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EXISTENCE OF SOLUTIONS FOR NONLINEAR SECOND-ORDER Q-DIFFERENCE EQUATIONS WITH THREE-POINT MULTI-TERM q-INTEGRAL BOUNDARY CONDITIONS

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Abstract: In this paper is concerned with the existence of solutions for nonlinear second-order q-difference equations with three-point multi-term q-integral boundary conditions. Some new existence results are obtained by using Banach's contraction mapping, Krasnoselskii's fixed point theorem and Leray-Schauder degree theory. As an application, we give two examples that illustrate our results.

AMS Subject Classification: 34B10, 39A13
Key Words: existence, q-difference equation, q-derivative, q-integral, boundary value problem

1. Introduction

The study of q-calculus was initiated in the beginning of the twentieth century by the pioneer works of Jackson [1], Carmichael [2], Mason [3], Adams [4], Trjitzinsky [5], etc. In the last few decades, the q-theory has involved into a variety of discipline areas of research not only in mathematics but also covering other fields of science and its applications as well. See [6-14]. For some recent

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work, we refer the reader to [15-26] and references therein. However, there are only a few papers that have studied the existence of solutions for boundary value problems of nonlinear q-difference equations, see [27-29].

In [30], Ahmad studied the existence of solutions for nonlinear third-order q-difference equation with boundary conditions

$$D_q^3 u(t) = f(t, u(t)), \quad 0 \leq t \leq 1,$$
$$u(0) = 0, \quad D_q u(0) = 0, \quad u(1) = 0.$$

He obtained some existence results by using standard fixed point theorems and Leray-Schauder degree theory. He also showed that if $q \to 1$ then his results corresponded to the classical results. In [31], Ahmad, Alsaedi and Ntouyas established the existence criteria for nonlinear second-order q-difference equation with non-separated boundary conditions

$$D_q^2 u(t) = f(t, u(t)), \quad t \in [0, T],$$
$$u(0) = \eta u(T), \quad D_q u(0) = \eta D_q u(T).$$

In [32], Pongam, Asawasumrit, Tariboon investigated the sequential derivatives of the nonlinear q-difference equation with three-point q-integral boundary condition of the form

$$D_q(D_p + \lambda)u(t) = f(t, u(t)), \quad t \in [0, T],$$
$$u(0) = 0, \quad \beta \int_0^\eta u(s) d_r s = u(T), \quad \eta \in (0, T).$$

Using fixed point theorems and Leray-Schauder degree theory, some existence were obtained.

In this paper, we consider the following nonlinear second-order q-difference equation with three-point multi-term q-integral boundary conditions

$$D_q^2 u(t) = f(t, u(t)), \quad 0 \leq t \leq T, \tag{1}$$

$$u(0) = \sum_{i=1}^{n} \alpha_i \int_0^\eta u(s) d_{p_i} s, \quad u(T) = \sum_{j=1}^{m} \beta_j \int_0^\eta u(s) d_{r_j} s, \tag{2}$$

where $0 < q, p_i, r_j < 1$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ and $\alpha_i, \beta_j \in R$ are such that $\eta^2 \Big(\sum_{j=1}^m \frac{\beta_j}{1+r_j} - \sum_{i=1}^n \frac{\alpha_i (1 - \eta \sum_{j=1}^m \beta_j)}{1+p_i} \Big) \neq T$, $\eta \in (0, T)$ is a fixed constant. We note that the boundary conditions (2) have different values of the q-numbers of q-integral at the boundary points. The existence results of problem (1)-(2) have not been established, previously.

2. Preliminaries

In this section, we introduce notation, some basic concepts of q-calculus and prove a lemma before starting our main results.

For $0 < q < 1$, we define the q-derivative of a real valued function f as

$$D_q f(t) := \frac{f(t) - f(qt)}{(1-q)t},$$

and $D_q f(0) = \lim_{t\to 0} D_q f(t)$ provided $f'(0)$ exists. The higher order q-derivatives are given by

$$D_q^0 f(t) = f(t), \qquad D_q^n f(t) = D_q D_q^{n-1} f(t), \qquad n \in N.$$

The q-integral of a function f defined in the interval $[a, b]$ is given by

$$\int_a^t f(s) d_q s := \sum_{n=0}^{\infty} (1-q)q^n [tf(tq^n) - af(q^n a)], \qquad t \in [a, b].$$

For $a = 0$, we denote

$$I_q f(t) = \int_0^t f(s) d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n),$$

provided the series converges. If $a \in [0, b]$ and f is defined in the interval $[0, b]$, then

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Note that

$$D_q I_q f(t) = f(t), \tag{3}$$

and if f is continuous at $t = 0$, then

$$I_q D_q f(t) = f(t) - f(0).$$

In q-calculus, the product rule and integration by parts formula are

$$D_q(gh)(t) = (D_q g(t))h(t) + q(qt)D_q h(t), \tag{4}$$

$$\int_0^t h(s) D_q g(s) d_q s = \Big[h(s)g(s)\Big]_0^t - \int_0^t D_q h(s) g(qs) d_q s. \tag{5}$$

Further, reversing the order of integration is given by

$$\int_0^t \int_0^s f(r) d_q r d_q s = \int_0^t \int_{qr}^t f(r) d_q s d_q r.$$

In the limit $q \to 1$ the q-calculus corresponds to the classical calculus.

