a subset A of X is I_{rw}-closed if and only if $cl^*(A) \subseteq A^{\wedge}_{rs}$.

Corollary 3.3.. *Let A be a $\wedge_{rs}$-set in (X, τ, I). Then A is I_{rw}-closed if and only if A is $*$-closed.*

If $I = \{\phi\}$, in Remark 3.1. and Corollary 3.3., we get the following Corollary 3.4. which gives characterizations of rw-closed sets.

Corollary 3.4.. *Let (X, τ) be a space and $A \subseteq X$. Then the following hold.*

(a) A is rw-closed if and only if $cl(A) \subseteq A^{\wedge}_{rs}$.

(b) If A is a $\wedge_{rs}$-set, then A is rw-closed if and only if A is closed.

Theorem 3.4.. *Let (X, τ, I) be an ideal space and $A \subseteq X$. If $A^{\wedge}_{rs}$ is I_{rw}-closed, then A is also I_{rw}-closed.*

Proof. Suppose that $A^{\wedge}_{rs}$ is I_{rw}-closed. If $A \subseteq U$ such that U is regular semiopen, then $A^{\wedge}_{rs} \subseteq U$. Since $A^{\wedge}_{rs}$ is I_{rw}-closed, $cl^*(A^{\wedge}_{rs}) \subseteq U$. Since $A \subseteq A^{\wedge}_{rs}$, it follows that $cl^*(A) \subseteq U$ and so A is I_{rw}-closed. □

The following Example 3.2 shows that the converse of Theorem 3.4. is not true.

Example 3.2. Let $X = \{a, b, c, d, e\}$, $\tau = \{\phi, \{c\}, \{e\}, \{c, e\}, \{a, b, c\}, \{c, d, e\}, \{a, b, c, e\}, X\}$ and $I = \{\phi\}$. $A = \{a, b\}$ is $*$-closed and hence I_{rw}-closed. $A^{\wedge}_{rs} = \{a, b, c\}$ is regular semiopen but it is not $*$-closed. Therefore, $A^{\wedge}_{rs}$ is not an I_{rw}-closed set.

In an ideal space (X, τ, I), a subset B of X is said to be an $I.\wedge_{rs}$-set if $B^{\wedge}_{rs} \subseteq F$ whenever $B \subseteq F$ and F is $*$-closed. A subset B of X is called $I.\vee_{rs}$-set if $X - B$ is an $I.\wedge_{rs}$-set. Every $\vee_{rs}$-set is an $I.\vee_{rs}$-set and every $\wedge_{rs}$-set is an

$I.\wedge_{rs}$-set. The following Example 3.3 shows that an $I.\wedge_{rs}$-set is not a $\wedge_{rs}$-set. Theorem 3.5. below gives a characterization of $I.\vee_{rs}$-sets.

Example 3.3. Let (X,τ) and I be as in Example 2.4. Let $A = \{a,b,c\}$. Then the only $*$-closed set containing A is X and so A is an $I.\wedge_{rs}$-set. Since $A^{\wedge}_{rs} = X$, A is not a $\wedge_{rs}$-set.

Theorem 3.5.. *A subset A of an ideal space (X,τ,I) is an $I.\vee_{rs}$-set if and only if $U \subseteq A^{\vee}_{rs}$ whenever $U \subseteq A$ and U is $*$-open.*

Proof. The proof follows from the definitions and Theorem 3.3.. □

Theorem 3.6.. *Let (X,τ,I) be an ideal space. Then for each $x \in X$, $\{x\}$ is either $*$-open or an $I.\vee_{rs}$-set.*

Proof. Suppose $\{x\}$ is not $*$-open for some $x \in X$. Then $X - \{x\}$ is not $*$-closed and so the only $*$-closed set containing $X - \{x\}$ is X. Therefore, $X - \{x\}$ is an $I.\wedge_{rs}$-set and hence $\{x\}$ is an $I.\vee_{rs}$-set. □

Theorem 3.7.. *Let B be an $I.\vee_{rs}$-set in (X,τ,I). Then for every $*$-closed set F such that $B^{\vee}_{rs} \cup (X - B) \subseteq F$, $F = X$ holds.*

Proof. Let B be an $I.\vee_{rs}$-set. Suppose F is a $*$-closed set such that $B^{\vee}_{rs} \cup (X - B) \subseteq F$. Then $X - F \subseteq X - (B^{\vee}_{rs} \cup (X - B)) = (X - B^{\vee}_{rs}) \cap B$. Since B is an $I.\vee_{rs}$-set and the $*$-open set $X - F \subseteq B$, by Theorem 3.5., $X - F \subseteq B^{\vee}_{rs}$. Also, $X - F \subseteq X - B^{\vee}_{rs}$. Therefore, $X - F \subseteq B^{\vee}_{rs} \cap (X - B^{\vee}_{rs}) = \phi$ and hence $F = X$. □

Corollary 3.5.. *Let B be an $I.\vee_{rs}$-set in an ideal space (X,τ,I). Then $B^{\vee}_{rs} \cup (X - B)$ is $*$-closed if and only if B is a $\vee_{rs}$-set.*

