EXISTENCE OF SOLUTIONS FOR NONLINEAR SECOND-ORDER $q$-DIFFERENCE EQUATIONS WITH THREE-POINT MULTI-TERM $q$-INTEGRAL BOUNDARY CONDITIONS

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Abstract: In this paper is concerned with the existence of solutions for nonlinear second-order $q$-difference equations with three-point multi-term $q$-integral boundary conditions. Some new existence results are obtained by using Banach’s contraction mapping, Krasnoselskii’s fixed point theorem and Leray-Schauder degree theory. As an application, we give two examples that illustrate our results.

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Key Words: existence, $q$-difference equation, $q$-derivative, $q$-integral, boundary value problem

1. Introduction

The study of $q$-calculus was initiated in the beginning of the twentieth century by the pioneer works of Jackson [1], Carmichael [2], Mason [3], Adams [4], Trjitzinsky [5], etc. In the last few decades, the $q$-theory has involved into a variety of discipline areas of research not only in mathematics but also covering other fields of science and its applications as well. See [6-14]. For some recent

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work, we refer the reader to [15-26] and references therein. However, there are only a few papers that have studied the existence of solutions for boundary value problems of nonlinear $q$-difference equations, see [27-29].

In [30], Ahmad studied the existence of solutions for nonlinear third-order $q$-difference equation with boundary conditions

$$D^3_q u(t) = f(t, u(t)), \quad 0 \leq t \leq 1,$$
$$u(0) = 0, \quad D_q u(0) = 0, \quad u(1) = 0.$$ 

He obtained some existence results by using standard fixed point theorems and Leray-Schauder degree theory. He also showed that if $q \to 1$ then his results corresponded to the classical results. In [31], Ahmad, Alsaedi and Ntouyas established the existence criteria for nonlinear second-order $q$-difference equation with non-separated boundary conditions

$$D^2_q u(t) = f(t, u(t)), \quad t \in [0, T],$$
$$u(0) = \eta u(T), \quad D_q u(0) = \eta D_q u(T).$$

In [32], Pongam, Asawasumrit, Tariboon investigated the sequential derivatives of the nonlinear $q$-difference equation with three-point $q$-integral boundary condition of the form

$$D_q(D_p + \lambda)u(t) = f(t, u(t)), \quad t \in [0, T],$$
$$u(0) = 0, \quad \beta \int_0^\eta u(s)d_r s = u(T), \quad \eta \in (0, T).$$

Using fixed point theorems and Leray-Schauder degree theory, some existence were obtained.

In this paper, we consider the following nonlinear second-order $q$-difference equation with three-point multi-term $q$-integral boundary conditions

$$D^2_q u(t) = f(t, u(t)), \quad 0 \leq t \leq T, \quad u(0) = \sum_{i=1}^n \alpha_i \int_0^\eta u(s)d_{p_i} s, \quad u(T) = \sum_{j=1}^m \beta_j \int_0^\eta u(s)d_{r_j} s, \quad (1)$$

$$\text{where } 0 < q, p_i, r_j < 1, \ i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m \text{ and } \alpha_i, \beta_j \in R \text{ are such that } \eta^2 \left( \sum_{j=1}^m \frac{\beta_j}{1 + r_j} - \sum_{i=1}^n \frac{\alpha_i (1 - \eta) \sum_{j=1}^m \beta_j}{1 + p_i} \right) \neq T, \ \eta \in (0, T) \text{ is a fixed constant.}$$

We note that the boundary conditions (2) have different values of the $q$-numbers of $q$-integral at the boundary points. The existence results of problem (1)-(2) have not been established, previously.
2. Preliminaries

In this section, we introduce notation, some basic concepts of \(q\)-calculus and prove a lemma before starting our main results.

For \(0 < q < 1\), we define the \(q\)-derivative of a real valued function \(f\) as

\[
D_q f(t) := \frac{f(t) - f(qt)}{(1 - q)t},
\]

and \(D_q f(0) = \lim_{t \to 0} D_q f(t)\) provided \(f'(0)\) exists. The higher order \(q\)-derivatives are given by

\[
D_0^q f(t) = f(t), \quad D_n^q f(t) = D_q D_{q}^{n-1} f(t), \quad n \in \mathbb{N}.
\]

The \(q\)-integral of a function \(f\) defined in the interval \([a, b]\) is given by

\[
\int_a^t f(s)d_q s := \sum_{n=0}^{\infty} (1 - q)q^n [tf(tq^n) - af(q^n a)], \quad t \in [a, b].
\]

For \(a = 0\), we denote

\[
I_q f(t) = \int_0^t f(s)d_q s = \sum_{n=0}^{\infty} t(1 - q)q^n f(tq^n),
\]

provided the series converges. If \(a \in [0, b]\) and \(f\) is defined in the interval \([0, b]\), then

\[
\int_a^b f(t)d_q t = \int_0^b f(t)d_q t - \int_0^a f(t)d_q t.
\]

Note that

\[
D_q I_q f(t) = f(t), \quad (3)
\]

and if \(f\) is continuous at \(t = 0\), then

\[
I_q D_q f(t) = f(t) - f(0).
\]

In \(q\)-calculus, the product rule and integration by parts formula are

\[
D_q(gh)(t) = (D_q g(t))h(t) + q(t)D_q h(t), \quad (4)
\]

\[
\int_0^t h(s)D_q g(s)d_q s = \left[h(s)g(s)\right]_0^t - \int_0^t D_q h(s)g(qs)d_q s. \quad (5)
\]

