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σ-STRUCTURES AND QUASI-ENLARGING OPERATIONS

Young Key Kim[1], Won Keun Min[2] §

[1]Department of Mathematics
MyongJi University
Youngin, 449-728, KOREA
[2]Department of Mathematics
Kangwon National University
Chuncheon, 200-701, KOREA

Abstract: The purpose of this paper is to introduce the notion of σ-structures and to investigate some properties for the structures. In particular, we investigate enlarging and quasi-enlarging operations induced by σ-structures.

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1. Introduction

Let X be a non-empty set with the power set 2^X. A function $\gamma : 2^X \to 2^X$ is said to be *monotonic* [2] iff $A \subseteq B \subseteq X$ implies $\gamma A \subseteq \gamma B$. The collection of all monotonic functions is denoted by $\Gamma(X)$ and the elements of $\Gamma(X)$ are said to be *operations*. If $\gamma, \phi \in \Gamma(X)$ and $A \subseteq X$, we write $\gamma\phi$ instead of $\gamma \circ \phi$. If $\gamma \in \Gamma(X)$ then a set $A \subseteq X$ is said to be γ-*open* [2] iff $A \subseteq \gamma A$.

Let X be a nonempty set and μ be a collection of subsets of X. Then μ is called a *generalized topology* (briefly GT) on X iff $\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in \mu$. The elements of μ are called μ-*open* sets and the complements are called μ-*closed* sets.

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§Correspondence author

For $A \subseteq X$, we denote by $i_\mu A$ the union of all μ-open sets contained in A, i.e. the largest μ-open set contained in A. The complement of a μ-open set is said to be *μ-closed*. Any intersection of μ-closed sets is μ-closed, and for $A \subseteq X$, we denote by $c_\mu A$ the intersection of all μ-closed sets containing A, i.e. the smallest μ-closed set containing A. The purpose of this paper is to introduce the notion of σ-structures, which are extended notions of topology and generalized topology , and to investigate some properties for the structures. In particular, we investigate enlarging and quasi-enlarging operations $(\gamma i_s, i_s\gamma i_s, i_s\gamma,\ \gamma i_s\gamma)$ induced by σ-structures.

2. σ-Structures

Definition 2.1. Let X be a nonempty set and $s \subseteq 2^X$. Then s is called a *σ-structure* on X if for $i \in I \neq \emptyset$, $U_i \in s$ implies $\cup_{i\in I} U_i \in s$. The elements of s are called *σ-open* sets and the complements are called *σ-closed* sets.

If s is a topology (or generalized topology), then it is obviously σ-structure. In [3], Császár introduced the notion of weak structure as the following: A family $w \in 2^X$ is called a *weak structure* (briefly WS) on X if $\emptyset \in w$. So we can get the next result.

Lemma 2.2. *Let X be a nonempty set. Then a family $s \subseteq 2^X$ is a GT on X iff s is a WS and a σ-structure on X.*

Definition 2.3. Let X be a nonempty set, and let s be a σ-structure on X. Then the two operators i_s and c_s are defined as the following:
$i_s A = \cup\{S \subseteq X : S \subseteq A,\quad S \text{ is } \sigma\text{-open }\}$;
$c_s A = \cap\{F \subseteq X : A \subseteq F,\quad F \text{ is } \sigma\text{-closed }\}$.

Theorem 2.4. *Let s be a σ-structure on a nonempty set X and $A, B \subseteq X$. Then:*

(1) $i_s\emptyset = \emptyset$;

(2) $i_s A \subseteq A$;

(3) if $A \subseteq B$, then $i_s A \subseteq i_s B$;

(4) $i_s i_s A = i_s A$;

(5) A is σ-open iff $A = i_s A$ for $A \neq \emptyset$.

