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# GRADIENT FORM OF THE ENERGY OF COLLECTIVE INTELLIGENCE

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**Abstract:** Considering an energy-based self-organizing group [10], we extend in this paper the given discrete model to a gradient one. Where the gradient system characterize the collective intelligence of a continuous mass and we present some related results. Moreover, the theoretical and computational analysis of the presented gradient model provide interesting open Framework.

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## 1. Introduction

In the real life, Emergence is the way that complex systems and patterns arising out of a multiplicity of relatively individual and collective interactions. The Collective Intelligence of such complex and natural systems present different behavior and a perfect results. Therefore, the mathematics of emergence is a interesting contribution to analyze natural and complex dynamical systems. This recent topic of research in the modeling of population dynamics interact many mathematicians for developing new models of such phenomena, see for instance [6, 7] among others. Moreover, in the same context, The challenge for analyzing the collective behavior of agents intelligent aggregation such as

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flocking, schooling, swarming..., have been presented for different population, see for instance [2, 3, 4, 5, 8, 9]. These models have described the collective moves in groups by many continuous and discrete systems. The model presented here gives a new idea of rules of self-organizing groups based on the local and global energies of autonomous population agents.

The present study analyze a kind of consensus dynamics, where agents simultaneously move to the barycenter of all agents in an epsilon neighborhood. The final state may be consensus, where all agents meet at the same position or grouping several classes of agents such that all agents in the same class maintain the same position and agents in different classes are at a distance greater than or equal to epsilon.

In this work, we are interested to extend the discrete energy model analyzed for example by [1, 10] to a continuous one. Namely to a gradient and we purpose an interesting question in the context of groups dynamics and collective intelligence of populations.

## 2. Energy-Based Move

Let $M_+(\mathbb{R}^n)$ be the set of nonnegative Randon measures on $X = \mathbb{R}^n$. A measure function is given as

$$m := \sum_{x \in S(m)} m(x)\delta_x = \int_{\mathbb{R}^n} m(dx),$$

where $S(m)$ denotes the support of $m$ and $\delta_x$ the Kronecker symbol. Fix a real number $\varepsilon > 0$. The local $\varepsilon$-energy of a positive measure $m$ is given as

$$e_\varepsilon(a, m) = \sum_{d(a,y) \le \varepsilon} m(y) d^2(a, y) = \int_{d(x,y) \le \varepsilon} d^2(x, y) m(dy), \tag{1}$$

where $d(\cdot, \cdot)$ is any differentiable metric on $X$.

Our main concern in the following is to analyse the global and local energies at each move step. Therefore, for a given pair $(a, a^*) \in X \times X$, we consider the moves operation defined on the set $M_+(X)$ of nonnegative measures as:

$$m \mapsto m^* = (a, a^*, m), \quad m^*(x) := \begin{cases} m(x); & \text{if} \quad x \notin \{a, a^*\}, \\ 0; & \text{if} \quad x = a, \\ m(a) + m(a^*); & \text{if} \quad x = a^*. \end{cases} \tag{2}$$

The mapping above is a mass translating map. Where, the move of $a$ to $a^*$ means that the mass of $a^*$ will adjusted by a new mass, namely the mass of $a$.

Thus as consequence the measure $m$ will be transformed to $m^*$. Moreover, we define this measure transformation in the following definition.

**Definition 1.** *A non-negative pair measure$(m, m^*)$ is called an $\varepsilon-$move, if there is a pair $(a, a^*) \in X \times X$ such that:*

*(i)* $m^* = (a, a^*, m)$,

*(ii)* $d(a, a^*) \leq \varepsilon$,

*(iii)* $e_\varepsilon(a, m) > e_\varepsilon(a^*, m^*)$.

For such a measure function, we define the energy map $E$ as

$$\begin{aligned} E \quad &: \quad M_+(X) \to I\!R^+; \qquad (3) \\ E(m) &= \sum_{d(x,y)\leq\varepsilon} m(x)m(y)d^2(x,y) = \int_X \int_{d(x,y)\leq\varepsilon} d^2(x,y)m(dy)m(dx) \end{aligned}$$

It is important to note that $E$ is in general a not continuous function of $m$.

Set for example $X = I\!R$, $S(m) = \{1, 2\}$, $\varepsilon = 1$ and given $m$ with

$$m = \sum_{x\in\{1,2\}} m(x)\delta_x = \delta_1 + \delta_2,$$

it follows that $E(m) = 2$. Now let $m_j$ be a sequence of positive measures defined as

$$m_j = \delta_1 + \delta_{2+\frac{1}{j}},$$

it follows

$$\lim_j m_j = m, \;\; E(m_j) = 0 \text{ and } E(\lim_j m_j) = 2 \neq \lim_j E(m_j) = 0$$

Hence, the map $E$ with definition (1.1) is not continuous in $m$.