Further, reversing the order of integration is given by

\[
\int_0^t \int_0^s f(r)d_q r d_q s = \int_0^t \int_0^r f(r)d_q s d_q r.
\]

In the limit \(q \to 1\) the \(q\)-calculus corresponds to the classical calculus.
Lemma 1. Let \( \eta^2 \left( \sum_{j=1}^{m} \frac{\beta_j}{1+r_j} - \sum_{i=1}^{n} \frac{\alpha_i (1-\eta \sum_{j=1}^{m} \beta_j)}{1+p_i} \right) \neq T \) and \( 0 < q, p_i, r_j < 1 \) for \( i = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, m \). Then for \( h \in C[0,T] \), the boundary value problem

\[
D_q^2 u(t) = h(t), \quad 0 < t < T, \tag{6}
\]

\[
u(0) = \sum_{i=1}^{n} \alpha_i \int_{0}^{\eta} u(s) d_{p_i} s, \quad u(T) = \sum_{j=1}^{m} \beta_j \int_{0}^{\eta} u(s) d_{r_j} s, \tag{7}
\]

has the unique solution

\[
u(t) = \int_{0}^{t} (t - qs) h(s) d_q s + c_1, \tag{10}
\]

where \( \alpha_1 \) is a constant of \( q \)-integration. Taking \( q \)-integral to (10) for \( t \in [0,T] \), we obtain

\[
u(t) = \int_{0}^{t} \left( \int_{0}^{\nu} h(s) d_q s \right) d_q \nu + c_1 t + c_2. \tag{11}
\]
Changing the order of $q$-integration, we get

$$u(t) = \left( \int_0^t h(s) \, dq \right) ds + c_1 t + c_2. \quad (12)$$

Therefore, (12) can be written as

$$u(t) = \int_0^t (t - qs) h(s) \, dq \, ds + c_1 t + c_2. \quad (13)$$

It follows that $c_2 = u(0)$. Taking the $p_i$-integral of (13) from 0 to $\eta$, we get

$$\int_0^\eta u(s) \, dp_i \, s = \int_0^\eta \left( \int_0^s (s - q\nu) h(\nu) \, dq \, \nu + c_1 s + c_2 \right) dp_i \, s$$

$$= \int_0^\eta \left( \int_0^s (s - q\nu) h(\nu) \, dq \, \nu \right) dp_i \, s + c_1 \int_0^\eta sd p_i \, s + c_2 \eta$$

$$= \int_0^\eta \left( \int_0^s (s - q\nu) h(\nu) \, dq \, \nu \right) dp_i \, s + c_1 \frac{\eta^2}{1 + p_i} + c_2 \eta.$$

The first condition of (7) implies that

$$-c_1 \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1 + p_i} + c_2 \left( 1 - \eta \sum_{i=1}^n \alpha_i \right) = \sum_{i=1}^n \alpha_i \int_0^\eta \left( \int_0^s (s - q\nu) h(\nu) \, dq \, \nu \right) dp_i \, s.$$

Form (13), we obtain

$$u(T) = \int_0^T (T - qs) h(s) \, dq \, ds + c_1 T + c_2.$$

Then, $r_j$-integrating of (13) gives

$$\int_0^\eta u(s) \, dr_j \, s = \int_0^\eta \left( \int_0^s (s - q\nu) h(\nu) \, dq \, \nu \right) dr_j \, s + c_1 \frac{\eta^2}{1 + r_j} + c_2 \eta.$$

The second condition of (7) implies that

$$c_1 \left( T - \eta^2 \sum_{j=1}^m \frac{\beta_j}{1 + r_j} \right) - c_2 \left( 1 - \eta \sum_{j=1}^m \beta_j \right)$$

$$= \sum_{j=1}^m \beta_j \int_0^\eta \left( \int_0^s (s - q\nu) h(\nu) \, dq \, \nu \right) dr_j \, s - \int_0^T (T - qs) h(s) \, dq \, s.$$
Solving the system of linear equations on the unknown constants $c_1$ and $c_2$, we obtain

$$c_1 = \left(1 - \eta \sum_{j=1}^{m} \beta_j \right) \sum_{i=1}^{n} \frac{\alpha_i}{\Phi} \int_0^{\eta} \left( \int_0^{s} (s - q \nu) h(\nu) d_q \nu \right) d_p, s$$

$$- \left(1 - \eta \sum_{i=1}^{n} \alpha_i \right) \sum_{j=1}^{m} \frac{\beta_j}{\Phi} \int_0^{\eta} \left( \int_0^{s} (s - q \nu) h(\nu) d_q \nu \right) d_r, s$$

$$+ \left(1 - \eta \sum_{i=1}^{n} \alpha_i \right) \int_0^{T} \frac{(T - qs)}{\Phi} h(s) d_q s,$$

$$c_2 = - \left( T - \eta^2 \sum_{j=1}^{m} \beta_j \right) \sum_{i=1}^{n} \frac{\alpha_i}{\Phi} \int_0^{\eta} \left( \int_0^{s} (s - q \nu) h(\nu) d_q \nu \right) d_p, s$$

$$- \left( \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i} \right) \sum_{j=1}^{m} \frac{\beta_j}{\Phi} \int_0^{\eta} \left( \int_0^{s} (s - q \nu) h(\nu) d_q \nu \right) d_r, s$$

$$+ \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i} \int_0^{T} \frac{(T - qs)}{\Phi} h(s) d_q s.$$

Substituting values of $c_1$ and $c_2$ in (13), we obtain the solution (8).