Theorem 2.5. *Let s be a σ-structure on a nonempty set X and $A, B \subseteq X$. Then:*

(1) $c_s X = X$*;*

(2) $A \subseteq c_s A$*;*

(3) if $A \subseteq B$*, then* $c_s A \subseteq c_s B$*;*

(4) $c_s c_s A = c_s A$*;*

(5) A *is* σ*-closed iff* $A = c_s A$ *for* $A \neq X$*.*

Example 2.6. Let $X = \{a, b, c\}$ and a σ-structure $s = \{\{a\}, \{b\}, \{a, b\}\}$. Then $i_s X = \{a, b\} \neq X$ and $c_s \emptyset = \{c\} \neq \emptyset$.

Theorem 2.7. *Let* s *be a* σ*-structure on a nonempty set* X *and* $A \subseteq X$*. Then:*

(1) The collection $\mu = \{A \subseteq X : i_s A = A\}$ *is a generalized topology on* X*.*

(2) $x \in i_s A$ *iff there exists a* σ*-open set* S *containing* x *such that* $S \subseteq A$*.*

(3) $x \in c_s A$ *iff* $S \cap A \neq \emptyset$ *for every* σ*-open set* S *containing* x*.*

(4) $c_s(A) = X - i_s(X - A)$*.*

(5) $i_s(A) = X - c_s(X - A)$*.*

Proof. (1) By Theorem 2.4, $\emptyset \in \mu$ and every element in μ is σ-open, and so μ is a generalized topology.

(2) Obvious.

(3) For $A \subseteq X$, $x \notin c_s A$ iff there exists a σ-closed set F such that $A \subseteq F$ and $x \notin F$ iff for the σ-open set $G = X - F$ containing x, $G \cap A = \emptyset$ iff there exists a σ-open set S containing x, $S \cap A = \emptyset$.

(4) By (3), $x \notin c_s A$ iff there exists a σ-open set S containing x, $S \cap A = \emptyset$ iff $x \in i_s(X - A)$.

Similarly, we can show the statement (5). $\square$

Definition 2.8. Let s, s' be σ-structures on X and Y, respectively. Then a function $f : X \to Y$ is said to be

(1) *σ-continuous* if $f^{-1}(G) \in s$ for every $G \in s'$;

(2) *σ-open* if $f(S) \in s'$ for every $S \in s$.

Theorem 2.9. *Let* $f : (X, s) \to (Y, s')$ *be a function, let* s *and* s' *be* σ*-structures on* X, Y*, respectively. Then*

(1) f *is* σ*-continuous iff* $f^{-1}(i'_s A) \subseteq i_s f^{-1}(A)$ *for every* $A \subseteq Y$*.*

(2) f *is* σ*-open iff* $f(i_s B) \subseteq i'_s f(B)$ *for every* $B \subseteq X$*.*

Proof. (1) Let f be σ-continuous. For $A \subseteq Y$, since $i'_s A$ is σ-open, $f^{-1}(i'_s A) = i_s f^{-1}(i'_s A) \subseteq i_s f^{-1}(A)$.

For the converse, let $G \in s'$; then $i'_s G = G$ and from hypothesis, it follows $f^{-1}(G) = f^{-1}(i'_s G) \subseteq i_s f^{-1}(G)$. So by Theorem 2.4, $f^{-1}(G)$ is σ-open.

(2) It is similar to the proof of (1). □

Let X be a nonempty set and s be a σ-structure. Then s is called a *strong σ-structure* on X if $X \in s$.

Obviously we have the following theorem:

Theorem 2.10. *Let $f : (X, s) \to (Y, s')$ be a function, let s and s' be σ-structures on X, Y, respectively. Then:*

(1) if f is σ-continuous and s' is strong, then s is strong;

(2) if f is σ-open, surjective and s is strong, then s' is strong.

3. Enlarging and Quasi-Enlarging Operations on σ-Structures

Let X be a non-empty set and $\gamma \in \Gamma(X)$ (see [4]).

(1) γ is said to be *enlarging* if $A \subseteq \gamma A$ for $A \subseteq X$.

(2) γ is said to be *quasi-enlarging* if $\gamma A \subseteq \gamma(A \cap \gamma A)$ for $A \subseteq X$.

