σ-STRUCTURES AND QUASI-ENLARGING OPERATIONS

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Abstract: The purpose of this paper is to introduce the notion of σ-structures and to investigate some properties for the structures. In particular, we investigate enlarging and quasi-enlarging operations induced by σ-structures.

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1. Introduction

Let X be a non-empty set with the power set 2^X. A function γ: 2^X → 2^X is said to be monotonic [2] iff A ⊆ B ⊆ X implies γA ⊆ γB. The collection of all monotonic functions is denoted by Γ(X) and the elements of Γ(X) are said to be operations. If γ, φ ∈ Γ(X) and A ⊆ X, we write γφ instead of γ ◦ φ. If γ ∈ Γ(X) then a set A ⊆ X is said to be γ-open [2] iff A ⊆ γA.

Let X be a nonempty set and μ be a collection of subsets of X. Then μ is called a generalized topology (briefly GT) on X iff ∅ ∈ μ and G_i ∈ μ for i ∈ I ≠ ∅ implies G = ∪_{i ∈ I} G_i ∈ μ. The elements of μ are called μ-open sets and the complements are called μ-closed sets.

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For $A \subseteq X$, we denote by $i_\mu A$ the union of all $\mu$-open sets contained in $A$, i.e. the largest $\mu$-open set contained in $A$. The complement of a $\mu$-open set is said to be $\mu$-closed. Any intersection of $\mu$-closed sets is $\mu$-closed, and for $A \subseteq X$, we denote by $c_\mu A$ the intersection of all $\mu$-closed sets containing $A$, i.e. the smallest $\mu$-closed set containing $A$. The purpose of this paper is to introduce the notion of $\sigma$-structures, which are extended notions of topology and generalized topology, and to investigate some properties for the structures. In particular, we investigate enlarging and quasi-enlarging operations ($\gamma i_s, i_s \gamma i_s, i_s \gamma, \gamma i_s \gamma$) induced by $\sigma$-structures.

2. $\sigma$-Structures

**Definition 2.1.** Let $X$ be a nonempty set and $s \subseteq 2^X$. Then $s$ is called a $\sigma$-structure on $X$ if for $i \in I \neq \emptyset$, $U_i \in s$ implies $\bigcup_{i \in I} U_i \in s$. The elements of $s$ are called $\sigma$-open sets and the complements are called $\sigma$-closed sets.

If $s$ is a topology (or generalized topology), then it is obviously $\sigma$-structure. In [3], Császár introduced the notion of weak structure as the following: A family $w \in 2^X$ is called a weak structure (briefly WS) on $X$ if $\emptyset \in w$. So we can get the next result.

**Lemma 2.2.** Let $X$ be a nonempty set. Then a family $s \subseteq 2^X$ is a GT on $X$ iff $s$ is a WS and a $\sigma$-structure on $X$.

**Definition 2.3.** Let $X$ be a nonempty set, and let $s$ be a $\sigma$-structure on $X$. Then the two operators $i_s$ and $c_s$ are defined as the following:

- $i_sA = \bigcup\{S \subseteq X : S \subseteq A, \ S \ \text{is} \ \sigma\text{-open}\};$
- $c_sA = \bigcap\{F \subseteq X : A \subseteq F, \ F \ \text{is} \ \sigma\text{-closed}\}.$

**Theorem 2.4.** Let $s$ be a $\sigma$-structure on a nonempty set $X$ and $A, B \subseteq X$. Then:

1. $i_s \emptyset = \emptyset$;
2. $i_s A \subseteq A$;
3. if $A \subseteq B$, then $i_s A \subseteq i_s B$;
4. $i_s i_s A = i_s A$;
5. $A$ is $\sigma$-open iff $A = i_s A$ for $A \neq \emptyset$.

**Theorem 2.5.** Let $s$ be a $\sigma$-structure on a nonempty set $X$ and $A, B \subseteq X$. Then:
\(\sigma\)-STRUCTURES AND QUASI-ENLARGING OPERATIONS

(1) \(c_\sigma X = X\);
(2) \(A \subseteq c_\sigma A\);
(3) if \(A \subseteq B\), then \(c_\sigma A \subseteq c_\sigma B\);
(4) \(c_\sigma c_\sigma A = c_\sigma A\);
(5) \(A\) is \(\sigma\)-closed iff \(A = c_\sigma A\) for \(A \neq X\).