Lemma 1. *Let* $\eta^2\Big(\sum_{j=1}^m \frac{\beta_j}{1+r_j} - \sum_{i=1}^n \frac{\alpha_i(1-\eta\sum_{j=1}^m\beta_j)}{1+p_i}\Big) \neq T$ *and* $0 < q, p_i, r_j < 1$ *for* $i = 1,2,\dots,n$, $j = 1,2,\dots,m$. *Then for* $h \in C[0,T]$, *the boundary value problem*

$$D_q^2 u(t) = h(t), \quad 0 < t < T, \tag{6}$$

$$u(0) = \sum_{i=1}^n \alpha_i \int_0^\eta u(s) d_{p_i} s, \quad u(T) = \sum_{j=1}^m \beta_j \int_0^\eta u(s) d_{r_j} s, \tag{7}$$

has the unique solution

$$\begin{aligned} u(t) = & \int_0^t (t-qs)h(s)d_qs \\ & + \left[(1-\eta\sum_{i=1}^n \alpha_i)t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1+p_i}\right] \int_0^T \frac{(T-qs)}{\Phi} h(s) d_qs \\ & - \left[(1-\eta\sum_{i=1}^n \alpha_i)t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1+p_i}\right] \\ & \times \sum_{j=1}^m \frac{\beta_j}{\Phi} \int_0^\eta \int_0^s (s-q\nu)h(\nu) d_q\nu d_{r_j} s \\ & \quad + \left[(1-\eta\sum_{j=1}^m \beta_j)t - T + \eta^2 \sum_{j=1}^m \frac{\beta_j}{1+r_j}\right] \\ & \quad \times \sum_{i=1}^n \frac{\alpha_i}{\Phi} \int_0^\eta \int_0^s (s-q\nu)h(\nu) d_q\nu d_{p_i} s, \end{aligned} \tag{8}$$

where

$$\Phi = \Big(1-\eta\sum_{i=1}^n \alpha_i\Big)\Big(\eta^2 \sum_{j=1}^m \frac{\beta_j}{1+r_j} - T\Big) - \eta^2 \sum_{i=1}^n \frac{\alpha_i(1-\eta\sum_{j=1}^m \beta_j)}{1+p_i}. \tag{9}$$

Proof. For $t \in [0,T]$, we get by q-integrating for (6) that,

$$D_q u(t) = \int_0^t h(s) d_qs + c_1, \tag{10}$$

where a_1 is a constant of q-integration. Taking q-integral to (10) for $t \in [0,T]$, we obtain

$$u(t) = \int_0^t \Big(\int_0^\nu h(s) d_qs\Big) d_q\nu + c_1 t + c_2. \tag{11}$$

Changing the order of q-integration, we get

$$u(t) = \int_0^t \Big(\int_{qs}^t h(s) d_q\nu \Big) d_q s + c_1 t + c_2. \tag{12}$$

Therefore, (12) can be written as

$$u(t) = \int_0^t (t - qs) h(s) d_q s + c_1 t + c_2. \tag{13}$$

It follows that $c_2 = u(0)$. Taking the p_i-integral of (13) from 0 to η, we get

$$\begin{aligned}\int_0^\eta u(s) d_{p_i} s &= \int_0^\eta \Big(\int_0^s (s - q\nu) h(\nu) d_q\nu + c_1 s + c_2 \Big) d_{p_i} s \\ &= \int_0^\eta \Big(\int_0^s (s - q\nu) h(\nu) d_q\nu \Big) d_{p_i} s + c_1 \int_0^\eta s d_{p_i} s + c_2 \eta \\ &= \int_0^\eta \Big(\int_0^s (s - q\nu) h(\nu) d_q\nu \Big) d_{p_i} s + c_1 \frac{\eta^2}{1 + p_i} + c_2 \eta.\end{aligned}$$

The first condition of (7) implies that

$$-c_1 \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1 + p_i} + c_2 \Big(1 - \eta \sum_{i=1}^n \alpha_i \Big) = \sum_{i=1}^n \alpha_i \int_0^\eta \Big(\int_0^s (s - q\nu) h(\nu) d_q\nu \Big) d_{p_i} s.$$

Form (13), we obtain

$$u(T) = \int_0^T (T - qs) h(s) d_q s + c_1 T + c_2.$$

Then, r_j-integrating of (13) gives

$$\int_0^\eta u(s) d_{r_j} s = \int_0^\eta \Big(\int_0^s (s - q\nu) h(\nu) d_q\nu \Big) d_{r_j} s + c_1 \frac{\eta^2}{1 + r_j} + c_2 \eta.$$

The second condition of (7) implies that

$$\begin{aligned}&c_1 \Big(T - \eta^2 \sum_{j=1}^m \frac{\beta_j}{1 + r_j} \Big) - c_2 \Big(1 - \eta \sum_{j=1}^m \beta_j \Big) \\ &\quad = \sum_{j=1}^m \beta_j \int_0^\eta \Big(\int_0^s (s - q\nu) h(\nu) d_q\nu \Big) d_{r_j} s - \int_0^T (T - qs) h(s) d_q s.\end{aligned}$$