For the forthcoming analysis, let $C = C([0, T], R)$ denote the Banach space of all continuous functions from $[0, T]$ to $R$ endowed with the norm defined by $\|u\| = \sup\{|u(t)|, t \in [0, T]\}$. In view of Lemma 1 and (9), we consider the operator $F : C \to C$ defined by

$$(Fu)(t) = \int_0^{t} (t - qs)f(s, u(s)) d_q s$$

$$+ \left[1 - \eta \sum_{i=1}^{n} \alpha_i t + \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i} \right] \int_0^{T} \frac{(T - qs)}{\Phi} f(s, u(s)) d_q s$$

$$- \left[1 - \eta \sum_{i=1}^{n} \alpha_i t + \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i} \right]$$

$$\times \sum_{j=1}^{m} \frac{\beta_j}{\Phi} \int_0^{\eta} \int_0^{s} (s - q \nu) f(\nu, u(\nu)) d_q \nu d_r, s$$

$$+ \left[1 - \eta \sum_{j=1}^{m} \beta_j t - \eta^2 \sum_{j=1}^{m} \frac{\beta_j}{1 + r_j} \right]$$
Note that the problem (1)-(2) has solutions if and only if the operator equation $Fx = x$ has fixed points.

In the following, for the sake of convenience, we set

$$
\Omega = T^2 + \left( (1 + \eta) \sum_{i=1}^{n} |\alpha_i| \right) T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1 + p_i} \frac{T^2}{\Phi} \\
+ \left( (1 + \eta) \sum_{i=1}^{n} |\alpha_i| \right) T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1 + p_i} \sum_{j=1}^{m} \frac{|\beta_j| \eta^3}{\Phi |r_j^2 + r_j + 1|} \\
+ \left( \eta^2 \sum_{j=1}^{m} \frac{|\beta_j|}{1 + r_j} + \eta \sum_{j=1}^{m} |\beta_j| \right) \sum_{i=1}^{n} \frac{|\alpha_i| \eta^3}{\Phi (p_i^2 + p_i + 1)},
$$

(15)

where $\Phi$ is defined by (9).

3. Main Results

Now we are in position to establish the main results. Our first result is based on Banach’s fixed point theorem.

**Theorem 2.** Let $f : [0, T] \times R \rightarrow R$ be a jointly continuous function. In addition, suppose that there exists a positive constant $L$ such that:

$(H_1)$ $|f(t, u(t)) - f(t, v(t))| \leq L |u - v|$, for all $t \in [0, T]$, $u, v \in R$.

$(H_2)$ $\Lambda := L \Omega < 1$, where $\Omega$ is defined by (15).

Then the boundary value problem (1)-(2) has a unique solution.

**Proof.** Assume that $\sup_{t \in [0,T]} |f(t, 0)| = M$, and define a constant $r$ as

$$
r \geq \frac{M \Omega}{1 - \delta}.
$$

(16)

Now, we shall show that $FB_r \subset B_r$, where $B_r = \{ u \in C : \|u\| \leq r \}$. For any $u \in B_r$, we get