In order to obtain an energy function which depends continuously on $m$, we extend the definition (3) into the following:

$$\begin{aligned} E(m) \quad &= \quad \int_X \int_X \varphi(x,y)d^2(x,y)m(dx)m(dy) \qquad (4) \\ &= \sum_{(x,y)\in S^2(m)} \varphi(x,y)d^2(x,y)m(x)m(y), \qquad (5) \end{aligned}$$

where $\varphi : X \times X \longrightarrow [0,1]$ is a continuous function, which satisfies :

$$\varphi(x,y) := \begin{cases} 1, & \text{if } \ d(x,y) \leq \varepsilon; \\ \phi(x,y), & \text{if } \ \varepsilon \leq d(x,y) \leq \varepsilon+\theta; \\ 0, & \text{if } \ d(x,y) \geq \varepsilon+\theta. \end{cases}$$

for $\varepsilon > 0$ and $\theta > 0$ and a continuous function $\phi : X \times X \longrightarrow [0,1]$. The parameters $\varepsilon > 0$ and $\theta > 0$ will be fixed throughout this paper and the function $\varphi$ will be called **intensity function**.

**Example 1.** Let $(X,d)$ a metric space with metric $d$, The following map

$$\varphi(x,y) := \begin{cases} 1, & \text{if } \ \mid x-y \mid \leq \varepsilon; \\ \left(\frac{\varepsilon+\theta-|x-y|)}{\theta}\right) & \text{if } \varepsilon \leq \mid x-y \mid \leq \varepsilon+\theta; \\ 0, & \text{if } \ \mid x-y \mid \geq \varepsilon+\theta. \end{cases}$$

is an intensity function.

According to Definition 1, we consider the the measure operator $f$ defined as

$$\begin{aligned} f \quad &: \quad M_+(X) \longrightarrow M_+(X); \\ f(m)(x) \quad &:= \quad \begin{cases} m(f(x)), & \text{if } f(x) \notin S(m), \\ m(f(x)) + m(z), & \text{if } f(x) = z \in S(m). \end{cases} \end{aligned} \tag{6}$$

Where for every mass points $x, z \in M_+(X)$, the operator $f$ satisfies the following propriety

$$f(x)(z) \quad := \quad \begin{cases} (f(x))(z), & \text{if } x = z, \\ 0, & \text{if } x \neq z. \end{cases} \tag{7}$$

**Remark.** It is important to note that if $x$ is a positive mass point then $f(x)$ is a positive mass point too. But the operator $f$ also operates on positive measure $m$ of $M_+(X)$, therefore is ,in this case, $f(m)$ a positive measure. To be clear, the operator $f$ can be considered as a rule of moving particles on a measure space: A mass point $x$ moves either to a massless point $y$ or to a mass point with positive mass. Hence, the measure $m$ moves to the image measure $f(m)$ defined above.

**Lemma 1.** *The total energy of this mass transformation is given by*

$$\begin{aligned} E(f(m)) \quad &= \quad \int_X \int_X \varphi(f(x),f(y)) d^2(f(x),f(y)) f(m)(dx) f(m)(dy) \\ &= \quad \sum_{x,y} m(f(x)) m(f(y)) \varphi(f(x),f(y)) d^2(f(x),f(y)). \end{aligned} \tag{8}$$

*Proof.* The proof is left to the reader.

**Lemma 2.** *Let $(X, d)$ be a metric space with metric $d$. The mapping $f$ conserve the total mass of a measure $m$ such that*

$$m(X) = f(m)(X) = f^2(m)(X) = \ldots = f^n(m)(X); \quad \forall n \in I\!N; \ \forall m \in M_+(X).$$

*Proof.* According to the definition of $f$ given by equation (7), we have

$$\begin{aligned} f(m)(X) &= \sum_{f(x)\in X} f(m)(x) \\ &= \sum_{f(x)\notin S(m)} f(m)(x) + \sum_{f(x)\in S(m)} f(m)(x) \\ &= \sum_{f(x)\notin S(m)} m(f(x)) + \sum_{z=f(x), f(x)\in S(m)} (m(f(x)) + m(z)) \\ &= (I) + (II), \end{aligned}$$

where

$$\begin{aligned} (I) &= \sum_{x\notin S(m), f(x)\notin S(m)} f(m)(x) + \sum_{x\in S(m), f(x)\notin S(m)} f(m)(x) \\ &= 0 + \sum_{x\in S(m), f(x)\notin S(m)} m(x)\delta_x, \end{aligned}$$

similar for the second term of the summation, we have

$$(II) = \sum_{x\notin S(m), z=f(x)\in S(m)} m(z) + \sum_{x\in S(m), f(x)\in S(m)} (m(f(x)) + m(z))$$

$$= \sum_{x\notin S(m), z=f(x)\in S(m)} m(z) + \sum_{x\in S(m), z=f(x)\in S(m)} m(x) + \sum_{x\in S(m), z=f(x)\in S(m)} m(z).$$

From $(I)$ and $(II)$, it follows that

$$f(m)(X) = m(X).$$

By induction over $n$, we conclude the result of the lemma.