Solving the system of linear equations on the unknown constants c_1 and c_2, we obtain

$$\begin{aligned}
c_1 &= \Big(1-\eta\sum_{j=1}^{m}\beta_j\Big)\sum_{i=1}^{n}\frac{\alpha_i}{\Phi}\int_0^{\eta}\Big(\int_0^{s}(s-q\nu)h(\nu)d_q\nu\Big)d_{p_i}s\\
&\quad-\Big(1-\eta\sum_{i=1}^{n}\alpha_i\Big)\sum_{j=1}^{m}\frac{\beta_j}{\Phi}\int_0^{\eta}\Big(\int_0^{s}(s-q\nu)h(\nu)d_q\nu\Big)d_{r_j}s\\
&\quad+\Big(1-\eta\sum_{i=1}^{n}\alpha_i\Big)\int_0^{T}\frac{(T-qs)}{\Phi}h(s)d_qs,\\
c_2 &= -\Big(T-\eta^2\sum_{j=1}^{m}\frac{\beta_j}{1+r_j}\Big)\sum_{i=1}^{m}\frac{\alpha_i}{\Phi}\int_0^{\eta}\Big(\int_0^{s}(s-q\nu)h(\nu)d_q\nu\Big)d_{p_i}s\\
&\quad-\Big(\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\Big)\sum_{j=1}^{m}\frac{\beta_j}{\Phi}\int_0^{\eta}\Big(\int_0^{s}(s-q\nu)h(\nu)d_q\nu\Big)d_{r_j}s\\
&\quad+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\int_0^{T}\frac{(T-qs)}{\Phi}h(s)d_qs.
\end{aligned}$$

Substituting values of c_1 and c_2 in (13), we obtain the solution (8).

For the forthcoming analysis, let $\mathcal{C}=C([0,T],R)$ denote the Banach space of all continuous functions from $[0,T]$ to R endowed with the norm defined by $\|u\|=\sup\{|u(t)|,\ t\in[0,T]\}$. In view of Lemma 1 and (9), we consider the operator $F:\mathcal{C}\to\mathcal{C}$ defined by

$$\begin{aligned}
(Fu)(t) &= \int_0^{t}(t-qs)f(s,u(s))d_qs\\
&\quad+\Bigg[(1-\eta\sum_{i=1}^{n}\alpha_i)t+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\Bigg]\int_0^{T}\frac{(T-qs)}{\Phi}f(s,u(s))d_qs\\
&\quad-\Bigg[(1-\eta\sum_{i=1}^{n}\alpha_i)t+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\Bigg]\\
&\quad\times\sum_{j=1}^{m}\frac{\beta_j}{\Phi}\int_0^{\eta}\int_0^{s}(s-q\nu)f(\nu,u(\nu))d_q\nu d_{r_j}s\\
&\quad+\Bigg[(1-\eta\sum_{j=1}^{m}\beta_j)t-T+\eta^2\sum_{j=1}^{m}\frac{\beta_j}{1+r_j}\Bigg]
\end{aligned}$$

$$\times \sum_{i=1}^{n} \frac{\alpha_i}{\Phi} \int_0^{\eta} \int_0^{s} (s - q\nu) f(\nu, u(\nu)) d_q \nu d_{p_i} s. \tag{14}$$

Note that the problem (1)-(2) has solutions if and only if the operator equation $Fx = x$ has fixed points.

In the following, for the sake of convenience, we set

$$\begin{aligned}\Omega = T^2 &+ \left((1 + \eta \sum_{i=1}^{n} |\alpha_i|)T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1 + p_i} \right) \frac{T^2}{|\Phi|} \\ &+ \left((1 + \eta \sum_{i=1}^{n} |\alpha_i|)T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1 + p_i} \right) \sum_{j=1}^{m} \frac{|\beta_j| \eta^3}{|\Phi|(r_j^2 + r_j + 1)} \\ &+ \left(\eta^2 \sum_{j=1}^{m} \frac{|\beta_j|}{1 + r_j} + \eta \sum_{j=1}^{m} |\beta_j| \right) \sum_{i=1}^{n} \frac{|\alpha_i| \eta^3}{|\Phi|(p_i^2 + p_i + 1)},\end{aligned} \tag{15}$$

where Φ is defined by (9).

3. Main Results

Now we are in position to establish the main results. Our first result is based on Banach's fixed point theorem.

Theorem 2. *Let $f : [0, T] \times R \to R$ be a jointly continuous function. In addition, suppose that there exists a positive constant L such that:*

(H_1) $|f(t, u(t)) - f(t, v(t))| \le L|u - v|$, *for all* $t \in [0, T]$, $u, v \in R$.

(H_2) $\Lambda := L\Omega < 1$, *where* Ω *is defined by* (15).

Then the boundary value problem (1)-(2) *has a unique solution.*

Proof. Assume that $\sup_{t \in [0,T]} |f(t, 0)| = M$, and define a constant r as

$$r \ge \frac{M\Omega}{1 - \delta}. \tag{16}$$

Now, we shall show that $FB_r \subset B_r$, where $B_r = \{u \in \mathcal{C} : \|u\| \le r\}$. For any $u \in B_r$, we get

$$\|Fu\| = \sup_{t \in [0,T]} \left| \int_0^t (t - qs) f(s, u(s)) d_q s \right.$$

