ON WEAKLY CONCIRCULAR SYMMETRIES OF THREE-DIMENSIONAL TRANS-SASAKIAN MANIFOLDS

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Abstract: The purpose of this paper is to study weakly concircular symmetric and weakly concircular Ricci symmetric three-dimensional trans-Sasakian manifolds.

AMS Subject Classification: 53C15, 53C25

Key Words: weakly symmetric, weakly concircular symmetric, weakly Ricci symmetric, weakly concircular Ricci symmetric manifold, trans Sasakian manifold

1. Introduction

The notion of weakly symmetric manifolds was introduced by Tamassy and Binh[16]. A non flat Riemannian manifold \((M^n, g)(n > 2)\) is called weakly symmetric if its curvature tensor \(R\) of type \((0, 4)\) satisfies the condition

\[
\]

for all vector fields \(X, Y, Z, U, V \in \chi(M^n)\), where \(A, B, H, D\) and \(E\) are

Received: March 21, 2013

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1-forms (not simultaneously zero) and $\nabla$ denotes the operator of covariant differentiation with respect to the Riemannian metric $g$. The 1-forms are called the associated 1-forms of the manifold and an $n$-dimensional manifold of this kind is denoted by $(WS)_n$. In 1999 De and Bandyopadhyay\[4\] studied a $(WS)_n$ and proved that in such a manifold the associated 1-forms $B = H$ and $D = E$. Hence (1.1) reduces to the following form:


A transformation of an 3-dimensional Riemannian manifold $M$, which transforms every geodesic circle of $M$ into a geodesic circle, is called a concircular transformation\[18\] and is defined by

$$C(Y, Z)U = R(Y, Z)U - \frac{r}{6} [g(Z, U)Y - g(Y, U)Z] \quad (3)$$

where $r$ is the scalar curvature of the manifold.

Recently Shaikh and Hui\[14\] introduced the notion of weakly concircular symmetric manifolds. A Riemannian manifold $(M^n, g)(n > 2)$ is called weakly concircular symmetric manifold if its concircular curvature tensor $C$ satisfies the condition


for all vector fields $X, Y, Z, U \in \chi(M^3)$, where $A, B, H$, and $D$ are 1-forms (not simultaneously zero) and 3-dimensional manifold of this kind is denoted by $(WCS)_3$. Also it is shown that in a $(WCS)_3$ the associated 1-forms $B = H$, and hence the defining condition (1.4) of a $(WCS)_3$ reduces to the following form:


$A, B$ and $D$ are 1-forms (not simultaneously zero).

Again Tamassy and Binh\[17\] introduced the notion of weakly Ricci symmetric manifolds. A Riemannian manifold $(M^n, g)(n > 2)$ is called weakly Ricci symmetric manifold if its Ricci tensor $S$ of type $(0, 2)$ is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(X, Z) + D(Z)S(Y, X) \quad (6)$$
where $A$, $B$ and $D$ are three non-zero $1-$forms, called the associated $1-$forms of the manifold , and $\nabla$ denotes the operator of covariant differentiation with respect to the metric $g$. Such an $3-$dimensional manifold is denoted by $(WRS)_3$.

Let $e_1, e_2, e_3$ be an orthonormal basis of the tangent space at each point of the manifold and let

$$P(Y, V) = \sum_{i=1}^{3} C(Y, e_i, e_i, V),$$

then from (1.3), we get

$$P(Y, V) = S(Y, V) - \frac{r}{3}g(Y, V).$$

The tensor $P$ is called the concircular Ricci symmetric tensor [5], which is a symmetric tensor of type $(0, 2)$. In[5] De and Ghosh introduced the notion of weakly concircular Ricci symmetric manifolds. A Riemannian manifold $(M^n, g)(n > 2)$ is called weakly concircular Ricci symmetric manifold [5] if its concircular Ricci tensor $P$ of type $(0, 2)$ is not identically zero and satisfies the condition


where $A$, $B$ and $D$ are $1$-forms(not simultaneously zero).

