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NEW EFFICIENT ALGORITHMS FOR MINIMIZATION OF NON-LINEAR FUNCTIONS

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Abstract: In this paper, we suggest two new algorithms for minimization of nonlinear functions. Then the comparative study of the new algorithms with the Newton's algorithm is established by means of examples.

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1. Introduction

Optimization problems with or without constraints arise in various fields such as science, engineering, economics, management sciences, etc., where numerical information is processed. An unconstrained minimization problem is the one where a value of the vector x is sought that minimizes the objective function $f(x)$. This problem can be considered as particular case of the general constrained non-linear programming problem. Several methods (see for example [7], [8], [10], [11]) are available for solving unconstrained minimization problems. Coppe (see [3]) proposed a two-stage feasible directions algorithm for nonlinear constrained optimization. Newton's method (see [12]) is used for

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various classes of optimization problems, such as unconstrained minimization problems, equality constrained minimization problems.

Many iterative methods have been developed to find a simple root of a nonlinear equation $f(x) = 0$ such as Newton's method and its variants, Secant method, Halley's method, Chebychev method and Jarratt's method (see [6]) and many researchers developed efficient modifications of Newton's method (see [1], [4]) and others on variants of Chebyshev-Halley's method free from second derivative. Hou et al (see [5]) presented a new twelfth order family of methods which improves an eighth order method, see [1]. Several efficient modifications of Jarratt's method have been developed and applied in [2], [4], [9], [13] to improve the order of convergence. Recently Young-II kim and Changbum Chun (see [14]) introduced modification of Jarratt's method for solving nonlinear equation with the comparative study of the new iterative method with Newton's Algorithm, Jarratt's method and Hou-Li method. In this paper, we suggest two new algorithms for minimization of nonlinear functions. Then, we present the comparative study among the new algorithms and Newton's algorithm.

2. New Algorithm

Consider the nonlinear optimization problem: Minimize $\{f(x),\, x \in R, f : R \to R\}$ where f is a nonlinear twice differentiable function. Consider the function $G(x) = x - (g(x)/g'(x))$ where $g(x) = f'(x)$. Here $f(x)$ is the function to be minimized. $G'(x)$ is defined around the critical point x^* of $f(x)$ if $g'(x^*) = f''(x^*) \neq 0$ and is given by $G'(x) = g(x)g''(x)/g'(x)$. If we assume that $g''(x^*) \neq 0$, we have $G'(x^*) = 0$ iff $g(x^*) = 0$. Consider the equation

$$g(x) = 0 \tag{1}$$

whose one or more roots are to be found. $y = g(x)$ represents the graph of the function $g(x)$ and assume that an initial estimate x_0 is known for the desired root of the equation $g(x) = 0$.

2.1. New Method-I

We introduce new method-I which is based on Jarrat's method (see [10]) for $g(x)$ and given by

$$x_{n+1} = x_n - J_g(x_n)\frac{g(x_n)}{g'(x_n)} \text{ where } J_g(x_n) = \frac{3g'(y_n) + g'(x_n)}{6g'(y_n) - 2g'(x_n)} \tag{2}$$

and $y_n = x_n - \frac{2g(x_n)}{3g'(x_n)}$ whose order of convergence is four.

New Algorithm-I. Since $g(x) = f'(x)$, the equation (2) becomes

$$x_{n+1} = x_n - J_{f'}(x_n)\frac{f'(x_n)}{f''(x_n)} \text{where} J_{f'}(x_n) = \frac{3f''(y_n) + f''(x_n)}{6f''(y_n) - 2f''(x_n)} \tag{3}$$

and $y_n = x_n - \frac{2f'(x_n)}{3f''(x_n)}$ whose order of convergence is also four.

2.2. New Method-II

Several efficient modifications of Jarrat's method have been developed and applied to improve the order of convergence. Young-II kim and Changbum Chun (see [14]) introduced a new twelfth order modifications version of Jarrat's method for solving nonlinear equations. Here in this section, we introduce new algorithm-II which is based on the above method. From the equation (2) let we have

$$z_n = x_n - J_g(x_n)\frac{g(x_n)}{g'(x_n)} \tag{4}$$

Using an elementary calculation, the circle of curvature at $(z_n, g'(z_n))$ found to be

$$\left(x - z_n + \frac{g'(z_n)[1 + g'(z_n)^2]}{g''(z_n)}\right)^2 + \left(y - g(z_n) - \frac{1 + g'(z_n)^2}{g''(z_n)}\right)^2 = \frac{\left(1 + g'(z_n)^2\right)^3}{g''(z_n)^2} \tag{5}$$

At the intersection point $(x_{n+1}, 0)$ of the above equation (5) with the x-axis, we get

$$(x_{n+1} - z_n)^2 + 2\frac{g'(z_n)(1 + g'(z_n)^2)}{g''(z_n)}(x_{n+1} - z_n) + g(z_n)^2 + 2g(z_n)\frac{(1 + g'(z_n)^2)}{g''(z_n)} = 0 \tag{6}$$

Equation (5) can further be rewritten as follows

$$x_{n+1} = z_n - \frac{g(z_n)^2 + 2g(z_n)\frac{1+g'(z_n)^2}{g''(z_n)}}{x_{n+1} - z_n + 2\frac{g'(z_n)(1+g'(z_n)^2)}{g''(z_n)}} \tag{7}$$

By replacing x_{n+1} on the right hand side of (7) by the Newton iterate, we have

$$x_{n+1} = z_n - \frac{g'(z_n)g''(z_n)g(z_n)^2 + 2g'(z_n)g(z_n)(1 + g'(z_n)^2)}{2g'(z_n)^2(1 + g'(z_n)^2) - g(z_n)g''(z_n)} \tag{8}$$