$$
\|Fu\| = \sup_{t \in [0,T]} \left| \int_{0}^{t} (t - qs) f(s, u(s)) dq ds \right|
$$
\[
+ \left[ (1 - \eta \sum_{i=1}^{n} \alpha_i) t + \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i} \right] \\
\times \int_0^T \frac{(T - qs)}{\Phi} f(s, u(s)) d_q s \\
+ \left[ (1 - \eta \sum_{i=1}^{n} \alpha_i) t + \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i} \right] \sum_{j=1}^{m} \frac{\beta_j}{\Phi} \\
\times \int_0^\eta \int_0^{s} (s - q\nu) f(\nu, u(\nu)) d_q \nu d_r_j s \\
+ \left[ (1 - \eta \sum_{j=1}^{m} \beta_j) t - T + \eta^2 \sum_{j=1}^{m} \frac{\beta_j}{1 + r_j} \right] \sum_{i=1}^{n} \frac{\alpha_i}{\Phi} \\
\times \int_0^\eta \int_0^{s} (s - q\nu) f(\nu, u(\nu)) d_q \nu d_p_i s \\
\leq \sup_{t \in [0, T]} \left\{ \int_0^t (t - qs) (|f(s, u(s)) - f(s, 0)| + |f(s, 0)|) d_q s \\
+ \left| (1 - \eta \sum_{i=1}^{n} \alpha_i) t + \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i} \right| \\
\times \int_0^T \frac{(T - qs)}{|\Phi|} (|f(s, u(s)) - f(s, 0)| + |f(s, 0)|) d_q s \\
+ \left| (1 - \eta \sum_{i=1}^{n} \alpha_i) t + \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i} \right| \sum_{j=1}^{m} \frac{|\beta_j|}{|\Phi|} \\
\times \int_0^\eta \int_0^{s} (s - q\nu) (|f(\nu, u(\nu)) - f(\nu, 0)| + |f(\nu, 0)|) d_q \nu d_r_j s \\
+ \left| (1 - \eta \sum_{j=1}^{m} \beta_j) t - T + \eta^2 \sum_{j=1}^{m} \frac{\beta_j}{1 + r_j} \right| \sum_{i=1}^{n} \frac{|\alpha_i|}{|\Phi|} \\
\times \int_0^\eta \int_0^{s} (s - q\nu) (|f(\nu, u(\nu)) - f(\nu, 0)| + |f(\nu, 0)|) d_q \nu d_p_i s \right\} \\
\leq \sup_{t \in [0, T]} \left\{ \int_0^t (t - qs) (L|u(s)| + |f(s, 0)|) d_q s \\
+ \left| (1 - \eta \sum_{i=1}^{n} \alpha_i) t + \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i} \right| \int_0^T \frac{(T - qs)}{|\Phi|} (L|u(s)| + |f(s, 0)|) d_q s \right\}
\]
\begin{align*}
&+ \left| (1 - \eta \sum_{i=1}^{n} \alpha_i) t + \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i} \right| \sum_{j=1}^{m} \frac{\beta_j}{|\Phi|} \\
&\times \int_{0}^{\eta} \int_{0}^{s} (s - q\nu)(L|u(\nu)| + |f(\nu, 0)|)d_{q\nu}d_{r_j}\ s \\
&+ \left| (1 - \eta \sum_{j=1}^{m} \beta_j) t - T + \eta^2 \sum_{j=1}^{m} \frac{\beta_j}{1 + r_j} \right| \sum_{i=1}^{n} \frac{\alpha_i}{|\Phi|} \\
&\times \int_{0}^{\eta} \int_{0}^{s} (s - q\nu)(L|u(\nu)| + |f(\nu, 0)|)d_{q\nu}d_{p_i}\ s \right\} \\
&\leq \sup_{t \in [0, T]} \left\{ \int_{0}^{t} (t - qs)(L||u|| + M)d_{q\nu}s \\
&+ \left| (1 - \eta \sum_{i=1}^{n} \alpha_i) t + \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i} \right| \int_{0}^{T} \frac{(T - qs)}{|\Phi|}(L||u|| + M)d_{q\nu}s \\
&+ \left| (1 - \eta \sum_{i=1}^{n} \alpha_i) t + \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i} \right| \sum_{j=1}^{m} \frac{\beta_j}{|\Phi|} \\
&\times \int_{0}^{\eta} \int_{0}^{s} (s - q\nu)(L|u(\nu)| + M)d_{q\nu}d_{r_j}\ s \\
&+ \left| (1 - \eta \sum_{j=1}^{m} \beta_j) t - T + \eta^2 \sum_{j=1}^{m} \frac{\beta_j}{1 + r_j} \right| \sum_{i=1}^{n} \frac{\alpha_i}{|\Phi|} \\
&\times \int_{0}^{\eta} \int_{0}^{s} (s - q\nu)(L|u(\nu)| + M)d_{q\nu}d_{p_i}\ s \right\} \\
&\leq \sup_{t \in [0, T]} \left\{ (L||u|| + M) \left[ t^2 + \left| (1 - \eta \sum_{i=1}^{n} \alpha_i) t + \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i} \right| \frac{T^2}{|\Phi|} \\
&+ \left| (1 - \eta \sum_{i=1}^{n} \alpha_i) t + \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i} \right| \sum_{j=1}^{m} \frac{\beta_j}{|\Phi|} \frac{\eta^3}{r_j^2 + r_j + 1} \\
&+ \left| (1 - \eta \sum_{j=1}^{m} \beta_j) t - T + \eta^2 \sum_{j=1}^{m} \frac{\beta_j}{1 + r_j} \right| \sum_{i=1}^{n} \frac{\alpha_i}{|\Phi|} \frac{\eta^3}{p_i^2 + p_i + 1} \right\} \\
&\leq (Lr + M) \left[ T^2 + \left( 1 + \eta \sum_{i=1}^{n} |\alpha_i| T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1 + p_i} \right) \frac{T^2}{|\Phi|} \right]
\end{align*}
\[
\begin{align*}
&+ \left( 1 + \eta \sum_{i=1}^{n} |\alpha_i| T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1 + p_i} \right) \sum_{j=1}^{m} \frac{|\beta_j| \eta^3}{|\Phi|(r_j^2 + r_j + 1)} \\
&+ \left( \eta^2 \sum_{j=1}^{m} |\beta_j| + \eta \sum_{j=1}^{m} |\beta_j| \right) \sum_{i=1}^{n} \frac{|\alpha_i| \eta^3}{|\Phi|(p_i^2 + p_i + 1)} \\
&\leq (Lr + M)\Omega \leq r.
\end{align*}
\]