**Example 2.6.** Let \(X = \{a, b, c\}\) and a \(\sigma\)-structure \(s = \{\{a\}, \{b\}, \{a, b\}\}\). Then \(i_s X = \{a, b\} \neq X\) and \(c_\sigma \emptyset = \{c\} \neq \emptyset\).

**Theorem 2.7.** Let \(s\) be a \(\sigma\)-structure on a nonempty set \(X\) and \(A \subseteq X\).

Then:

(1) The collection \(\mu = \{A \subseteq X : i_s A = A\}\) is a generalized topology on \(X\).
(2) \(x \in i_s A\) iff there exists a \(\sigma\)-open set \(S\) containing \(x\) such that \(S \subseteq A\).
(3) \(x \in c_\sigma A\) iff \(S \cap A \neq \emptyset\) for every \(\sigma\)-open set \(S\) containing \(x\).
(4) \(c_\sigma (A) = X - i_s (X - A)\).
(5) \(i_s (A) = X - c_\sigma (X - A)\).

**Proof.** (1) By Theorem 2.4, \(\emptyset \in \mu\) and every element in \(\mu\) is \(\sigma\)-open, and so \(\mu\) is a generalized topology.

(2) Obvious.

(3) For \(A \subseteq X\), \(x \notin c_\sigma A\) iff there exists a \(\sigma\)-closed set \(F\) such that \(A \subseteq F\) and \(x \notin F\) iff for the \(\sigma\)-open set \(G = X - F\) containing \(x\), \(G \cap A = \emptyset\) iff there exists a \(\sigma\)-open set \(S\) containing \(x\), \(S \cap A = \emptyset\).

(4) By (3), \(x \notin c_\sigma A\) iff there exists a \(\sigma\)-open set \(S\) containing \(x\), \(S \cap A = \emptyset\) iff \(x \in i_s (X - A)\).

Similarly, we can show the statement (5).

**Definition 2.8.** Let \(s, s'\) be \(\sigma\)-structures on \(X\) and \(Y\), respectively. Then a function \(f : X \rightarrow Y\) is said to be

(1) \(\sigma\)-continuous if \(f^{-1} (G) \in s\) for every \(G \in s'\);
(2) \(\sigma\)-open if \(f(S) \in s'\) for every \(S \in s\).

**Theorem 2.9.** Let \(f : (X, s) \rightarrow (Y, s')\) be a function, let \(s\) and \(s'\) be \(\sigma\)-structures on \(X, Y\), respectively. Then

(1) \(f\) is \(\sigma\)-continuous iff \(f^{-1} (i'_{s'} A) \subseteq i_s f^{-1} (A)\) for every \(A \subseteq Y\).
(2) \(f\) is \(\sigma\)-open iff \(f(i_s B) \subseteq i'_s f (B)\) for every \(B \subseteq X\).
Proof. (1) Let \( f \) be \( \sigma \)-continuous. For \( A \subseteq Y \), since \( i'_s A \) is \( \sigma \)-open, \( f^{-1}(i'_s A) = i_s f^{-1}(i'_s A) \subseteq i_s f^{-1}(A) \).

For the converse, let \( G \in s' \); then \( i'_s G = G \) and from hypothesis, it follows \( f^{-1}(G) = f^{-1}(i'_s G) \subseteq i_s f^{-1}(G) \). So by Theorem 2.4, \( f^{-1}(G) \) is \( \sigma \)-open.

(2) It is similar to the proof of (1). \( \square \)

Let \( X \) be a nonempty set and \( s \) be a \( \sigma \)-structure. Then \( s \) is called a strong \( \sigma \)-structure on \( X \) if \( X \in s \).

Obviously we have the following theorem:

**Theorem 2.10.** Let \( f : (X, s) \to (Y, s') \) be a function, let \( s \) and \( s' \) be \( \sigma \)-structures on \( X, Y \), respectively. Then:

(1) if \( f \) is \( \sigma \)-continuous and \( s' \) is strong, then \( s \) is strong;

(2) if \( f \) is \( \sigma \)-open, surjective and \( s \) is strong, then \( s' \) is strong.

### 3. Enlarging and Quasi-Enlarging Operations on \( \sigma \)-Structures

Let \( X \) be a non-empty set and \( \gamma \in \Gamma(X) \) (see [4]).

