RIGHT AND LEFT ROUGH SETS INDUCED BY FUNCTIONS

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Abstract: In this paper, we investigate the properties of right and left rough sets induced by functions on a generalized residuated lattice. In particular, we construct right (left) closure (interior) operators.

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1. Introduction

Pawlak [8,9] introduced the rough set as a mathematical tool for dealing information as the uncertainty of data in engineering, physics, computer sciences and many other diverse field. Recently, Bělohlávek [1-3] investigate the properties of fuzzy rough set on a residuated lattice which supports part of foundation of theoretic computer science. Chen and Li [3] introduced fuzzy rough sets induced by functions. Georgescu and Popescue [4,5] introduced non-commutative fuzzy Galois connection in a generalized residuated lattice which is induced by two implications.

In this paper, we investigate the properties of right and left rough sets induced by functions on a generalized residuated lattice. In particular, we construct right (left) closure (interior) operators.

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2. Preliminaries

Definition 1. (see [4,5]) A structure \((L, \lor, \land, \circ, \rightarrow, \Rightarrow, \perp, \top)\) is called a generalized residuated lattice if it satisfies the following conditions:

(GR1) \((L, \lor, \land, \top, \perp)\) is a bounded lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(GR2) \((L, \circ, \top)\) is a monoid;

(GR3) it satisfies a residuation, i.e.
\[ a \circ b \leq c \text{ iff } a \leq b \rightarrow c \text{ iff } b \leq a \Rightarrow c. \]

We call that a generalized residuated lattice has the law of double negation if \(a = (a^*)^0 = (a^0)^*\) where \(a^0 = a \rightarrow \perp\) and \(a^* = a \Rightarrow \perp\).

Remark 2. (see [1-7]) (1) A generalized residuated lattice is a residuated lattice \((\rightarrow = \Rightarrow)\) iff \(\circ\) is commutative.

(2) A left-continuous t-norm \(([0, 1], \leq, \circ)\) defined by \(a \rightarrow b = \bigvee\{c \mid a \circ c \leq b\}\) is a residuated lattice xms

(3) A pseudo MV-algebra is a generalized residuated lattice with the law of double negation.

In this paper, we assume \((L, \land, \lor, \circ, \rightarrow, \Rightarrow, \perp, \top)\) is a complete generalized residuated lattice with the law of double negation.

Lemma 3. (see [4,5]) For each \(x, y, z, x_i, y_i \in L\), we have the following properties.

(1) If \(y \leq z\), \((x \circ y) \leq (x \circ z)\), \(x \rightarrow y \leq x \rightarrow z\) and \(z \rightarrow x \leq y \rightarrow x\) for \(\rightarrow \in \{\rightarrow, \Rightarrow\}\).

(2) \(x \circ y \leq x \land y \leq x \lor y\).

(3) \(x \rightarrow (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma} (x \rightarrow y_i)\) and \((\bigvee_{i \in \Gamma} x_i) \rightarrow y = \bigwedge_{i \in \Gamma} (x_i \rightarrow y)\) for \(\rightarrow \in \{\rightarrow, \Rightarrow\}\).

(4) \(x \rightarrow (\bigvee_{i \in \Gamma} y_i) \geq \bigvee_{i \in \Gamma} (x \rightarrow y_i)\), for \(\rightarrow \in \{\rightarrow, \Rightarrow\}\).

(5) \((\bigwedge_{i \in \Gamma} x_i) \rightarrow y \geq \bigvee_{i \in \Gamma} (x_i \rightarrow y)\), for \(\rightarrow \in \{\rightarrow, \Rightarrow\}\).

(6) \((x \circ y) \rightarrow z = x \rightarrow (y \rightarrow z)\) and \((x \circ y) \Rightarrow z = y \Rightarrow (x \Rightarrow z)\).

(7) \(x \rightarrow (y \Rightarrow z) = y \Rightarrow (x \rightarrow z)\) and \(x \Rightarrow (y \rightarrow z) = y \rightarrow (x \Rightarrow z)\).

(8) \(x \circ (x \rightarrow y) \leq y\) and \((x \Rightarrow y) \circ x \leq y\).

(9) \((x \Rightarrow y) \circ (y \Rightarrow z) \leq x \Rightarrow z\) and \((y \rightarrow z) \circ (x \rightarrow y) \leq x \rightarrow z\).
(10) \( (x \Rightarrow z) \leq (y \odot x) \Rightarrow (y \odot z) \) and \( (x \rightarrow z) \leq (x \odot y) \rightarrow (z \odot y) \).