Therefore, \( F B_r \subset B_r \). Next, we will show that \( F \) is contraction. In the following, from (\( H_1 \)) for any \( u, v \in C \) and for each \( t \in [0, T] \), we have

\[
\|F u - F v\| = \sup_{t \in [0, T]} |(F u)(t) - (F v)(t)|
\]

\[
= \sup_{t \in [0, T]} \left| \int_0^t (t - qs)(f(s, u(s)) - f(s, v(s)))d_q s \right|
\]

\[
+ \left[ (1 - \eta \sum_{i=1}^{n} \alpha_i) t + \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i} \right] \sum_{j=1}^{m} \frac{\beta_j}{\Phi}
\]

\[
\times \left[ \int_0^T \left( T - qs \right) (f(s, u(s)) - f(s, v(s)))d_q s \right]
\]

\[
+ \left[ (1 - \eta \sum_{j=1}^{m} \beta_j) t - T + \eta^2 \sum_{j=1}^{m} \frac{\beta_j}{1 + r_j} \right] \sum_{i=1}^{n} \frac{\alpha_i}{\Phi}
\]

\[
\times \left| \int_0^\eta \int_0^s (s - q\nu)(f(\nu, u(\nu)) - f(\nu, v(\nu)))d_q d_r d_\nu \right|
\]

\[
\leq L\|u - v\| \left[ T^2 + \left( 1 + \eta \sum_{i=1}^{n} |\alpha_i| T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1 + p_i} \right) \frac{T^2}{|\Phi|} \\
+ \left( 1 + \eta \sum_{i=1}^{n} |\alpha_i| T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1 + p_i} \right) \sum_{j=1}^{m} \frac{|\beta_j| \eta^3}{|\Phi|(r_j^2 + r_j + 1)} \\
+ \left( \eta^2 \sum_{j=1}^{m} \frac{|\beta_j|}{1 + r_j} + \eta \sum_{j=1}^{m} |\beta_j| \right) \sum_{i=1}^{n} \frac{|\alpha_i| \eta^3}{|\Phi|(p_i^2 + p_i + 1)} \right].
\]
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\[ L\Omega ||u - v|| \leq \Lambda ||u - v||. \]

From \((H_2)\), we have that \(F\) is a contraction. Hence, by the Banach’s fixed point theorem, we get that \(F\) has a fixed point which is a unique solution of the problem (1)-(2). This completes the proof.

Now, we state a result due to Krasnoselskii [33] which is needed to prove the existence of at least one solution of the problem (1)-(2).

**Theorem 3.** Let \(K\) be a bounded closed convex and nonempty subset of a Banach space \(X\). Let \(A, B\) be the operators such that:

(i) \(Ax + By \in K\) whenever \(x, y \in K\),

(ii) \(A\) is compact and continuous,

(iii) \(B\) is a contraction mapping.

Then there exists \(z \in K\) such that \(z = Az + Bz\).

Our second result is based on the following Krasnoselskii’s fixed point theorem.

**Theorem 4.** Assume that \((H_1)\) and \((H_2)\) hold with

\[ |f(t, u)| \leq \mu(t) \tag{17} \]

for all \((t, u) \in [0, T] \times R\), where \(\mu \in L^1([0, T], R^+)\). Then the boundary value problem (1)-(2) has at least one solution on \([0, T]\).

**Proof.** Setting \(\max_{t \in [0, T]} |\mu(t)| = ||\mu||\), and choosing a constant \(r\) such that

\[ r \geq ||\mu||\Omega, \tag{18} \]

where \(\Omega\) is given by (15). Consider \(B_r = \{u \in C : ||u|| \leq r\}\). In view of Lemma 2.1 we define the operators \(G_1\) and \(G_2\) as follows:

\[
(G_1 u)(t) = \int_0^t (t - qs)f(s, u(s))dqs, \\
(G_2 u)(t) = \left[(1 - \eta \sum_{i=1}^n \alpha_i)t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1 + p_i}\right] \int_0^t \frac{(T - qs)}{\Phi} f(s, u(s))dqs \\
- \left[(1 - \eta \sum_{i=1}^n \alpha_i)t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1 + p_i}\right] \\
\times \sum_{j=1}^m \frac{\beta_j}{\Phi} \int_0^{\eta} \int_0^s (s - q\nu)f(\nu, u(\nu))d\nu d\nu r_j s \\
\]

where \(\alpha_i\) and \(\beta_j\) are the coefficients associated with \(\alpha\) and \(\beta\) respectively.

\[ = \int_0^t \left[(1 - \eta \sum_{i=1}^n \alpha_i)t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1 + p_i}\right] \int_0^t \frac{(T - qs)}{\Phi} f(s, u(s))dqs \\
- \left[(1 - \eta \sum_{i=1}^n \alpha_i)t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1 + p_i}\right] \\
\times \sum_{j=1}^m \frac{\beta_j}{\Phi} \int_0^{\eta} \int_0^s (s - q\nu)f(\nu, u(\nu))d\nu d\nu r_j s \\
\]

where \(\Omega\) is given by (15). Consider \(B_r = \{u \in C : ||u|| \leq r\}\). In view of Lemma 2.1 we define the operators \(G_1\) and \(G_2\) as follows:

\[
(G_1 u)(t) = \int_0^t (t - qs)f(s, u(s))dqs, \\
(G_2 u)(t) = \left[(1 - \eta \sum_{i=1}^n \alpha_i)t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1 + p_i}\right] \int_0^t \frac{(T - qs)}{\Phi} f(s, u(s))dqs \\
- \left[(1 - \eta \sum_{i=1}^n \alpha_i)t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1 + p_i}\right] \\
\times \sum_{j=1}^m \frac{\beta_j}{\Phi} \int_0^{\eta} \int_0^s (s - q\nu)f(\nu, u(\nu))d\nu d\nu r_j s \\
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- \left[(1 - \eta \sum_{i=1}^n \alpha_i)t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1 + p_i}\right] \\
\times \sum_{j=1}^m \frac{\beta_j}{\Phi} \int_0^{\eta} \int_0^s (s - q\nu)f(\nu, u(\nu))d\nu d\nu r_j s \\
\]