(3) γ is said to be *weakly-quasi-enlarging* if $A \cap \gamma A \subseteq \gamma(A \cap \gamma A)$ for $A \subseteq X$.

(4) For $\mu \subseteq 2^X$, γ is said to be *μ-enlarging* if $L \subseteq \gamma L$ for all $L \in \mu$.

Clearly every enlarging operation is quasi-enlarging. For a σ-structure s, since the operation c_s is enlarging, it is also quasi-enlarging.

Lemma 3.1. *Let s be a σ-structure on a nonempty set X. Then the operation i_s is quasi-enlarging.*

Proof. For $A \subseteq X$, since $i_s A \subseteq A \cap i_s A$, from (3) and (4) of Theorem 2.4, $i_s A = i_s i_s A \subseteq i_s(A \cap i_s A)$. So i_s is quasi-enlarging. □

Corollary 3.2. (see Proposition 1.6 of [4]) *Let s be a GT on a nonempty set X. Then the operation i_s is quasi-enlarging.*

Henceforth, we deal with the special operations γi_s, $i_s \gamma i_s$, $i_s \gamma$, $\gamma i_s \gamma$.

Lemma 3.3. *Let s be a σ-structure on a nonempty set X, and let γ be s-enlarging. Then for $A \in s$,*

(1) $A \subseteq \gamma i_s A$;

(2) $A \subseteq i_s \gamma i_s A$;

(3) $A \subseteq \gamma i_s \gamma A$;

(4) $A \subseteq i_s \gamma A$.

Proof. (1) For $A \in s$, $A = i_s A \in s$. Since γ is an s-enlargement of s, $A \subseteq \gamma A = \gamma i_s A$.

(2) For $A \in s$, by (1), $A \subseteq \gamma i_s A$ and so it implies $A = i_s A \subseteq i_s \gamma i_s A$.

(3) For $A \in s$, by (1) and $A \subseteq \gamma A$, $A \subseteq \gamma i_s A \subseteq \gamma i_s \gamma A$. So $A \subseteq \gamma i_s \gamma A$.

(4) For $A \in s$, from (2) and $i_s A = A$, $A \subseteq i_s \gamma i_s A \subseteq i_s \gamma A$. □

By Lemma 3.3, we have the following theorem.

Theorem 3.4. *Let s be a σ-structure on a nonempty set X, and let γ be s-enlarging. Let an operation $\lambda : 2^X \to 2^X$ be defined as $\lambda(A) = \gamma i_s A$ ($\gamma i_s \gamma A$, $i_s \gamma i_s A$, $i_s \gamma A$) for $A \in 2^X$. Then λ is s-enlarging.*

Corollary 3.5. *Let s be a σ-structure on a nonempty set X. Then for $A \in s$,*

(1) $A \subseteq c_s i_s A$;

(2) $A \subseteq i_s c_s i_s A$;

(3) $A \subseteq c_s i_s c_s A$;

(4) $A \subseteq i_s c_s A$.

Proof. Since c_s is an s-enlarging operation, by Lemma 3.3, the things are obtained. □

Theorem 3.6. *Let s be a σ-structure on a non empty set X, and let γ be s-enlarging. Then the operation γi_s is quasi-enlarging.*

Proof. For $A \subseteq X$, since i_s is quasi-enlarging, $i_s A \subseteq i_s(A \cap i_s A) \subseteq i_s(A \cap \gamma i_s A)$. From the monotonicity of γ, it follows $\gamma i_s A \subseteq \gamma i_s(A \cap \gamma i_s A)$. So γi_s is quasi-enlarging. □

Corollary 3.7. (see Corollary 1.8 of [4]) *Let s be a GT on a nonempty set X, and let γ be s-enlarging. Then the operation γi_s is quasi-enlarging.*

Theorem 3.8. *Let s be a σ-structure on a non empty set X, and let γ be s-enlarging. Then the operation $i_s\gamma i_s$ is quasi-enlarging.*