(11) \( (x \Rightarrow y) \leq (y \Rightarrow z) \Rightarrow (x \Rightarrow z) \) and \( (y \Rightarrow z) \leq (x \Rightarrow y) \Rightarrow (x \Rightarrow z) \).

(12) \( x_i \rightarrow y_i \leq (\bigwedge_{i \in I} x_i) \rightarrow (\bigwedge_{i \in I} y_i) \) for \( \rightarrow \in \{\rightarrow, \Rightarrow\} \).

(13) \( x_i \rightarrow y_i \leq (\bigvee_{i \in I} x_i) \rightarrow (\bigvee_{i \in I} y_i) \) for \( \rightarrow \in \{\rightarrow, \Rightarrow\} \).

(14) \( x \rightarrow y = \top \) iff \( x \leq y \).

(15) \( x \rightarrow y = y^0 \Rightarrow x^0 \) and \( x \Rightarrow y = y^* \rightarrow x^* \).

(16) \( \bigwedge_{i \in I} x_i^* = (\bigvee_{i \in I} x_i)^* \) and \( \bigvee_{i \in I} x_i^* = (\bigwedge_{i \in I} x_i)^* \).

(17) \( \bigwedge_{i \in I} x_i^0 = (\bigvee_{i \in I} x_i)^0 \) and \( \bigvee_{i \in I} x_i^0 = (\bigwedge_{i \in I} x_i)^0 \).

**Definition 4.** Let \( X \) be a set. A function \( e_X^r : X \times X \rightarrow L \) is called:

(E1) \( e_X^r(x, x) = \top \) for all \( x \in X \),

(E2) If \( e_X^r(x, y) = \top \) and \( e_X^r(y, x) = \top \), then \( x = y \),

(R) \( e_X^r(x, y) \odot e_X^r(y, z) \leq e_X^r(x, z) \), for all \( x, y, z \in X \).

Then \( e_X^r \) is called a right partial order. If \( e_X^l \) satisfies (E1), (E2) and (L) \( e_X^l(y, z) \odot e_X^l(x, y) \leq e_X^l(x, z) \), for all \( x, y, z \in X \).

Then \( e_X^l \) is called a left partial order. The triple \( (X, e_X^r, e_X^l) \) is called a bi-partial ordered set.

**Example 5.** (1) We define two functions \( e_L^r, e_L^l : L \times L ightarrow L \) as \( e_L^r(x, y) = x \Rightarrow y \) and \( e_L^l(x, y) = x \rightarrow y \). Then \( e_L^r \) is a right partial order and \( e_L^l \) is a left partial order.

(2) We define two functions \( e_{LX}^r, e_{LX}^l : L^X \times L^X \rightarrow L \) as

\[
e_{LX}^r(A, B) = \bigwedge_{x \in X} (A(x) \Rightarrow B(x)), \quad e_{LX}^l(A, B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x)).
\]

By Lemma 3(9), \( e_{LX}^r \) is a right partial order and \( e_{LX}^l \) is a left partial order.

**Definition 6.** An operator \( I^r : L^X \rightarrow L^X \) is called a right interior operator on \( X \) if it satisfies the following conditions:

(I1) \( I^r(A) \leq A \), xms

(I2) \( I^r(I^r(A)) \geq I^r(A) \),

(R1) \( e_{LX}^r(A, B) \leq e_{LX}^r(I^r(A), I^r(B)) \).

The pair \( (X, I^r) \) is called a right interior space. An operator \( I^l : L^X \rightarrow L^X \) is called a left interior operator on \( X \) if it satisfies the conditions (I1),(I2), and
(LI) $e^l_{LX}(A, B) \leq e^l_{LX}(I^l(A), I^l(B))$.

An operator $C^r: L^X \rightarrow L^X$ is called a right closure operator on $X$ if it satisfies the following conditions:

(C1) $C^r(A) \geq A$,

(C2) $C^r(C^r(A)) \leq C^r(A)$,

(RC) $e^l_{LX}(A, B) \leq e^l_{LX}(C^r(A), C^r(B))$.

The pair $(X, C^r)$ is called an $r$-closure space.

An operator $C^l: L^X \rightarrow L^X$ is called a left closure operator on $X$ if it satisfies the conditions (C1),(C2) and

(RC) $e^l_{LX}(A, B) \leq e^l_{LX}(C^l(A), C^l(B))$.

**Definition 7.** Let $(X, e^r_X, e^l_X)$ be a bi-partial ordered set and $A \in L^X$.

(1) A point $x_0$ is called a right join of $A$, denoted by $x_0 = \sqcup_r A$, if it satisfies

(RJ1) $A(x) \leq e^r_X(x, x_0)$,

(RJ2) $\bigwedge_{x \in X}(A(x) \Rightarrow e^r_X(x, y)) \leq e^r_X(x_0, y)$.