$$
\begin{aligned}
&+\left[(1-\eta\sum_{i=1}^{n}\alpha_i)t+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\right]\\
&\times\int_0^T\frac{(T-qs)}{\Phi}f(s,u(s))d_qs\\
&+\left[(1-\eta\sum_{i=1}^{n}\alpha_i)t+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\right]\sum_{j=1}^{m}\frac{\beta_j}{\Phi}\\
&\times\int_0^\eta\int_0^s(s-q\nu)f(\nu,u(\nu))d_q\nu d_{r_j}s\\
&+\left[(1-\eta\sum_{j=1}^{m}\beta_j)t-T+\eta^2\sum_{j=1}^{m}\frac{\beta_j}{1+r_j}\right]\sum_{i=1}^{n}\frac{\alpha_i}{\Phi}\\
&\times\int_0^\eta\int_0^s(s-q\nu)f(\nu,u(\nu))d_q\nu d_{p_i}s\Bigg|\\
\le\sup_{t\in[0,T]}&\Bigg\{\int_0^t(t-qs)(|f(s,u(s))-f(s,0)|+|f(s,0)|)d_qs\\
&+\left|(1-\eta\sum_{i=1}^{n}\alpha_i)t+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\right|\\
&\times\int_0^T\frac{(T-qs)}{|\Phi|}(|f(s,u(s))-f(s,0)|+|f(s,0)|)d_qs\\
&+\left|(1-\eta\sum_{i=1}^{n}\alpha_i)t+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\right|\sum_{j=1}^{m}\frac{|\beta_j|}{|\Phi|}\\
&\times\int_0^\eta\int_0^s(s-q\nu)(|f(\nu,u(\nu))-f(\nu,0)|+|f(\nu,0)|)d_q\nu d_{r_j}s\\
&+\left|(1-\eta\sum_{j=1}^{m}\beta_j)t-T+\eta^2\sum_{j=1}^{m}\frac{\beta_j}{1+r_j}\right|\sum_{i=1}^{n}\frac{|\alpha_i|}{|\Phi|}\\
&\times\int_0^\eta\int_0^s(s-q\nu)(|f(\nu,u(\nu))-f(\nu,0)|+|f(\nu,0)|)d_q\nu d_{p_i}s\Bigg\}\\
\le\sup_{t\in[0,T]}&\Bigg\{\int_0^t(t-qs)(L|u(s)|+|f(s,0)|)d_qs\\
&+\left|(1-\eta\sum_{i=1}^{n}\alpha_i)t+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\right|\int_0^T\frac{(T-qs)}{|\Phi|}(L|u(s)|+|f(s,0)|)d_qs
\end{aligned}
$$

$$+\left|(1-\eta\sum_{i=1}^{n}\alpha_i)t+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\right|\sum_{j=1}^{m}\frac{|\beta_j|}{|\Phi|}$$

$$\times\int_0^{\eta}\int_0^{s}(s-q\nu)(L|u(\nu)|+|f(\nu,0)|)d_q\nu d_{r_j}s$$

$$+\left|(1-\eta\sum_{j=1}^{m}\beta_j)t-T+\eta^2\sum_{j=1}^{m}\frac{\beta_j}{1+r_j}\right|\sum_{i=1}^{n}\frac{|\alpha_i|}{|\Phi|}$$

$$\left.\times\int_0^{\eta}\int_0^{s}(s-q\nu)(L|u(\nu)|+|f(\nu,0)|)d_q\nu d_{p_i}s\right\}$$

$$\leq\sup_{t\in[0,T]}\left\{\int_0^{t}(t-qs)(L||u||+M)d_qs\right.$$

$$+\left|(1-\eta\sum_{i=1}^{n}\alpha_i)t+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\right|\int_0^{T}\frac{(T-qs)}{|\Phi|}(L||u||+M)d_qs$$

$$+\left|(1-\eta\sum_{i=1}^{n}\alpha_i)t+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\right|\sum_{j=1}^{m}\frac{|\beta_j|}{|\Phi|}$$

$$\times\int_0^{\eta}\int_0^{s}(s-q\nu)(L||u||+M)d_q\nu d_{r_j}s$$

$$+\left|(1-\eta\sum_{j=1}^{m}\beta_j)t-T+\eta^2\sum_{j=1}^{m}\frac{\beta_j}{1+r_j}\right|\sum_{i=1}^{n}\frac{|\alpha_i|}{|\Phi|}$$

$$\left.\times\int_0^{\eta}\int_0^{s}(s-q\nu)(L||u||+M)d_q\nu d_{p_i}s\right\}$$

$$\leq\sup_{t\in[0,T]}\left\{(L||u||+M)\left[t^2+\left|(1-\eta\sum_{i=1}^{n}\alpha_i)t+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\right|\frac{T^2}{|\Phi|}\right.\right.$$

$$+\left|(1-\eta\sum_{i=1}^{n}\alpha_i)t+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\right|\sum_{j=1}^{m}\frac{|\beta_j|}{|\Phi|}\frac{\eta^3}{r_j^2+r_j+1}$$

$$\left.\left.+\left|(1-\eta\sum_{j=1}^{m}\beta_j)t-T+\eta^2\sum_{j=1}^{m}\frac{\beta_j}{1+r_j}\right|\sum_{i=1}^{n}\frac{|\alpha_i|}{|\Phi|}\frac{\eta^3}{p_i^2+p_i+1}\right]\right\}$$

$$\leq(Lr+M)\left[T^2+\left((1+\eta\sum_{i=1}^{n}|\alpha_i|)T+\eta^2\sum_{i=1}^{n}\frac{|\alpha_i|}{1+p_i}\right)\frac{T^2}{|\Phi|}\right.$$

$$+\left((1+\eta\sum_{i=1}^{n}|\alpha_i|)T+\eta^2\sum_{i=1}^{n}\frac{|\alpha_i|}{1+p_i}\right)\sum_{j=1}^{m}\frac{|\beta_j|\eta^3}{|\Phi|(r_j^2+r_j+1)}$$
$$+\left(\eta^2\sum_{j=1}^{m}\frac{|\beta_j|}{1+r_j}+\eta\sum_{j=1}^{m}|\beta_j|\right)\sum_{i=1}^{n}\frac{|\alpha_i|\eta^3}{|\Phi|(p_i^2+p_i+1)}\Bigg]$$
$$\leq (Lr+M)\Omega \leq r.$$