(1) \( \gamma \) is said to be **enlarging** if \( A \subseteq \gamma A \) for \( A \subseteq X \).

(2) \( \gamma \) is said to be **quasi-enlarging** if \( \gamma A \subseteq \gamma(A \cap \gamma A) \) for \( A \subseteq X \).

(3) \( \gamma \) is said to be **weakly-quasi-enlarging** if \( A \cap \gamma A \subseteq \gamma(A \cap \gamma A) \) for \( A \subseteq X \).

(4) For \( \mu \subseteq 2^X \), \( \gamma \) is said to be **\( \mu \)-enlarging** if \( L \subseteq \gamma L \) for all \( L \in \mu \).

Clearly every enlarging operation is quasi-enlarging. For a \( \sigma \)-structure \( s \), since the operation \( c_s \) is enlarging, it is also quasi-enlarging.

**Lemma 3.1.** Let \( s \) be a \( \sigma \)-structure on a nonempty set \( X \). Then the operation \( i_s \) is quasi-enlarging.

**Proof.** For \( A \subseteq X \), since \( i_s A \subseteq A \cap i_s A \), from (3) and (4) of Theorem 2.4, \( i_s A = i_s i_s A \subseteq i_s (A \cap i_s A) \). So \( i_s \) is quasi-enlarging. \( \square \)

**Corollary 3.2.** (see Proposition 1.6 of [4]) Let \( s \) be a GT on a nonempty set \( X \). Then the operation \( i_s \) is quasi-enlarging.

Henceforth, we deal with the special operations \( \gamma i_s, i_s \gamma i_s, i_s \gamma, \gamma i_s \gamma \).
Lemma 3.3. Let $s$ be a $\sigma$-structure on a nonempty set $X$, and let $\gamma$ be $s$-enlarging. Then for $A \in s$,

1. $A \subseteq \gamma i_s A$;
2. $A \subseteq i_s \gamma i_s A$;
3. $A \subseteq \gamma i_s \gamma A$;
4. $A \subseteq i_s \gamma A$.

Proof. (1) For $A \in s$, $A = i_s A \in s$. Since $\gamma$ is an $s$-enlargement of $s$, $A \subseteq \gamma A = \gamma i_s A$.

(2) For $A \in s$, by (1), $A \subseteq \gamma i_s A$ and so it implies $A = i_s A \subseteq i_s \gamma i_s A$.

(3) For $A \in s$, by (1) and $A \subseteq \gamma A$, $A \subseteq \gamma i_s A \subseteq \gamma i_s \gamma A$. So $A \subseteq \gamma i_s \gamma A$.

(4) For $A \in s$, from (2) and $i_s A = A$, $A \subseteq i_s \gamma i_s A \subseteq i_s \gamma A$.

By Lemma 3.3, we have the following theorem.

Theorem 3.4. Let $s$ be a $\sigma$-structure on a nonempty set $X$, and let $\gamma$ be $s$-enlarging. Let an operation $\lambda : 2^X \to 2^X$ be defined as $\lambda(A) = \gamma i_s A \ (\gamma i_s \gamma A, i_s \gamma i_s A, i_s \gamma A)$ for $A \in 2^X$. Then $\lambda$ is $s$-enlarging.

Corollary 3.5. Let $s$ be a $\sigma$-structure on a nonempty set $X$. Then for $A \in s$,

1. $A \subseteq c_s i_s A$;
2. $A \subseteq i_s c_s i_s A$;
3. $A \subseteq c_s i_s c_s A$;
4. $A \subseteq i_s c_s A$.

Proof. Since $c_s$ is an $s$-enlarging operation, by Lemma 3.3, the things are obtained.

Theorem 3.6. Let $s$ be a $\sigma$-structure on a non empty set $X$, and let $\gamma$ be $s$-enlarging. Then the operation $\gamma i_s$ is quasi-enlarging.

Proof. For $A \subseteq X$, since $i_s$ is quasi-enlarging, $i_s A \subseteq i_s(A \cap i_s A) \subseteq i_s(A \cap \gamma i_s A)$. From the monotonicity of $\gamma$, it follows $\gamma i_s A \subseteq \gamma i_s(A \cap \gamma i_s A)$. So $\gamma i_s$ is quasi-enlarging.