In 1985, Oubina[11] introduced the notion of trans-Sasakian manifolds, which contains both the class of Sasakian and cosympletic structures, closely related to the locally conformal Kähler manifolds. Trans-Sasakian manifolds of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are the cosympletic, $\alpha-$Sasakian and $\beta-$Kenmotsu manifolds respectively. In particular, if $\alpha = 1$, $\beta = 0$; and $\alpha = 0$, $\beta = 1$, then trans-Sasakian manifold reduces to a Sasakian and Kenmotsu manifolds respectively. Thus trans-Sasakian structures provide a large class of generalized quasi-Sasakian structures. The Local structure of trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by J. C. Marrero[9]. He proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosympletic or $\beta-$Kenmotsu or $\alpha-$Sasakian manifold. But when $n > 3$ trans-Sasakian manifold does not exist In this paper we consider the three dimensional trans-Sasakian manifold.

symmetric weakly Ricci symmetric Kenmotsu manifolds and proved that in such a manifold the sum of the associated 1–forms is zero everywhere and hence such a manifold does not exist unless the sum of the associated 1–forms is everywhere zero. Weakly symmetric and weakly Ricci symmetric properties for trans-Sasakian manifolds, Lorentzian $\alpha$–Sasakian manifolds were studied in [13], [1] and [15] respectively.

The object of the present paper is to study weakly concircular symmetric and weakly concircular Ricci symmetric trans-Sasakian manifolds. Section 2 deals with preliminaries of trans-Sasakian manifolds. In Section 3 of the paper we have obtained all the 1–forms of weakly concircular symmetric three dimensional trans-Sasakian manifold and hence such a structure exist. In the last section we study weakly concircular Ricci symmetric three dimensional trans-Sasakian manifolds and obtained all the 1–forms of weakly concircular Ricci symmetric three dimensional trans-Sasakian manifold and consequently such a structure exist.

2. Preliminaries

A $(2n + 1)$-dimensional smooth manifold $M$ is said to be an almost contact metric manifold [2] if it admits a $(1,1)$ tensor field $\phi$, a vector field $\xi$, a 1–form $\eta$, and a Riemannian metric $g$, which satisfy

\[ \phi^2 X = -X + \eta(X)\xi, \]
\[ g(\phi X, Y) = -g(X, \phi Y), \quad \eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \]
\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \]

for all vector fields $X, Y$ on $M$.

An almost contact metric manifold is said to be trans-Sasakian manifold[11] if $(M \times R, J, G)$ belongs to the class $W_4$ of the Hermitian manifolds, where $J$ is the almost complex structure on $M \times R$ defined by

\[ J(Z, f \frac{d}{dt}) = (\phi Z - f\xi, \eta(Z)\frac{d}{dt}) \]

for any vector field $Z$ on $M$ and smooth function $f$ on $M \times R$ and $G$ is product metric on $M \times R$. This may be stated by the condition[3]

\[ (\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \]

where $\alpha, \beta$ are smooth functions on $M$ and such a structure is said to be the trans-Sasakian structure of type $(\alpha, \beta)$. From (2.4) it follows that

\[ \nabla_X \xi = -\alpha \phi X + \beta\{X - \eta(X)\xi\}, \]
\[(\nabla_X \eta)(Y) = -\alpha g(\phi X, \phi Y).\] (15)