where \(\Omega\) is given by (15). Consider \(B_r = \{u \in C : ||u|| \leq r\}\). In view of Lemma 2.1 we define the operators \(G_1\) and \(G_2\) as follows:

\[
(G_1 u)(t) = \int_0^t (t - qs)f(s, u(s))dqs, \\
(G_2 u)(t) = \left[(1 - \eta \sum_{i=1}^n \alpha_i)t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1 + p_i}\right] \int_0^t \frac{(T - qs)}{\Phi} f(s, u(s))dqs \\
- \left[(1 - \eta \sum_{i=1}^n \alpha_i)t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1 + p_i}\right] \\
\times \sum_{j=1}^m \frac{\beta_j}{\Phi} \int_0^{\eta} \int_0^s (s - q\nu)f(\nu, u(\nu))d\nu d\nu r_j s \\
\]
\[+ \left[ (1 - \eta \sum_{j=1}^{m} \beta_j) t - T + \eta^2 \sum_{j=1}^{m} \frac{\beta_j}{1 + r_j} \right] \]
\[\times \left[ \sum_{i=1}^{n} \frac{\alpha_i}{\Phi} \int_{0}^{\eta} \int_{0}^{s} (s - q\nu) f(\nu, u(\nu)) d\nu d\rho_i s. \right] \]

Now we shall show that \( G_1 + G_2 \) has a fixed point in \( B_r \). For \( u, v \in B_r \), we have by computing directly, that

\[\|G_1 u + G_2 v\| \leq \|u\| \int_{0}^{T} (T - qs) d\nu ds \]
\[+ \left| (1 - \eta \sum_{i=1}^{n} \alpha_i) t + \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i} \right| \int_{0}^{T} (T - qs) d\nu ds \]
\[+ \left| (1 - \eta \sum_{j=1}^{m} \beta_j) t - T + \eta^2 \sum_{j=1}^{m} \frac{\beta_j}{1 + r_j} \right| \sum_{i=1}^{n} \frac{|\alpha_i|}{\Phi} \]
\[\times \left[ \int_{0}^{\eta} \int_{0}^{s} (s - q\nu) d\nu d\rho_i s \right. \]
\[\leq \|u\| \int_{0}^{T} \left( T^2 + \left( (1 + \eta \sum_{i=1}^{n} |\alpha_i|) T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1 + p_i} \right) \frac{T^2}{\Phi} \right) \]
\[+ \left( (1 + \eta \sum_{i=1}^{n} |\alpha_i|) T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1 + p_i} \right) \sum_{j=1}^{m} \frac{|\beta_j| \eta^3}{\Phi(p_j^2 + p_i + 1)} \]
\[+ \left( \eta^2 \sum_{j=1}^{m} \frac{|\beta_j|}{1 + r_j} + \eta \sum_{j=1}^{m} |\beta_j| \right) \sum_{i=1}^{n} \frac{|\alpha_i| \eta^3}{\Phi(p_i^2 + p_i + 1)} \]
\[\leq \|u\| \Omega \leq r. \]

Therefore, \( G_1 u + G_2 v \in B_r \). From assumption \((H_1)\) and \((H_2)\) for \( u, v \in C \), we get that

\[\|G_2 u - G_2 v\| \leq \Lambda \|u - v\|_C. \]

Hence, \( G_2 \) is a contraction mapping. Next, we shall show that \( G_1 \) is compact and continuous. The continuity of \( f \) implies that the operator \( G_1 \) is continuous.
By using condition (17), we have that $G_1$ is uniformly bounded on $B_r$ since
\[ \|G_1 u\| \leq \|\mu\|T^2. \] (19)
Furthermore, in view of $(H_1)$, we define $\sup_{(t,u)\in[0,T]\times B_r} |f(t,u)| = f_{\text{max}} < \infty$, and consequently we get that
\[ |(G_1 u)(t_2) - (G_1 v)(t_1)| \]
\[ \leq \sup_{(t,u)\in[0,T]\times B_r} \left| \int_0^{t_2} (t_2 - qs) f(s,u(s)) dq s - \int_0^{t_1} (t_1 - qs) f(s,u(s)) dq s \right| \]
\[ \leq \int_0^{t_1} (t_2 - t_1) |f(s,u(s))| dq s + \int_0^{t_2} (t_2 - qs) |f(s,u(s))| dq s \]
\[ \leq f_{\text{max}} \left( \int_0^{t_1} (t_2 - t_1) dq s + \int_0^{t_2} (t_2 - qs) dq s \right). \]
Actually, as $t_2 \to t_1$, the right-hand side of the above inequality, which is independent of $u$, tends to zero. Therefore, $G_1$ is equicontinuous. Since $G_1$ is uniformly bounded and equicontinuous on $B_r$, we get that $G_1$ is relatively compact on $B_r$. Hence, by the Arzelà-Ascoli Theorem, $G_1$ is compact on $B_r$. Thus, all the assumptions of Theorem 3 are satisfied. By Theorem 4, we have that the boundary value problem (1)-(2) has at least one solution on $[0,T]$. This completes the proof.

Next, we prove the existence of solutions for the problem (1)-(2) by using Leray-Schauder degree theory.

**Theorem 5.** Let $f : [0,T] \times R \to R$. Suppose that there exist constants $0 \leq \kappa < \Omega^{-1}$, where $\Omega$ is given by (15) and $M > 0$ such that $|f(t,u)| \leq \kappa|u| + M$ for all $t \in [0,T], u \in R$. Then the boundary value problem (1)-(2) has at least one solution on $[0,T]$.