Corollary 3.7. (see Corollary 1.8 of [4]) Let $s$ be a GT on a nonempty set $X$, and let $\gamma$ be $s$-enlarging. Then the operation $\gamma i_s$ is quasi-enlarging.
Theorem 3.8. Let \( s \) be a \( \sigma \)-structure on a nonempty set \( X \), and let \( \gamma \) be \( s \)-enlarging. Then the operation \( i_s\gamma i_s \) is quasi-enlarging.

Proof. For \( A \subseteq X \), from Theorem 2.4 (4) and Theorem 3.6, it follows
\[
\gamma i_s A \subseteq \gamma i_s(A \cap \gamma i_s A)
\]
\[
= \gamma i_s i_s(A \cap \gamma i_s A)
\]
\[
\subseteq \gamma i_s(i_s A \cap i_s \gamma i_s A)
\]
\[
\subseteq \gamma i_s(A \cap i_s \gamma i_s A)
\]
This implies \( i_s \gamma i_s A \subseteq i_s \gamma i_s(A \cap i_s \gamma i_s A) \). Hence \( i_s \gamma i_s \) is quasi-enlarging. \( \Box \)

Corollary 3.9. Let \( s \) be a GT on a nonempty set \( X \), and let \( \gamma \) be \( s \)-enlarging. Then the operation \( i_s \gamma i_s \) is quasi-enlarging.

Theorem 3.10. Let \( s \) be a \( \sigma \)-structure on a nonempty set \( X \), and let \( \gamma \) be \( s \)-enlarging. If \( \gamma(A \cap B) = \gamma A \cap \gamma B \) for \( A, B \subseteq X \), then the operation \( \gamma i_s \gamma \) is quasi-enlarging.

Proof. For \( A \subseteq X \), since \( \gamma A \subseteq X \) and \( i_s \gamma A \in s \), from Theorem 2.4 (4) and Theorem 3.6, it follows
\[
\gamma i_s \gamma A \subseteq \gamma i_s(\gamma A \cap \gamma i_s \gamma A)
\]
\[
\subseteq \gamma i_s(\gamma A \cap \gamma i_s \gamma A)
\]
\[
= \gamma i_s \gamma(A \cap \gamma i_s \gamma A)
\]
Hence \( \gamma i_s \gamma \) is quasi-enlarging. \( \Box \)

Corollary 3.11. Let \( s \) be a GT on \( X \), and let \( \gamma \) be \( s \)-enlarging. If \( \gamma(A \cap B) = \gamma A \cap \gamma B \) for \( A, B \subseteq X \), then the operation \( \gamma i_s \gamma \) is quasi-enlarging.

Let \( X \) be a non-empty set and \( \gamma \in \Gamma(X) \). Then for \( \mu \in 2^X \), \( \gamma \) is said to be \( \mu \)-friendly [4] if \( \gamma A \cap L \subseteq \gamma(A \cap L) \) for \( A \subseteq X \), \( L \in \mu \).

Theorem 3.12. Let \( s \) be a \( \sigma \)-structure on a nonempty set \( X \), and let \( \gamma \) be \( s \)-friendly. Then the operation \( i_s \gamma \) is quasi-enlarging.

Proof. For \( A \subseteq X \), \( i_s \gamma A = \gamma A \cap i_s \gamma A \). Since \( i_s \gamma A \in s \) and \( \gamma \) is \( s \)-friendly, \( i_s \gamma A = \gamma A \cap i_s \gamma A \subseteq \gamma(A \cap i_s \gamma A) \). This implies \( i_s i_s \gamma A \subseteq i_s \gamma(A \cap i_s \gamma A) \). Finally, from \( i_s i_s \gamma A = i_s \gamma A \), we have the result that \( i_s \gamma \) is quasi-enlarging. \( \Box \)
Corollary 3.13. Let \( s \) be a \( \sigma \)-structure on a non empty set \( X \), and let \( \gamma \) be \( s \)-friendly. Then the operation \( \gamma i_s \gamma \) is quasi-enlarging.

Proof. For \( A \subseteq X \), from Theorem 3.12, it follows \( i_s \gamma A \subseteq i_s \gamma (A \cap i_s \gamma A) \subseteq i_s \gamma (A \cap \gamma i_s \gamma A) \). So \( \gamma i_s \gamma A \subseteq \gamma i_s \gamma (A \cap \gamma i_s \gamma A) \). \( \square \)

Corollary 3.14. Let \( s \) be a GT on \( X \), and let \( \gamma \) be \( s \)-friendly. Then the operations \( i_s \gamma, \gamma i_s \gamma \) are quasi-enlarging.