Proof. For $A \subseteq X$, from Theorem 2.4 (4) and Theorem 3.6, it follows

$$\begin{aligned} \gamma i_s A &\subseteq \gamma i_s(A \cap \gamma i_s A) \\ &= \gamma i_s i_s(A \cap \gamma i_s A) \\ &\subseteq \gamma i_s(i_s A \cap i_s\gamma i_s A) \\ &\subseteq \gamma i_s(A \cap i_s\gamma i_s A) \end{aligned}$$

This implies $i_s\gamma i_s A \subseteq i_s\gamma i_s(A \cap i_s\gamma i_s A)$. Hence $i_s\gamma i_s$ is quasi-enlarging. □

Corollary 3.9. *Let s be a GT on a nonempty set X, and let γ be s-enlarging. Then the operation $i_s\gamma i_s$ is quasi-enlarging.*

Theorem 3.10. *Let s be a σ-structure on a nonempty set X, and let γ be s-enlarging. If $\gamma(A \cap B) = \gamma A \cap \gamma B$ for $A, B \subseteq X$, then the operation $\gamma i_s\gamma$ is quasi-enlarging.*

Proof. For $A \subseteq X$, since $\gamma A \subseteq X$ and $i_s\gamma A \in s$, from Theorem 2.4 (4) and Theorem 3.6, it follows

$$\begin{aligned} \gamma i_s\gamma A &\subseteq \gamma i_s(\gamma A \cap \gamma i_s\gamma A) \\ &\subseteq \gamma i_s(\gamma A \cap \gamma\gamma i_s\gamma A) \\ &= \gamma i_s\gamma(A \cap \gamma i_s\gamma A) \end{aligned}$$

Hence $\gamma i_s\gamma$ is quasi-enlarging. □

Corollary 3.11. *Let s be a GT on X, and let γ be s-enlarging. If $\gamma(A \cap B) = \gamma A \cap \gamma B$ for $A, B \subseteq X$, then the operation $\gamma i_s\gamma$ is quasi-enlarging.*

Let X be a non-empty set and $\gamma \in \Gamma(X)$. Then for $\mu \in 2^X$, γ is said to be μ-$friendly$ [4] if $\gamma A \cap L \subseteq \gamma(A \cap L)$ for $A \subseteq X$, $L \in \mu$.

Theorem 3.12. *Let s be a σ-structure on a non empty set X, and let γ be s-friendly. Then the operation $i_s\gamma$ is quasi-enlarging.*

Proof. For $A \subseteq X$, $i_s\gamma A = \gamma A \cap i_s\gamma A$. Since $i_s\gamma A \in s$ and γ is s-friendly, $i_s\gamma A = \gamma A \cap i_s\gamma A \subseteq \gamma(A \cap i_s\gamma A)$. This implies $i_s i_s\gamma A \subseteq i_s\gamma(A \cap i_s\gamma A)$. Finally, from $i_s i_s\gamma A = i_s\gamma A$, we have the result that $i_s\gamma$ is quasi-enlarging. □

Corollary 3.13. *Let s be a σ-structure on a non empty set X, and let γ be s-friendly. Then the operation $\gamma i_s\gamma$ is quasi-enlarging.*

Proof. For $A \subseteq X$, from Theorem 3.12, it follows $i_s\gamma A \subseteq i_s\gamma(A \cap i_s\gamma A) \subseteq i_s\gamma(A \cap \gamma i_s\gamma A)$. So $\gamma i_s\gamma A \subseteq \gamma i_s\gamma(A \cap \gamma i_s\gamma A)$. □

Corollary 3.14. *Let s be a GT on X, and let γ be s-friendly. Then the operations $i_s\gamma, \gamma i_s\gamma$ are quasi-enlarging.*

Lemma 3.15. *Let s be a σ-structure on a non empty set X, and let γ be s-enlarging and $A \subseteq X$.*

(1) $A \subseteq i_s\gamma i_s A \subseteq \gamma i_s A \subseteq \gamma A$;

(2) $A \subseteq i_s\gamma i_s A \subseteq \gamma i_s A \subseteq \gamma i_s\gamma A \subseteq \gamma\gamma A$;

(3) $A \subseteq i_s\gamma i_s A \subseteq i_s\gamma A \subseteq \gamma A$;

(4) $A \subseteq i_s\gamma i_s A \subseteq i_s\gamma A \subseteq \gamma i_s\gamma A \subseteq \gamma\gamma A$.