**Proof.** We define an operator $F : C \to C$ as in (14). In view of the fixed point problem
\[ u = Fu. \] (20)
We are going to prove the existence of at least one solution $u \in C[0,T]$ satisfying (20). Set a ball $B_R \subset C[0,T]$, as
\[ B_R = \{ u \in C : \max_{t\in[0,T]} |u(t)| < R \}, \]
where a constant radius $R > 0$. Hence, we will show that $F : \overline{B}_R \to C[0,T]$ satisfies a condition
\[ u \neq \lambda Fu, \quad \forall u \in \partial B_R, \quad \forall \lambda \in [0,1]. \] (21)
We set
\[ H(\lambda, u) = \lambda F u, \quad u \in \mathcal{C}, \quad \lambda \in [0, 1]. \]

As shown in Theorem 2 and 4, we have that the operator \( F \) that is continuous, uniformly bounded and equicontinuous. Then, by the Arzelá-Ascoli theorem, a continuous map \( h_\lambda \) defined by \( h_\lambda(u) = u - H(\lambda, u) = u - \lambda F u \) is completely continuous. If (21) holds, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that
\[
\text{deg}(h_\lambda, B_R, 0) = \text{deg}(I - \lambda F, B_R, 0) = \text{deg}(h, B_R, 0) = 1 \neq 0, \quad 0 \in B_R,
\]
where \( I \) denotes the unit operator. By the nonzero property of Leray-Schauder degree, \( h_1(u) = u - F u = 0 \) for at least one \( u \in B_R \). Let us assume that \( u = \lambda F u \) for some \( \lambda \in [0, 1] \) and for all \( t \in [0, T] \) so that
\[
|u(t)| = |\lambda(Fu)(t)|
\]
\[
\leq \int_0^t (t - qs)|f(s, u(s))|d_q s
\]
\[
+ \left| (1 - \eta \sum_{i=1}^n \alpha_i) t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1 + p_i} \right| \left( \int_0^T \frac{(T - qs)}{|\Phi|} |f(s, u(s))|d_q s \right)
\]
\[
+ \left| (1 - \eta \sum_{j=1}^m \beta_j) t - \eta^2 \sum_{j=1}^m \frac{\beta_j}{1 + r_j} \right| \sum_{i=1}^n \frac{\alpha_i}{|\Phi|} \left( \int_0^T \frac{|f(s, u(s))|d_q s}{|\Phi|} \right)
\]
\[
\leq (\kappa|u| + M) \left[ \int_0^t (t - qs)d_q s \right]
\]
\[
+ \left| (1 - \eta \sum_{i=1}^n \alpha_i) t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1 + p_i} \right| \left( \int_0^T \frac{(T - qs)}{|\Phi|} |f(s, u(s))|d_q s \right)
\]
\[
+ \left| (1 - \eta \sum_{j=1}^m \beta_j) t - \eta^2 \sum_{j=1}^m \frac{\beta_j}{1 + r_j} \right| \sum_{i=1}^n \frac{\alpha_i}{|\Phi|} \left( \int_0^T \frac{|f(s, u(s))|d_q s}{|\Phi|} \right).
\]
\begin{align*}
&\times \sum_{j=1}^{m} \frac{|\beta_j|}{\Phi} \int_{0}^{\eta} \int_{0}^{s} (s - q\nu)dq\nu dr_j s \\
&+ \left| (1 - \eta \sum_{j=1}^{m} \beta_j) t - T + \eta^2 \sum_{j=1}^{m} \frac{\beta_j}{1 + r_j} \right| \\
&\times \sum_{i=1}^{n} \frac{|\alpha_i|}{\Phi} \int_{0}^{\eta} \int_{0}^{s} (s - q\nu)dq\nu dp_i s \\
&\leq \left( \kappa|u| + M \right) \left[ T^2 + \left( (1 + \eta \sum_{i=1}^{n} |\alpha_i|)T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1 + p_i} \right) T^2 - \right. \\
&\left. \left( (1 + \eta \sum_{i=1}^{n} |\alpha_i|)T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1 + p_i} \right) \sum_{j=1}^{m} \frac{|\beta_j|}{\Phi} \left[ r_j^2 + r_j + 1 \right] \right] \\
&\left. + \left( \eta^2 \sum_{j=1}^{m} \frac{|\beta_j|}{1 + r_j} + \eta \sum_{j=1}^{m} |\beta_j| \right) \sum_{i=1}^{n} \frac{|\alpha_i|}{\Phi} \left[ (p_i^2 + p_i + 1) \right] \right] \\
&= (\kappa|u| + M)\Omega,
\end{align*}

which on taking norm sup_{t \in [0,T]} |u(t)| = \|u\| and solving for \|u\|, yields
\[
\|u\| \leq \frac{M\Omega}{1 - \kappa\Omega}.
\]

If \( R = \frac{M\Omega}{1 - \kappa\Omega} + 1 \), inequality (21) holds. This completes the proof.