Proof. Let B be an $I.\vee_{rs}$-set in (X,τ,I). If $B^{\vee}_{rs} \cup (X - B)$ is $*$-closed, then by Theorem 3.7., $B^{\vee}_{rs} \cup (X - B) = X$ and so $B \subseteq B^{\vee}_{rs}$. Therefore, $B = B^{\vee}_{rs}$ which implies that B is a $\vee_{rs}$-set. Conversely, suppose B is an $\vee_{rs}$-set. Then $B = B^{\vee}_{rs}$ and so $B^{\vee}_{rs} \cup (X - B) = B \cup (X - B) = X$ is $*$-closed. □

Theorem 3.8.. *Let B be a subset of an ideal space (X,τ,I) such that $B^{\vee}_{rs}$ is $*$-closed. If X is the only $*$-closed set containing $B^{\vee}_{rs} \cup (X - B)$, then B is an $I.\vee_{rs}$-set.*

Proof. Let U be a $*$-open set contained in B. Since $B^{\vee}_{rs}$ is $*$-closed, $B^{\vee}_{rs} \cup (X - U)$ is $*$-closed. Also, $B^{\vee}_{rs} \cup (X - B) \subseteq B^{\vee}_{rs} \cup (X - U)$. By hypothesis, $B^{\vee}_{rs} \cup (X - U) = X$. Therefore, $U \subseteq B^{\vee}_{rs}$ which implies by Theorem 3.5., that B is an $I.\vee_{rs}$-set. □

An ideal space (X,τ,I) is said to be an rsT_I-space if every I_{rw}-closed set is a $*$-closed set. Clearly, every rsT_I-space is a T_I-space but the converse is not true, since an I_{rw}-closed set need not be an I_g-closed set by Example 2.1(a). The following Theorem 3.9. gives characterizations of rsT_I-spaces.

Theorem 3.9.. *In an ideal space (X, τ, I), the following statements are equivalent.*

(a) (X, τ, I) is an rsT_I-space.

(b) Every $I.\vee_{rs}$-set is a $\vee_{rs}$-set.

(c) Every $I.\wedge_{rs}$-set is a $\wedge_{rs}$-set.

Proof. $(a) \Rightarrow (b)$ If B is an $I.\vee_{rs}$-set which is not a $\vee_{rs}$-set, then $B^{\vee}_{rs} \subsetneq B$. So, there exists an element $x \in B$ such that $x \notin B^{\vee}_{rs}$. Then $\{x\}$ is not regular semiclosed. Therefore, $X - \{x\}$ is not regular semiopen and so it follows that $X - \{x\}$ is I_{rw}-closed. By hypothesis, $X - \{x\}$ is $*$-closed. Since $x \in B$ and $x \notin B^{\vee}_{rs}$, $B^{\vee}_{rs} \cup (X - B) \subseteq X - \{x\}$. Since $X - \{x\}$ is $*$-closed, by Theorem 3.7., $X - \{x\} = X$, a contradiction.

$(b) \Rightarrow (a)$ Suppose that there exists an I_{rw}-closed set B which is not $*$-closed. Then, there exists $x \in cl^*(B)$ such that $x \notin B$. By Theorem 3.6., $\{x\}$ is either $*$-open or an $I.\vee_{rs}$-set. If $\{x\}$ is $*$-open, then $\{x\} \cap B = \phi$ is a contradiction to the fact that $x \in cl^*(B)$. If $\{x\}$ is an $I.\vee_{rs}$-set, then $\{x\}$ is a $\vee_{rs}$-set and hence it follows that $\{x\}$ is regular semiclosed. Since $B \subseteq X - \{x\}$, $X - \{x\}$ is regular semiopen and B is I_{rw}-closed, $cl^*(B) \subseteq X - \{x\}$, a contradiction to the fact that $x \in cl^*(B)$. Therefore, (X, τ, I) is an rsT_I-space.

$(b) \Leftrightarrow (c)$ The proof follows from the definition of an $I.\vee_{rs}$-set and from Corollary 3.2. □

Theorem 3.10.. *An ideal space (X, τ, I) is an rsT_I-space if and only if every singleton set in X is either $*$-open or regular semiclosed.*

Proof. If $x \in X$ such that $\{x\}$ is not regular semiclosed, then $X - \{x\}$ is not regular semiopen and so it follows that $X - \{x\}$ is I_{rw}-closed. By hypothesis, $X - \{x\}$ is $*$-closed and so $\{x\}$ is $*$-open. Conversely, let A be an I_{rw}-closed set and $x \in cl^*(A)$. Consider the following two cases:

Case (i): Suppose $\{x\}$ is regular semiclosed. Since A is I_{rw}-closed, by Theorem 2.3., $cl^*(A) - A$ does not contain a non-empty regular semiclosed set which implies that $x \notin cl^*(A) - A$ and so $x \in A$.

Case (ii): Suppose $\{x\}$ is $*$-open. Then $\{x\} \cap A \neq \phi$ and so $x \in A$.

Thus in both cases $x \in A$. Therefore, $A = cl^*(A)$ which implies that A is $*$-closed. So (X, τ, I) is an rsT_I-space.

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