Therefore, $FB_r \subset B_r$. Next, we will show that F is contraction. In the following, from (H_1) for any $u, v \in \mathcal{C}$ and for each $t \in [0,T]$, we have

$$\|Fu-Fv\| = \sup_{t\in[0,T]}|(Fu)(t)-(Fv)(t)|$$
$$= \sup_{t\in[0,T]}\Bigg|\int_0^t (t-qs)(f(s,u(s))-f(s,v(s)))d_qs$$
$$+\left[(1-\eta\sum_{i=1}^{n}\alpha_i)t+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\right]$$
$$\times\int_0^T\frac{(T-qs)}{\Phi}(f(s,u(s))-f(s,v(s)))d_qs$$

$$+\left[(1-\eta\sum_{i=1}^{n}\alpha_i)t+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\right]\sum_{j=1}^{m}\frac{\beta_j}{\Phi}$$
$$\times\int_0^\eta\int_0^s (s-q\nu)(f(\nu,u(\nu))-f(\nu,v(\nu)))d_q\nu d_{r_j}s$$
$$+\left[(1-\eta\sum_{j=1}^{m}\beta_j)t-T+\eta^2\sum_{j=1}^{m}\frac{\beta_j}{1+r_j}\right]\sum_{i=1}^{n}\frac{\alpha_i}{\Phi}$$
$$\times\int_0^\eta\int_0^s (s-q\nu)(f(\nu,u(\nu))-f(\nu,v(\nu)))d_q\nu d_{p_i}s\Bigg|$$
$$\leq L||u-v||\Bigg[T^2+\left((1+\eta\sum_{i=1}^{n}|\alpha_i|)T+\eta^2\sum_{i=1}^{n}\frac{|\alpha_i|}{1+p_i}\right)\frac{T^2}{|\Phi|}$$
$$+\left((1+\eta\sum_{i=1}^{n}|\alpha_i|)T+\eta^2\sum_{i=1}^{n}\frac{|\alpha_i|}{1+p_i}\right)\sum_{j=1}^{m}\frac{|\beta_j|\eta^3}{|\Phi|(r_j^2+r_j+1)}$$
$$+\left(\eta^2\sum_{j=1}^{m}\frac{|\beta_j|}{1+r_j}+\eta\sum_{j=1}^{m}|\beta_j|\right)\sum_{i=1}^{n}\frac{|\alpha_i|\eta^3}{|\Phi|(p_i^2+p_i+1)}\Bigg]$$

$$= L\Omega||u - v|| \leq \Lambda||u - v||.$$

From (H_2), we have that F is a contraction. Hence, by the Banach's fixed point theorem, we get that F has a fixed point which is a unique solution of the problem (1)-(2). This completes the proof.

Now, we state a result due to Krasnoselskii [33] which is needed to prove the existence of at least one solution of the problem (1)-(2).

Theorem 3. *Let K be a bounded closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that:*

(i) $Ax + By \in K$ whenever $x, y \in K$,

(ii) A is compact and continuous,

(iii) B is a contraction mapping.

Then there exists $z \in K$ such that $z = Az + Bz$.

Our second result is based on the following Krasnoselskii's fixed point theorem.

Theorem 4. *Assume that (H_1) and (H_2) hold with*

$$|f(t, u)| \leq \mu(t) \tag{17}$$

for all $(t, u) \in [0, T] \times R$, where $\mu \in L^1([0, T], R^+)$. Then the boundary value problem (1)-(2) has at least one solution on $[0, T]$.

Proof. Setting $\max_{t\in[0,T]} |\mu(t)| = ||\mu||$, and choosing a constant r such that

$$r \geq \|\mu\|\Omega, \tag{18}$$

where Ω is given by (15). Consider $B_r = \{u \in \mathcal{C} : \|u\| \leq r\}$. In view of Lemma 2.1 we define the operators G_1 and G_2 as follows:

$$\begin{aligned}
(G_1u)(t) &= \int_0^t (t - qs) f(s, u(s)) d_q s, \\
(G_2u)(t) &= \left[(1 - \eta \sum_{i=1}^{n} \alpha_i)t + \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i}\right] \int_0^T \frac{(T - qs)}{\Phi} f(s, u(s)) d_q s \\
&\quad - \left[(1 - \eta \sum_{i=1}^{n} \alpha_i)t + \eta^2 \sum_{i=1}^{n} \frac{\alpha_i}{1 + p_i}\right] \\
&\quad \times \sum_{j=1}^{m} \frac{\beta_j}{\Phi} \int_0^\eta \int_0^s (s - q\nu) f(\nu, u(\nu)) d_q \nu d_{r_j} s
\end{aligned}$$

$$+\left[(1-\eta\sum_{j=1}^{m}\beta_j)t-T+\eta^2\sum_{j=1}^{m}\frac{\beta_j}{1+r_j}\right]$$
$$\times\sum_{i=1}^{n}\frac{\alpha_i}{\Phi}\int_0^{\eta}\int_0^{s}(s-q\nu)f(\nu,u(\nu))d_q\nu d_{p_i}s.$$