4. Some Examples

**Example 6.** Consider the following three-point multi-term \( q \)-integral boundary value problem

\[
D^2_{\frac{t}{2}} u(t) = \frac{1}{(t + 4)^2} \cdot \frac{|u|}{|u| + 1}, \quad t \in [0, \sqrt{2}].
\]  

\[
u(0) = \frac{\sqrt{3}}{2} \int_{0}^{\sqrt{2}} u(s)d\frac{1}{4}s - \int_{0}^{\sqrt{2}} u(s)d\frac{3}{4}s,
\]

\[
u(\sqrt{2}) = -\sqrt{7} \int_{0}^{\sqrt{2}} u(s)d\frac{1}{2}s + \sqrt{\pi} \int_{0}^{\sqrt{2}} u(s)d\frac{3}{2}s - \frac{3}{2} \int_{0}^{\sqrt{2}} u(s)d\frac{5}{4}s.
\]
Set \( q = 1/2, T = \sqrt{2}, \eta = \sqrt{2}/2, n = 2, \alpha_1 = \sqrt{3}/2, \alpha_2 = -1, p_1 = 1/4, p_2 = 2/3, m = 3, \beta_3 = -3/2, r_1 = 1/2, r_2 = 3/4, r_3 = 4/5, f(t, u) = (1/(t + 4)^2)(|u|/(1 + |u|)) \). Since \(|f(t, u) - f(t, v)| \leq (1/16)|u - v|\), (H1) is satisfied with \( \eta^2 \left( \sum_{j=1}^{m} \frac{\beta_j}{1+r_j} - \sum_{i=1}^{n} \frac{\alpha_i(1-\eta \sum_{j=1}^{m} \beta_j)}{1+p_i} \right) - T \approx -4.595866 \),

\[
\Phi \approx -2.845975,
\]

\[
\Omega = T^2 + \left( 1 + \eta \sum_{i=1}^{n} |\alpha_i| \right) T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1+p_i} \right) \left( T^2 \frac{|\beta_j|}{1+r_j} - \frac{|\alpha_i|}{1+p_i} \right) \left( \frac{|\beta_j|}{(r_j^2 + p_i + 1)} \right)
\]

\[
\approx 11.093692,
\]

\( L = 1/16 \). Hence \( \Lambda := L \Omega \approx 0.693356 < 1 \). Therefore, by Theorem 2, the boundary value problem (22)-(23) has a unique solution on \([0, 2]\).

**Example 7.** Consider the following three-point multi-term \( q \)-integral boundary value problem

\[
D_{\frac{3}{5}}^2 u(t) = \frac{\sin(4\pi u)}{16\pi} + \frac{|u|}{|u| + 1}, \quad t \in [0, 1]. \tag{24}
\]

\[
u(0) = \sqrt{3} \int_{0}^{\frac{1}{2}} \frac{1}{2} u(s) d_{\frac{1}{2}} s - \frac{7}{2} \int_{0}^{\frac{1}{2}} u(s) d_{\frac{3}{5}} s - 5 \int_{0}^{\frac{1}{2}} u(s) d_{\frac{1}{3}} s,
\]

\[
u(1) = \frac{1}{4} \int_{0}^{\frac{1}{2}} u(s) d_{\frac{1}{2}} s - \frac{1}{5} \int_{0}^{\frac{1}{2}} u(s) d_{\frac{3}{5}} s - \frac{1}{2} \int_{0}^{\frac{1}{2}} u(s) d_{\frac{1}{3}} s. \tag{25}
\]

Set \( q = 2/3, T = 1, \eta = 1/2, n = 3, \alpha_1 = \sqrt{3}, \alpha_2 = -7/2, \alpha_3 = -5, p_1 = 1/2, p_2 = 2/3, p_3 = 3/4, m = 3, \beta_1 = -1/5, \beta_2 = 1/4, \beta_3 = -1/2, r_1 = 1/3, r_2 = 1/4, r_3 = 1/5 \). Here, \(|f(t, u)| = |\sin(4\pi u)/16\pi + |u|/(1 + |u|)| \leq (|u|/4) + 1\). So, \( M = 1, \eta^2 \left( \sum_{j=1}^{m} \frac{\beta_j}{1+r_j} - \sum_{i=1}^{n} \frac{\alpha_i(1-\eta \sum_{j=1}^{m} \beta_j)}{1+p_i} \right) - T \approx 0.072831 \), and

\[
\Phi = \left( 1 - \eta \sum_{i=1}^{n} \alpha_i \right) \left( \eta^2 \sum_{j=1}^{m} \frac{\beta_j}{1+r_j} - T \right) - \eta^2 \sum_{i=1}^{n} \frac{\alpha_i(1-\eta \sum_{j=1}^{m} \beta_j)}{1+p_i}
\]
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\[ \Omega \approx 3.621341, \]

\[ \Omega = T^2 + \left( (1 + \eta \sum_{i=1}^{n} |\alpha_i|)T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1 + p_i} \right) \frac{T^2}{|\Phi|} \]

\[ + \left( (1 + \eta \sum_{i=1}^{n} |\alpha_i|)T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1 + p_i} \right) \left( \sum_{j=1}^{m} \frac{|\beta_j|}{1 + r_j} \right) \left( \sum_{i=1}^{n} \frac{|\alpha_i| |\eta^3|}{|\Phi| (r_j^2 + p_i + 1)} \right) \]

\[ \approx 3.970666, \]

\[ \kappa = \frac{1}{4} < \frac{1}{\Omega} \approx 0.251847. \]

Hence, by Theorem 5, the boundary value problem (24)-(25) has at least one solution on \([0, 1]\).

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References