In a trans-Sasakian manifold \(M^3(\phi, \xi, \eta, g)\) the following relations hold [11]:

\[R(X, Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - (X\alpha)\phi Y - (X\beta)\phi^2(Y) + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] + (Y\alpha)\phi X + (Y\beta)\phi^2(X),\] (16)

\[\eta(R(X, Y)Z) = (\alpha^2 - \beta^2)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] - 2\alpha\beta[g(\phi X, Z)\eta(Y) - g(\phi Y, Z)\eta(X)] - (Y\alpha)g(\phi X, Z) - (X\beta)\{g(Y, Z) - \eta(Y)\eta(Z)\} + (X\alpha)g(\phi Y, Z) - (Y\beta)\{g(X, Z) - \eta(X)\eta(Z)\},\] (17)

\[S(X, \xi) = [2(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) - ((\phi X)\alpha) - (X\beta),\] (18)

\[R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)[\eta(X)\xi - X],\] (19)

\[S(\xi, \xi) = 2(\alpha^2 - \beta^2 - \xi\beta),\] (20)

\[(\xi\alpha) + 2\alpha\beta = 0\] (21)

\[Q\xi = [2(\alpha^2 - \beta^2) - (\xi\beta)]\xi + \phi(\text{grad}\alpha) - (\text{grad}\beta),\] (22)

where \(R\) is the curvature tensor of type(1,3) of the manifold and \(Q\) is the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor \(S\), that is, \(g(QX, Y) = S(X, Y)\) for any vector fields \(X, Y\) on \(M\).

### 3. Weakly Concircular Symmetric Three-Dimensional Trans-Sasakian Manifolds

**Definition 1.** A trans-Sasakian manifold \(M^3(\phi, \xi, \eta, g)\) is said to be weakly concircular symmetric if its concircular curvature tensor \(C\) satisfies (1.5).

Setting \(Y = V = e_i\) in (1.5) and taking summation over \(i, 1 \leq i \leq 3\), we get

\[(\nabla_X S)(Z, U) - \frac{dr(X)}{3}g(Z, U) = A(X)[S(Z, U) - \frac{r}{3}g(Z, U)] + B(Z)[S(X, U) - \frac{r}{3}g(X, U)] + D(U)[S(X, Z) - \frac{r}{3}g(X, Z)] - \frac{r}{6}[\{B(X) + D(X)\}g(Z, U) - B(Z)g(X, U) - D(U)g(Z, X)] + B(R(X, Z)U) + D(R(X, U)Z)\] (23)
Putting $X = Z = U = \xi$ in (3.1) and then using (2.7) and (2.11) we obtain

$$A(\xi) + B(\xi) + D(\xi) = \frac{\text{grad}F.\xi}{F} \tag{24}$$

where $F = 6(\alpha^2 - \beta^2 - \xi\beta) - r$

We can see that if $\text{grad}F$ is orthogonal to $\xi$ then

$$A(\xi) + B(\xi) + D(\xi) = 0 \tag{25}$$

Since $A(X) = g(X, \rho), A(\xi) = B(\xi) = D(\xi) = g(\rho, \xi)$

In view of (3.3) we obtain that $A(\xi) = B(\xi) = D(\xi) = 0$.

If $\text{grad}F$ and $\xi$ are not inclined orthogonal then $\text{grad}F.\xi \neq 0$. Hence

$$A(\xi) + B(\xi) + D(\xi) \neq 0 \tag{26}$$

that is $A(\xi) = B(\xi) = D(\xi) \neq 0$.

**Theorem 1.** In a weakly concircular trans-Sasakian manifold $M^3(\phi, \xi, \eta, g)$ the relation (3.2) holds.

Next substituting $X$ and $Z$ by $\xi$ in (3.1) and then using (2.9) and (2.12) we obtain

$$(\nabla_\xi S)(\xi, U) = [A(\xi) + B(\xi)]S(U, \xi) + [\alpha^2 - \beta^2 - (\xi\beta)][D(U) + \eta(U)D(\xi)] \tag{27}$$