Now we shall show that G_1+G_2 has a fixed point in B_r. For u, $v\in B_r$, we have by computing directly, that

$$\begin{aligned}\|G_1u+G_2v\| &\le ||u||\int_0^t(t-qs)d_qs\\ &\quad+\left|(1-\eta\sum_{i=1}^{n}\alpha_i)t+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\right|\frac{||u||}{|\Phi|}\int_0^T(T-qs)d_qs\\ &\quad+\left|(1-\eta\sum_{i=1}^{n}\alpha_i)t+\eta^2\sum_{i=1}^{n}\frac{\alpha_i}{1+p_i}\right|\sum_{j=1}^{m}\frac{|\beta_j|}{|\Phi|}\\ &\quad\times||u||\int_0^{\eta}\int_0^{s}(s-q\nu)d_q\nu d_{r_j}s\\ &\quad+\left|(1-\eta\sum_{j=1}^{m}\beta_j)t-T+\eta^2\sum_{j=1}^{m}\frac{\beta_j}{1+r_j}\right|\sum_{i=1}^{n}\frac{|\alpha_i|}{|\Phi|}\end{aligned}$$

$$\begin{aligned}&\times||u||\int_0^{\eta}\int_0^{s}(s-q\nu)d_q\nu d_{p_i}s\\ &\le||u||\left[T^2+\left((1+\eta\sum_{i=1}^{n}|\alpha_i|)T+\eta^2\sum_{i=1}^{n}\frac{|\alpha_i|}{1+p_i}\right)\frac{T^2}{|\Phi|}\right.\\ &\quad+\left((1+\eta\sum_{i=1}^{n}|\alpha_i|)T+\eta^2\sum_{i=1}^{n}\frac{|\alpha_i|}{1+p_i}\right)\sum_{j=1}^{m}\frac{|\beta_j|\eta^3}{|\Phi|(r_j^2+r_j+1)}\\ &\quad\left.+\left(\eta^2\sum_{j=1}^{m}\frac{|\beta_j|}{1+r_j}+\eta\sum_{j=1}^{m}|\beta_j|\right)\sum_{i=1}^{n}\frac{|\alpha_i|\eta^3}{|\Phi|(p_i^2+p_i+1)}\right]\\ &\le||u||\Omega\le r.\end{aligned}$$

Therefore, $G_1u+G_2v\in B_r$. From assumption (H_1) and (H_2) for $u,v\in\mathcal{C}$, we get that

$$\|G_2u-G_2v\|\le\Lambda\|u-v\|_{\mathcal{C}}.$$

Hence, G_2 is a contraction mapping. Next, we shall show that G_1 is compact and continuous. The continuity of f implies that the operator G_1 is continuous.

By using condition (17), we have that G_1 is uniformly bounded on B_r since

$$\|G_1 u\| \leq \|\mu\| T^2. \tag{19}$$

Furthermore, in view of (H_1), we define $\sup_{(t,u)\in[0,T]\times B_r} |f(t,u)| = f_{\max} < \infty$, and consequently we get that

$$\begin{aligned}
&|(G_1 u)(t_2) - (G_1 v)(t_1)| \\
&\leq \sup_{(t,u)\in[0,T]\times B_r} \left| \int_0^{t_2} (t_2 - qs) f(s,u(s)) d_q s - \int_0^{t_1} (t_1 - qs) f(s,u(s)) d_q s \right| \\
&\leq \int_0^{t_1} (t_2 - t_1) |f(s,u(s))| d_q s + \int_{t_1}^{t_2} (t_2 - qs) |f(s,u(s))| d_q s \\
&\leq f_{\max} \left(\int_0^{t_1} (t_2 - t_1) d_q s + \int_{t_1}^{t_2} |t_2 - qs| d_q s \right).
\end{aligned}$$

Actually, as $t_2 \to t_1$, the right-hand side of the above inequality, which is independent of u, tends to zero. Therefore, G_1 is equicontinuous. Since G_1 is uniformly bounded and equicontinuous on B_r, we get that G_1 is relatively compact on B_r. Hence, by the Arzelá-Ascoli Theorem, G_1 is compact on B_r. Thus, all the assumptions of Theorem 3 are satisfied. By Theorem 4, we have that the boundary value problem (1)-(2) has at least one solution on $[0,T]$. This completes the proof.

Next, we prove the existence of solutions for the problem (1)-(2) by using Leray-Schauder degree theory.

Theorem 5. *Let $f : [0,T] \times R \to R$. Suppose that there exist constants $0 \leq \kappa < \Omega^{-1}$, where Ω is given by (15) and $M > 0$ such that $|f(t,u)| \leq \kappa|u| + M$ for all $t \in [0,T], u \in R$. Then the boundary value problem (1)-(2) has at least one solution on $[0,T]$.*

Proof. We define an operator $F : \mathcal{C} \to \mathcal{C}$ as in (14). In view of the fixed point problem

$$u = Fu. \tag{20}$$

We are going to prove the existence of at least one solution $u \in C[0,T]$ satisfying (20). Set a ball $B_R \subset C[0,T]$, as

$$B_R = \{u \in \mathcal{C} : \max_{t\in[0,T]} |u(t)| < R\},$$

where a constant radius $R > 0$. Hence, we will show that $F : \overline{B}_R \to C[0,T]$ satisfies a condition

$$u \neq \lambda F u, \qquad \forall u \in \partial B_R, \qquad \forall \lambda \in [0,1]. \tag{21}$$

We set

$$H(\lambda, u) = \lambda F u, \qquad u \in \mathcal{C}, \qquad \lambda \in [0,1].$$