**Lemma 3.** *The energy (8) can be written in integral form as*

$$E(f(m)) = \int_X \int_X \Phi(f^2(x), f^2(y)) m(dx) m(dy), \tag{9}$$

*where $\Phi(x, y) := d^2(x, y)\varphi(x, y)$ and $f^2 = f \circ f$.*

*Proof.* By definition of the total energy, we have

$$E(f(m)) = \int_X \int_X d^2(f(x), f(y))\varphi(x,y)d^2(f(x), f(y))f(m)(dx)f(m)(dy).$$

Since $\Phi(x,y)$ is a continuous in $x$ and $y$, then from (8) it follows:

$$\begin{aligned} E(f(m)) &= \int_X \int_X \Phi(f(x), f(y))f(m)(dx)f(m)(dy) \\ &= \int_X \Big( \int_X \Phi(f(x), f(y))f(m)(dy) \Big) f(m)(dx) \\ &= \int_X \Big( \int_X \Phi(f(x), f^2(y))m(dy) \Big) f(m)(dx) \\ &= \int_X \Big( \int_X \Phi(f(x), f^2(y))f(m)(dx) \Big) m(dy) \\ &= \int_X \int_X \Phi(f^2(x), f^2(y))m(dx)m(dy). \end{aligned}$$

## 3. Gradient Form of the Energy of Collective Intelligence

In order to extend the discrete move (2) to a continuous one, we define the class of positive measure $m$ satisfying the following properties :

$$e(x+dx, m+dm) = \frac{1}{2}\Big(e(x+dx, m) + e(x, m+dm)\Big), \tag{10}$$

where $dx$ and $dm$ are considered as usual differential operators. The condition (10) will be called the **Equilibrium** condition and every positive measure satisfying the Equilibrium condition will be called a **Regular** measure.

In the following, we purpose the main theorem of this work, namely the continuous version of the discrete mass transformation model:

**Theorem 1.** *For every subset $X$ of $\mathbb{R}^n$ and a Regular measure $m \in M_+(X)$ exists a positive functional $F : X \times M_+(X) \to \mathbb{R}^+$ such that the following gradient is satisfied:*

$$-\nabla e(x,m) = F(x,m). \tag{11}$$

*Proof.* For an infinitesimal move of a single mass, $x$ moves to $x^*$ and $m$ moves to $m^*$. Let us replace $x^*$ by $x+dx$ and $m^*$ by $m+dm$, such that$dx < \epsilon$ and $m+dm \in M_+(\mathbb{R})$ for all $dm$. Thus we write

$$e(x^*, m^*) = e(x+dx, m+dm). \tag{12}$$

From the definition of an admissible move given by (1) and (12), we have

$$e(x,m) - e(x^*,m^*) \quad = \quad e(x,m) - e(x+dx, m+dm) > 0.$$

So there exists a positive functional $F : X \times M_+(X) \to I\!R^+$ such that

$$e(x,m) - e(x+dx, m+dm) \quad = \quad \frac{1}{2}F(x,m). \tag{13}$$

Since $m$ is a regular measure satisfying the Equilibrium condition 10, it follows

$$\begin{aligned} \frac{1}{2}F(x,m) &= e(x,m) - e(x+dx, m+dm) \\ &= e(x,m) - \frac{1}{2}\Big(e(x+dx,m) + e(x, m+dm). \end{aligned}$$

Hence, by a passage to the limit $dx \to 0$ and $dm \to 0$ we get

$$F(x,m) = -\Big(e(x+dx,m) - e(x,m)) + (e(x+dx,m) - e(x,m)\Big) = -\nabla e(x,m).$$

## 4. Concluding Remarks

In this work we have extended the collective intelligence energy to a continuous one. Especially for the equilibrium class of measure, we find an interesting gradient representation of the model, which its analysis should be an interesting open question. Moreover, we have presented some interesting properties of this mapping. The collective dynamics of agents present the study of consensus and emergence phenomena as a gradient systems.

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