Again we have

$$(\nabla_\xi S)(\xi, U) = \nabla_\xi S(\xi, U) - S(\nabla_\xi \xi, U) - S(\xi, \nabla_\xi U)$$
$$= [2\{2\alpha(\xi\alpha) - 2\beta(\xi\beta)\} - \xi(\xi\beta)]\eta(U)$$
$$- (U(\xi\beta)) - (\phi U(\xi\alpha)), \tag{28}$$

where (2.9) has been used. In view of (3.5) and (3.6) we obtain from (3.2) that

$$D(U) = \frac{\{12\{2\alpha(\xi\alpha) - 2\beta(\xi\beta)\} - 6\xi(\xi\beta)\} \eta(U)}{F} -$$
$$\frac{6U[\xi\beta] + 6\phi U[\xi\alpha] + 2\xi[r]\eta(U)}{F^2} - \left\{ \frac{\xi[F]}{F^2} \right\}$$
$$\left\{ \{6[2(\alpha^2 - \beta^2) - \xi\beta] - 2r\} \eta(U) - 6\{U[\beta] + \phi U[\alpha]\} \right\} +$$
$$D(\xi) \left\{ \{6((\alpha^2 - \beta^2)\eta(U) - U[\beta]) - 6\phi U[\alpha] - r\eta(U) \right\} \right\} \right\} \tag{29}$$
for any vector field $U$.

If $\nabla F$ and $\xi$ are orthogonal then by virtue of (3.2) and (3.3) we get

$$D(U) = \frac{[12\{2\alpha(\xi\alpha) - 2\beta(\xi\beta)\} - 6\xi(\xi\beta)] \eta(U)}{F} - \frac{6U[\xi\beta] + 6\phi U[\xi\alpha] + 2\xi[r] \eta(u)}{F} \neq 0$$

If $\nabla F$ and $\xi$ are not orthogonal then by virtue of (3.4) we get $D(U) \neq 0$

Next setting $X = U = \xi$ in (3.1) and proceeding in a similar manner as above we get

$$B(Z) = \frac{[12\{2\alpha(\xi\alpha) - 2\beta(\xi\beta)\} - 6\xi(\xi\beta)] \eta(Z)}{F} - \frac{6Z[\xi\beta] + 6\phi Z[\xi\alpha] + 2\xi[r] \eta(Z)}{F} \left\{ \frac{\xi[F]}{F^2} \right\} - \frac{\{6(2\alpha^2 - \beta^2) - \xi\beta\| - 2r\} \eta(Z) - 6\{Z[\beta] + \phi Z[\alpha]\}}{F} + B(\xi) \left\{ \frac{6(\alpha^2 - \beta^2) \eta(Z) - Z[\beta]}{F} - 6\phi Z[\alpha] - r \eta(Z) \right\}$$

(30)

for any vector field $Z$

If $\nabla F$ and $\xi$ are orthogonal then by virtue of (3.2) and (3.3) we get

$$B(Z) = \frac{[12\{2\alpha(\xi\alpha) - 2\beta(\xi\beta)\} - 6\xi(\xi\beta)] \eta(Z)}{F} - \frac{6Z[\xi\beta] + 6\phi Z[\xi\alpha] + 2\xi[r] \eta(Z)}{F} \neq 0$$

If $\nabla F$ and $\xi$ are not orthogonal then by virtue of (3.4) we get $B(Z) \neq 0$

Again setting $Z = U = \xi$ in (3.1) we get

$$\left( \nabla_\xi S \right)(\xi, \xi) - \frac{dr(x)}{3} = A(X)[S(\xi, \xi) - \frac{dr(x)}{3}] + B(R(X, \xi) \xi) + D(R(X, \xi) \xi) + [B(\xi) + D(\xi)] \left[ S(X, \xi) - \frac{dr(x)}{3} \eta(X) \right] - \frac{r}{6}[B(X) + D(X) - B(\xi) \eta(X) - D(\xi) \eta(X)]$$