As shown in Theorem 2 and 4, we have that the operator F that is continuous, uniformly bounded and equicontinuous. Then, by the Arzelá-Ascoli theorem, a continuous map h_λ defined by $h_\lambda(u) = u - H(\lambda, u) = u - \lambda F u$ is completely continuous. If (21) holds, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(h_\lambda, B_R, 0) &= \deg(I - \lambda F, B_R, 0) = \deg(h, b_R, 0)\\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \qquad 0 \in B_R,\end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, $h_1(u) = u - Fu = 0$ for at least one $u \in B_R$. Let us assume that $u = \lambda F u$ for some $\lambda \in [0,1]$ and for all $t \in [0,T]$ so that

$$\begin{aligned}|u(t)| &= |\lambda(Fu)(t)|\\ &\le \int_0^t (t-qs)|f(s,u(s))|d_qs\\ &\quad + \left|(1-\eta\sum_{i=1}^n \alpha_i)t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1+p_i}\right| \int_0^T \frac{(T-qs)}{|\Phi|}|f(s,u(s))|d_qs\\ &\quad + \left|(1-\eta\sum_{i=1}^n \alpha_i)t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1+p_i}\right|\\ &\quad \times \sum_{j=1}^m \frac{|\beta_j|}{|\Phi|} \int_0^\eta \int_0^s (s-q\nu)|f(\nu,u(\nu))|d_q\nu d_{r_j}s\\ &\quad + \left|(1-\eta\sum_{j=1}^m \beta_j)t - T + \eta^2 \sum_{j=1}^m \frac{\beta_j}{1+r_j}\right|\\ &\quad \times \sum_{i=1}^n \frac{|\alpha_i|}{|\Phi|} \int_0^\eta \int_0^s (s-q\nu)|f(\nu,u(\nu))|d_q\nu d_{p_i}s\\ &\le (\kappa|u| + M)\Bigg[\int_0^t (t-qs)d_qs\\ &\quad + \left|(1-\eta\sum_{i=1}^n \alpha_i)t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1+p_i}\right| \int_0^T \frac{(T-qs)}{|\Phi|}d_qs\\ &\quad + \left|(1-\eta\sum_{i=1}^n \alpha_i)t + \eta^2 \sum_{i=1}^n \frac{\alpha_i}{1+p_i}\right|\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{j=1}^{m} \frac{|\beta_j|}{|\Phi|} \int_0^{\eta} \int_0^{s} (s - q\nu) d_q\nu d_{r_j} s \\
&+ \left| (1 - \eta \sum_{j=1}^{m} \beta_j) t - T + \eta^2 \sum_{j=1}^{m} \frac{\beta_j}{1 + r_j} \right| \\
&\times \sum_{i=1}^{n} \frac{|\alpha_i|}{|\Phi|} \int_0^{\eta} \int_0^{s} (s - q\nu) d_q\nu d_{p_i} s \Bigg] \\
\leq\ & \frac{(\kappa|u| + M)}{1 + q} \Bigg[T^2 + \left((1 + \eta \sum_{i=1}^{n} |\alpha_i|) T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1 + p_i} \right) \frac{T^2}{|\Phi|} \\
&+ \left((1 + \eta \sum_{i=1}^{n} |\alpha_i|) T + \eta^2 \sum_{i=1}^{n} \frac{|\alpha_i|}{1 + p_i} \right) \sum_{j=1}^{m} \frac{|\beta_j| \eta^3}{|\Phi| (r_j^2 + r_j + 1)} \\
&+ \left(\eta^2 \sum_{j=1}^{m} \frac{|\beta_j|}{1 + r_j} + \eta \sum_{j=1}^{m} |\beta_j| \right) \sum_{i=1}^{n} \frac{|\alpha_i| \eta^3}{|\Phi| (p_i^2 + p_i + 1)} \Bigg] \\
=\ & (\kappa|u| + M)\Omega,
\end{aligned}
$$

which on taking norm $\sup_{t \in [0,T]} |u(t)| = \|u\|$ and solving for $\|u\|$, yields

$$\|u\| \leq \frac{M\Omega}{1 - \kappa\Omega}.$$

If $R = \frac{M\Omega}{1-\kappa\Omega} + 1$, inequality (21) holds. This completes the proof.

4. Some Examples

Example 6. Consider the following three-point multi-term q-integral boundary value problem

$$D_{\frac{1}{2}}^2 u(t) = \frac{1}{(t+4)^2} \cdot \frac{|u|}{|u|+1}, \quad t \in [0, \sqrt{2}]. \tag{22}$$

$$
\begin{aligned}
u(0) &= \frac{\sqrt{3}}{2} \int_0^{\frac{\sqrt{2}}{2}} u(s) d_{\frac{1}{4}} s - \int_0^{\frac{\sqrt{2}}{2}} u(s) d_{\frac{2}{3}} s, \\
u(\sqrt{2}) &= -\sqrt{7} \int_0^{\frac{\sqrt{2}}{2}} u(s) d_{\frac{1}{2}} s + \frac{\sqrt{\pi}}{2} \int_0^{\frac{\sqrt{2}}{2}} u(s) d_{\frac{3}{4}} s - \frac{3}{2} \int_0^{\frac{\sqrt{2}}{2}} u(s) d_{\frac{4}{5}} s.
\end{aligned} \tag{23}
$$