Proof. (1) Since $i_s A \in s$, it is obvious.

(2) By (1), $A \subseteq \gamma A$ and $\gamma i_s A \subseteq \gamma i_s\gamma A$. So we have (2).

(3) and (4) are obvious. □

Corollary 3.16. *Let s be a σ-structure on a non empty set X, and let γ be s-enlarging and $A \subseteq X$. If $\gamma\gamma A = \gamma A$, then:*

(1) $A \subseteq i_s\gamma i_s A \subseteq \gamma i_s A \subseteq \gamma i_s\gamma A \subseteq \gamma A$;

(2) $A \subseteq i_s\gamma i_s A \subseteq i_s\gamma A \subseteq \gamma i_s\gamma A \subseteq \gamma A$.

Let us consider the s-enlarging operations λ defined in Theorem 3.4. For $A \subseteq X$, we will call A a *γ_σ-semiopen* (resp., *γ_σ-α-open*, *γ_σ-preopen*, *γ_σ-β-open*) set if A is λ-open, that is, $A \subseteq \gamma i_s A$ (resp., $A \subseteq i_s\gamma i_s A$, $A \subseteq i_s\gamma A$, $A \subseteq \gamma i_s\gamma A$). We denote $S(\gamma_\sigma)$ (resp., $\alpha(\gamma_\sigma)$, $P(\gamma_\sigma)$, $\beta(\gamma_\sigma)$) the set of all γ_σ-semiopen (resp., *γ_σ-α-open*, *γ_σ-preopen*, *γ_σ-β-open*) sets in X.

If the σ-structure s is a topology on X and the s-enlarging operation $\gamma = c_s$, then the *γ_σ-semiopen* (resp., *γ_σ-α-open*, *γ_σ-preopen*, *γ_σ-β-open*) set is the *semiopen* [6] (resp., *α-open* [8], *preopen* [7], *β-open* [1]). Furthermore, if the σ-structure s is a generalized topology on X and the s-enlarging operation $\gamma = c_s$, then the *γ_σ-semiopen* (resp., *γ_σ-α-open*, *γ_σ-preopen*, *γ_σ-β-open*) set is the *s-semiopen* (resp., *s-α-open*, *s-preopen*, *s-β-open*) [3]. So we can the following diagrams:

$$\begin{tikzcd}
 & & \gamma_\sigma\text{-semiopen} \arrow[dr] & \\
\sigma\text{-open} \arrow[r] & \gamma_\sigma\text{-}\alpha\text{-open} \arrow[ur] \arrow[dr] & & \gamma_\sigma\text{-}\beta\text{-open} \\
 & & \gamma_\sigma\text{-preopen} \arrow[ur] &
\end{tikzcd}$$

Theorem 3.17. *Let s be a σ-structure on a non empty set X, and let γ be s-enlarging and $A \subseteq X$. If $\gamma\gamma = \gamma$, then $\alpha(\gamma_\sigma) = S(\gamma_\sigma) \cap P(\gamma_\sigma)$.*

Proof. It is sufficient to show that $S(\gamma_\sigma) \cap P(\gamma_\sigma) \subseteq \alpha(\gamma_\sigma)$. For $A \in S(\gamma_\sigma) \cap P(\gamma_\sigma)$, $A \subseteq i_s\gamma A \subseteq i_s\gamma\gamma i_s A = i_s\gamma i_s A$. So $A \in \alpha(\gamma_\sigma)$. $\square$

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