(31)
Now we have
\[(\nabla_X S)(\xi, \xi) = \nabla_X S(\xi, \xi) - 2S(\nabla_X \xi, \xi),\]
which yields by using (2.5) and (2.9) that
\[
(\nabla_\xi S)(\xi, \xi) = 2[2\alpha(X\alpha) - 2\beta(X\beta) - (X(\xi\beta))] \\
+ 2\alpha[(X\alpha) - \eta(X)(\xi\alpha) - ((\phi X)\beta)] \\
+ 2\beta[(\phi X)\alpha + \{(X\beta) - (\xi\beta)\eta(X)\}] \\
(32)
\]

using (2.10) (2.11) and (3.10) in (3.9) we get
\[
A(X) = \frac{X(F + r)}{F} + \frac{6\alpha \{(X\alpha) - \eta(X)(\xi\alpha) - (\phi X)\beta\}}{F} + \\
\frac{6\beta[(\phi X)\alpha + \{X[\beta] - \xi\beta\eta(X)\}] - X[r]}{F} - \\
\frac{[B(\xi) + D(\xi)]\{3(\alpha^2 - \beta^2)\eta(X) - X[\beta]\} - 3\phi X[\alpha] - \frac{r}{2}\eta(X)}{F} - \\
\frac{[B(X) + D(X)]\{3(\alpha^2 - \beta^2) - \xi\beta\frac{r}{2}\}}{F} \\
(33)
\]

for any vector \(X\).

If \(\text{grad} F\) and \(\xi\) are orthogonal then by virtue of (3.2) and (3.3) we get
\[
A(X) = \frac{X(F + r)}{F} + \frac{6\alpha \{(X\alpha) - \eta(X)(\xi\alpha) - (\phi X)\beta\}}{F} + \\
\frac{6\beta[(\phi X)\alpha + \{X[\beta] - \xi\beta\eta(X)\}] - X[r]}{F} - \\
\frac{[B(X) + D(X)]\{3(\alpha^2 - \beta^2) - \xi\beta\frac{r}{2}\}}{F} \\
\neq 0
\]

If \(\text{grad} F\) and \(\xi\) are not orthogonal then by virtue of (3.4) we get \(A(X) \neq 0\)

This leads to the following:

**Theorem 2.** There exists no weakly Concircular symmetric trans-Sasakian manifold \(M^3\), if \(A + B + D\) is not everywhere zero.
4. Weakly Concircular Ricci Symmetric Three-Dimensional Trans-Sasakian Manifolds

Definition 2. A trans-Sasakian manifold $M^3(\phi, \xi, \eta, g)$ is said to be weakly concircular Ricci symmetric if its concircular Ricci tensor $P$ of type $(0, 2)$ satisfies (1.9).

In view of (1.8), (1.9) yields

$$
(\nabla_X S)(Y, Z) - \frac{dr(X)}{3} g(Y, Z) = A(X)[S(Y, Z) - \frac{r}{3} g(Y, Z)]
$$

$$
+ B(Y)[S(X, Z) - \frac{r}{3} g(X, Z)] + D(Z)[S(X, Y) - \frac{r}{3} g(X, Y)]
$$

(34)

Setting $X = Y = Z = \xi$ in (4.1), we get the relation (3.2) and hence we can state the following:

Theorem 3. In a weakly concircular Ricci symmetric trans-Sasakian manifold $M^3(\phi, \xi, \eta, g)$, the relation (3.2) holds.

Next, substituting $X$ and $Y$ by $\xi$ in (4.1) and using (2.9) and (3.2), we obtain

$$
D(Z) = \frac{\{6\xi[\alpha^2 - \beta^2] - (3)\xi[\xi\beta]\} \eta(Z)}{F} - 
$$

$$
\frac{3Z\xi[\beta] + 3\phi Z[\xi\alpha] + \xi\eta(Z)}{F} + 
$$

$$
D(\xi) \left\{ \frac{[6(\alpha^2 - \beta^2) - 3\xi\beta - r] \eta(Z) - 3\phi Z[\alpha] - 3Z[\beta]}{F} \right\} - 
$$

$$
\left\{ \frac{\xi[F]}{F^2} \right\} \left\{ [2n(\alpha^2 - \beta^2) - (\xi\beta)] \eta(Z) - Z[\beta] - \phi Z[\alpha] - \frac{r}{3} \eta(Z) \right\}
$$

(35)

for any vector $Z$.