Set $q = 1/2$, $T = \sqrt{2}$, $\eta = \sqrt{2}/2$, $n = 2$, $\alpha_1 = \sqrt{3}/2$, $\alpha_2 = -1$, $p_1 = 1/4$, $p_2 = 2/3$, $m = 3$, $\beta_1 = -\sqrt{7}$, $\beta_2 = \sqrt{\pi}/2$, $\beta_3 = -3/2$,$r_1 = 1/2$, $r_2 = 3/4$, $r_3 = 4/5$, $f(t,u) = (1/(t+4)^2)(|u|/(1+|u|))$. Since $|f(t,u) - f(t,v)| \le (1/16)|u-v|$, (H_1) is satisfied with $\eta^2\Big(\sum_{j=1}^m \frac{\beta_j}{1+r_j} - \sum_{i=1}^n \frac{\alpha_i(1-\eta\sum_{j=1}^m \beta_j)}{1+p_i}\Big) - T \approx -4.595866$,

$$\begin{aligned}\Phi &= \Big(1 - \eta\sum_{i=1}^n \alpha_i\Big)\Big(\eta^2\sum_{j=1}^m \frac{\beta_j}{1+r_j} - T\Big) - \eta^2\sum_{i=1}^n \frac{\alpha_i(1-\eta\sum_{j=1}^m \beta_j)}{1+p_i}\\ &\approx -2.845975,\end{aligned}$$

$$\begin{aligned}\Omega &= T^2 + \left((1+\eta\sum_{i=1}^n |\alpha_i|)T + \eta^2\sum_{i=1}^n \frac{|\alpha_i|}{1+p_i}\right)\frac{T^2}{|\Phi|}\\ &\quad + \left((1+\eta\sum_{i=1}^n |\alpha_i|)T + \eta^2\sum_{i=1}^n \frac{|\alpha_i|}{1+p_i}\right)\sum_{j=1}^m \frac{|\beta_j|\eta^3}{|\Phi|(r_j^2+r_j+1)}\\ &\quad + \left(\eta^2\sum_{j=1}^m \frac{|\beta_j|}{1+r_j} + \eta\sum_{j=1}^m |\beta_j|\right)\sum_{i=1}^n \frac{|\alpha_i|\eta^3}{|\Phi|(p_i^2+p_i+1)}\\ &\approx 11.093692,\end{aligned}$$

$L = 1/16$. Hence $\Lambda := L\Omega \approx 0.693356 < 1$. Therefore, by Theorem 2, the boundary value problem (22)-(23) has a unique solution on $[0, 2]$.

Example 7. Consider the following three-point multi-term q-integral boundary value problem

$$D_{\frac{2}{3}}^2 u(t) = \frac{\sin(4\pi u)}{16\pi} + \frac{|u|}{|u|+1}, \quad t \in [0,1]. \tag{24}$$

$$\begin{aligned}u(0) &= \sqrt{3}\int_0^{\frac{1}{2}} u(s)d_{\frac{1}{2}}s - \frac{7}{2}\int_0^{\frac{1}{2}} u(s)d_{\frac{2}{3}}s - 5\int_0^{\frac{1}{2}} u(s)d_{\frac{3}{4}}s\\ u(1) &= \frac{1}{4}\int_0^{\frac{1}{2}} u(s)d_{\frac{1}{4}}s - \frac{1}{5}\int_0^{\frac{1}{2}} u(s)d_{\frac{1}{3}}s - \frac{1}{2}\int_0^{\frac{1}{2}} u(s)d_{\frac{1}{5}}s.\end{aligned} \tag{25}$$

Set $q = 2/3$, $T = 1$, $\eta = 1/2$, $n = 3$, $\alpha_1 = \sqrt{3}$, $\alpha_2 = -7/2$, $\alpha_3 = -5$, $p_1 = 1/2$, $p_2 = 2/3$, $p_3 = 3/4$, $m = 3$, $\beta_1 = -1/5$, $\beta_2 = 1/4$, $\beta_3 = -1/2$,$r_1 = 1/3$, $r_2 = 1/4$, $r_3 = 1/5$, Here, $|f(t,u)| = |\sin(4\pi u)/16\pi + |u|/(1+|u|)| \le (|u|/4)+1$. So, $M = 1$, $\eta^2\Big(\sum_{j=1}^m \frac{\beta_j}{1+r_j} - \sum_{i=1}^n \frac{\alpha_i(1-\eta\sum_{j=1}^m \beta_j)}{1+p_i}\Big) - T \approx 0.072831$, and

$$\Phi = \Big(1 - \eta\sum_{i=1}^n \alpha_i\Big)\Big(\eta^2\sum_{j=1}^m \frac{\beta_j}{1+r_j} - T\Big) - \eta^2\sum_{i=1}^n \frac{\alpha_i(1-\eta\sum_{j=1}^m \beta_j)}{1+p_i}$$

$$\approx -3.621341,$$

$$\begin{aligned}\Omega &= T^2 + \left((1+\eta\sum_{i=1}^{n}|\alpha_i|)T + \eta^2\sum_{i=1}^{n}\frac{|\alpha_i|}{1+p_i}\right)\frac{T^2}{|\Phi|} \\ &\quad + \left((1+\eta\sum_{i=1}^{n}|\alpha_i|)T + \eta^2\sum_{i=1}^{n}\frac{|\alpha_i|}{1+p_i}\right)\sum_{j=1}^{m}\frac{|\beta_j|\eta^3}{|\Phi|(r_j^2+r_j+1)} \\ &\quad + \left(\eta^2\sum_{j=1}^{m}\frac{|\beta_j|}{1+r_j} + \eta\sum_{j=1}^{m}|\beta_j|\right)\sum_{i=1}^{n}\frac{|\alpha_i|\eta^3}{|\Phi|(p_i^2+p_i+1)} \\ &\approx 3.970666, \\ \kappa &= \frac{1}{4} < \frac{1}{\Omega} \approx 0.251847.\end{aligned}$$

Hence, by Theorem 5, the boundary value problem (24)-(25) has at least one solution on $[0,1]$.

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