Lemma 3.15. Let \( s \) be a \( \sigma \)-structure on a non empty set \( X \), and let \( \gamma \) be \( s \)-enlarging and \( A \subseteq X \).

1. \( A \subseteq i_s \gamma i_s A \subseteq \gamma i_s A \subseteq \gamma A \);
2. \( A \subseteq i_s \gamma i_s A \subseteq \gamma i_s A \subseteq i_s \gamma A \subseteq \gamma A \);
3. \( A \subseteq i_s \gamma i_s A \subseteq i_s \gamma A \subseteq \gamma i_s \gamma A \subseteq \gamma A \);
4. \( A \subseteq i_s \gamma i_s A \subseteq i_s \gamma A \subseteq \gamma i_s \gamma A \subseteq \gamma A \).

Proof. (1) Since \( i_s A \in s \), it is obvious.
(2) By (1), \( A \subseteq \gamma A \) and \( \gamma i_s A \subseteq \gamma i_s \gamma A \). So we have (2).
(3) and (4) are obvious. \( \square \)

Corollary 3.16. Let \( s \) be a \( \sigma \)-structure on a non empty set \( X \), and let \( \gamma \) be \( s \)-enlarging and \( A \subseteq X \). If \( \gamma \gamma A = \gamma A \), then:

1. \( A \subseteq i_s \gamma i_s A \subseteq \gamma i_s A \subseteq i_s \gamma A \subseteq \gamma A \);
2. \( A \subseteq i_s \gamma i_s A \subseteq i_s \gamma A \subseteq \gamma i_s \gamma A \subseteq \gamma A \).

Let us consider the \( s \)-enlarging operations \( \lambda \) defined in Theorem 3.4. For \( A \subseteq X \), we will call \( A \) a \( \gamma_\sigma \)-semiopen (resp., \( \gamma_\sigma \)-open, \( \gamma_\sigma \)-preopen, \( \gamma_\sigma \)-beta-open) set if \( A \) is \( \lambda \)-open, that is, \( A \subseteq i_s A \) (resp., \( A \subseteq i_s \gamma i_s A, A \subseteq i_s \gamma A, A \subseteq i_s i_s \gamma A \)). We denote \( S(\gamma_\sigma) \) (resp., \( \alpha(\gamma_\sigma), P(\gamma_\sigma), \beta(\gamma_\sigma) \)) the set of all \( \gamma_\sigma \)-semiopen (resp., \( \gamma_\sigma \)-open, \( \gamma_\sigma \)-preopen, \( \gamma_\sigma \)-beta-open) sets in \( X \).

If the \( \sigma \)-structure \( s \) is a topology on \( X \) and the \( s \)-enlarging operation \( \gamma = c_s \), then the \( \gamma_\sigma \)-semiopen (resp., \( \gamma_\sigma \)-open, \( \gamma_\sigma \)-preopen, \( \gamma_\sigma \)-beta-open) set is the semiopen [6] (resp., \( \alpha \)-open [8], preopen [7], beta-open [1]). Furthermore, if the \( \sigma \)-structure \( s \) is a generalized topology on \( X \) and the \( s \)-enlarging operation \( \gamma = c_s \), then the \( \gamma_\sigma \)-semiopen (resp., \( \gamma_\sigma \)-open, \( \gamma_\sigma \)-preopen, \( \gamma_\sigma \)-beta-open) set is the \( s \)-semiopen (resp., \( s \)-alpha-open, \( s \)-preopen, \( s \)-beta-open) [3]. So we can the following diagrams:
Theorem 3.17. Let $s$ be a $\sigma$-structure on a non empty set $X$, and let $\gamma$ be $s$-enlarging and $A \subseteq X$. If $\gamma \gamma = \gamma$, then $\alpha(\gamma \sigma) = S(\gamma \sigma) \cap P(\gamma \sigma)$.

Proof. It is sufficient to show that $S(\gamma \sigma) \cap P(\gamma \sigma) \subseteq \alpha(\gamma \sigma)$. For $A \in S(\gamma \sigma) \cap P(\gamma \sigma)$, $A \subseteq i_s \gamma A \subseteq i_s \gamma i_s A = i_s \gamma i_s A$. So $A \in \alpha(\gamma \sigma)$. \qed

References