If $\nabla F$ and $\xi$ are orthogonal then by virtue of (3.2) and (3.3) we get

$$
D(Z) = \frac{\{6\xi[\alpha^2 - \beta^2] - 3\xi[\xi\beta]\} \eta(Z)}{F} - 
$$

$$
\frac{3Z\xi[\beta] + 3\phi Z[\xi\alpha] + \xi\eta(Z)}{F} - 
$$

$$
\frac{3Z\xi[\beta] + 3\phi Z[\xi\alpha] + \xi\eta(Z)}{F}
$$

$\neq 0$
If $\text{grad}F$ and $\xi$ are not orthogonal then by virtue of (3.4) we get $D(Z) \neq 0$
Again setting $X = Z = \xi$ in (4.1) and proceeding in a similar manner as above we get

$$B(Y) = \frac{\{6\xi[\alpha^2 - \beta^2] - 3\xi[\xi\beta]\}\eta(Y)}{F} - \frac{3Y\xi[\beta] + 3\phi Y[\xi\alpha] + \xi[r]\eta(Y)}{F} +$$

$$B(\xi)\left\{\left[6(\alpha^2 - \beta^2) - 3\xi\beta - r\eta(Y) - 3\phi Y[\alpha] - 3Y[\beta]\right] - \left\{\frac{\xi[F]}{F^2}\right\}\left\{[2(\alpha^2 - \beta^2) - (\xi\beta)]\eta(Y) - Y[\beta] - \phi Y[\alpha] - \frac{r}{3}\eta(Y)\right\}\right\}$$

(36)

for any vector $Y$,
If $\text{grad}F$ and $\xi$ are orthogonal then by virtue of (3.2) and (3.3) we get

$$B(Y) = \frac{\{6\xi[\alpha^2 - \beta^2] - 3\xi[\xi\beta]\}\eta(Y)}{F} - \frac{3Y\xi[\beta] + 3\phi Y[\xi\alpha] + \xi[r]\eta(Y)}{F} \neq 0$$

If $\text{grad}F$ and $\xi$ are not orthogonal then by virtue of (3.4) we get $B(Y) \neq 0$
Again putting $Y = Z = \xi$ in (4.1) and using (2.11) and (3.2), we get

$$A(X) = \frac{X(F + r)}{F} + \frac{6\alpha\{(X\alpha) - \eta(X)(\xi\alpha) - (\phi X)\beta\}}{F} + \frac{6\beta[\phi X][\alpha] + \{X[\beta] - \xi\beta\eta(X)\}}{F} - X[\alpha] +$$

$$A(\xi)\left\{\left[6(\alpha^2 - \beta^2) - 3\xi\beta - r\eta(X) - 3\phi X[\alpha] - 3X[\beta]\right] - \left\{\frac{\xi[F]}{F^2}\right\}\left\{[2(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) - X[\beta] - \phi X[\alpha] - \frac{r}{3}\eta(X)\right\}\right\}$$

(37)

for any vector $X$,
If $\text{grad}F$ and $\xi$ are orthogonal then by virtue of (3.2) and (3.3) we get

$$A(X) = \frac{X(F + r)}{F} + \frac{6\alpha\{(X\alpha) - \eta(X)(\xi\alpha) - (\phi X)\beta\}}{F}$$
\[
\frac{6\beta[(\phi X)[\alpha] + \{X[\beta] - \xi \beta \eta(X)\}]}{F} - X[r] \\
\neq 0
\]

If \(\text{grad}F\) and \(\xi\) are not orthogonal then by virtue of (3.4) we get \(A(X) \neq 0\)

This leads the following:

**Theorem 4.** There exists no weakly Concircular Ricci symmetric trans-Sasakian manifold \(M^3\), if sum of the associated 1–forms, \(D\), \(B\) and \(A\) is not everywhere zero.

**References**