To derive its second derivative free variant, let us consider the following approximation

$$g''(z_n) \approx \frac{g'(w_n) - g'(z_n)}{w_n - z_n} \quad \text{where} \quad w_n = z_n - \frac{g(z_n)}{g'(z_n)} \tag{9}$$

The equation (8) becomes

$$x_{n+1} = z_n - \frac{g(z_n)[2 + 3g'^2(z_n) - g'(z_n)g'(w_n)]}{g'(z_n) + 2g'^2(z_n) + g'(w_n)} \tag{10}$$

By replacing the first term of (6), $(x_{n+1} - z_n)^2$ with $\left(\frac{g(z_n)}{g'(z_n)}\right)^2$ from Newton's iterate results the following

$$x_{n+1} = z_n - \frac{g(z_n)^2 g''(z_n) + 2g(z_n)g'(z_n)^2}{2g'(z_n)^3} \tag{11}$$

By using the approximation defined by (9), we obtain the second derivative free variant of (11)

$$x_{n+1} = z_n - \frac{1}{2}\left(3 - \frac{g'(w_n)}{g'(z_n)}\right)\frac{g(z_n)}{g'(z_n)} \tag{12}$$

New Algorithm-II. Since $g(x) = f'(x)$, the above equation (12) becomes

$$x_{n+1} = z_n - \frac{1}{2}\left(3 - \frac{f''(w_n)}{f''(z_n)}\right)\frac{f'(z_n)}{f''(z_n)} \tag{13}$$

The convergence analysis of the equation (13) is defined by the following theorem.

3. Convergence Analysis

Theorem 1. *Assume that the function $g : D \subset R \to R$ for an open interval D has a simple root x^* in D. If $f(x)$ is sufficiently smooth in a neighborhood of the root x^*, then the method given by (12) is of order twelve.*

Proof. The proof of this theorem follows as in convergence theorem (see [14]) and hence the order of convergence of the algorithm (13). □

4. Numerical Illustrations

Example 4.1. Consider the function $f(x) = xe^x - 1, x \in R$. Then, minimizing point of the function is equal to -1 which is obtained in 7 iterations by Newton's algorithm, in 3 iterations by New Algorithm-I and in 2 iterations by New Algorithm-II for the initial value $x_0 = 1$ and also seen the variation in iterations for the initial value of $x_0 = 2$ and $x_0 = 3$. The results of the Example 4.1 are shown in the tables 1-3.

Iterations	Newton's algorithm	New Algorithm-I	New Algorithm-II
0	1.000000	1.000000	1.000000
1	0.333333	-0.377329	-0.853119
2	-0.238095	-0.970322	-1.000000
3	-0.670528	-1.000000	
4	-0.918350		
5	-0.993836		
6	-0.999962		
7	-1		

Table 1

Iterations	Newton's algorithm	New Algorithm-I	New Algorithm-II
0	2.000000	2.000000	2.000000
1	1.250000	0.358414	-0.383209
2	0.557692	-0.740017	-0.999950
3	-0.051330	-0.998443	-1.000000
4	-0.538159	-1.000000	
5	-0.854409		
6	-0.981421		
7	-0.999661		
8	-0.999999		
9	-1.000000		

Table 2

Example 4.2: Consider the function $f(x) = x^5 + x^4 + 4x^2 - 15, x \in R$. Then, minimizing point of the function is equal to 0 which is obtained in 6 iterations by Newton's algorithm, 3 iterations by New Algorithm-I and in 2 iterations by New Algorithm-II for the initial value $x_0 = 2$ and also seen the

Iterations	Newton's algorithm	New Algorithm-I	New Algorithm-II
0	3.000000	3.000000	3.000000
1	2.200000	1.201916	0.306123
2	1.438095	-0.241894	-0.983696
3	0.728954	-0.946118	-1.000000
4	0.095395	-0.999996	
5	-0.427368	-1.000000	
6	-0.791491		
7	-0.964025		
8	-0.998751		
9	-0.999999		
10	-1.000000		

Table 3

variations in iteration for the initial value of $x_0 = 3$ and $x_0 = 1$. The results of the Example 4.2 are shown in the tables 4-6.

Iterations	Newton's algorithm	New Algorithm-I	New Algorithm-II
0	1.000000	1.000000	1.000000
1	0.575000	0.095748	0.000276
2	0.200420	0.000000	0.000000
3	0.010252		
4	0.000001		
5	0.000000		

Table 4

Iterations	Newton's algorithm	New Algorithm-I	New Algorithm-II
0	2.000000	2.000000	2.000000
1	1.407407	0.843329	0.314838
2	0.927224	0.001797	0.000000
3	0.509766	0.000000	
4	0.150546		
5	0.004197		
6	0.000000		

Table 5

Iterations	Newton's algorithm	New Algorithm-I	New Algorithm-II
0	3.000000	3.000000	3.000000
1	2.181402	1.452581	0.816819
2	1.549994	0.454100	0.000000
3	1.045487	-0.014166	
4	0.615469	0.000000	
5	0.233460		
6	0.016429		
7	0.000005		
8	0.000000		

Table 6

5. Conclusion

In this article we have introduced two new algorithms namely New Algorithm-I and New Algorithm-II for minimization of unconstrained nonlinear functions and comparison made with Newton's algorithm. It is clear from the illustrations that the rate of convergence of the new algorithms are faster than the Newton's algorithm. In future, we extend these new algorithms for minimization of constrained nonlinear functions.

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