(2) A point $x_0$ is called a left join of $A$, denoted by $x_0 = \sqcup_l A$, if it satisfies

(LJ1) $A(x) \leq e^l_X(x, x_0)$,

(LJ2) $\bigwedge_{x \in X}(A(x) \Rightarrow e^l_X(x, y)) \leq e^l_X(x_0, y)$.

An bi-partial ordered set $(X, e^r_X, e^l_X)$ is called a right (resp. left) join complete lattice if $\sqcup_r A$ (resp $\sqcup_l A$) exists for $A \in L^X$.

**Remark 8.** Let $(X, e^r_X, e^l_X)$ be a bi-partial ordered set and $A \in L^X$. If $x_0$ is a right join of $A$, then it is unique because $e^r_X(x_0, y) = e^r_X(y_0, y)$ for all $y \in X$, put $y = x_0$ or $y = y_0$, then $e^r_X(x_0, y_0) = e^r_X(y_0, x_0) = \top$ implies $x_0 = y_0$. Similarly, if left join of $A$ exist, then it is unique.

**Theorem 9.** Let $(X, e^r_X, e^l_X)$ be a bi-partial ordered set and $A \in L^X$.

(1) $x_0$ is a right join of $A$ iff $\bigwedge_{x \in X}(A(x) \Rightarrow e^r_X(x, y)) = e^r_X(x_0, y)$.

(2) $x_0$ is a left join of $A$ iff $\bigwedge_{x \in X}(A(x) \Rightarrow e^l_X(x, y)) = e^l_X(x_0, y)$.

**Proof.** (1) ($\Rightarrow$) Let $x_0$ be a right join of $A$. Then $A(x) \leq e^r_X(x, x_0)$. Thus, $A(x) \circ e^r_X(x, y) \leq e^r_X(x, x_0) \circ e^r_X(x, y) \leq e^r_X(x, y)$. Hence $e^r_X(x_0, y) \leq \bigwedge_{x \in X}(A(x) \Rightarrow e^r_X(x, y))$. By (RJ2), the equality holds.

($\Leftarrow$) Since $\bigwedge_{x \in X}(A(x) \Rightarrow e^r_X(x, x_0)) = e^r_X(x, x_0) = \top$, then $A(x) \leq e^r_X(x, x_0)$. Hence the result holds. \qed
Remark 10. Let \((L^X, e^r_{L^X}, e^l_{L^X})\) be a bi-partial ordered set and \(A \in L^L\).

(1) Since \(\sqcup_r \Phi\) is a right join of \(\Phi\) iff
\[
\bigwedge_{A \in L^X} (\Phi(A) \Rightarrow e^r_{L^X}(A, B)) = e^r_{L^X} \left( \bigvee_{A \in L^X} (A \circ \Phi(A), B) = e^r_{L^X}(\sqcup_r \Phi, B), \right.
\]
then \(\sqcup_r \Phi = \bigvee_{A \in L^X} (A \circ \Phi(A))\).

(2) Since \(\sqcup_l \Phi\) is a left join of \(\Phi\) iff
\[
\bigwedge_{A \in L^X} (\Phi(A) \rightarrow e^l_{L^X}(A, B)) = e^l_{L^X} \left( \bigvee_{A \in L^X} (\Phi(A) \circ A), B) = e^l_{L^X}(\sqcup_l \Phi, B), \right.
\]
then \(\sqcup_l \Phi = \bigvee_{A \in L^X} (\Phi(A) \circ A)\).

3. Right and Left Rough Sets Induced by Functions

Definition 11. Let \((L^X, e^r_{L^X}, e^l_{L^X})\) be a bi-partial ordered set and \(M = \{\{a/x\} \mid a \in L, a \neq \perp, x \in X\}\) be the set of all singletons. Let \(\phi: M \to L^X\)

\[
\rho_r(A, B) = \bigvee_{x \in X} (A(x) \circ B(x)), \rho_l(A, B) = \bigvee_{x \in X} (B(x) \circ A(x)).
\]

\[
N_r(A)(x) = \bigvee_{\{a/x\} \in M} (a \circ e^r_{L^X}(\phi(\{a/x\}), A)).
\]

\[
N_l(A)(x) = \bigvee_{\{a/x\} \in M} (e^l_{L^X}(\phi(\{a/x\}), A) \circ a).
\]

\[
H_r(A)(x) = \bigvee_{\{a/x\} \in M} (a \circ \rho_r(\phi(\{a/x\}), A)).
\]

\[
N_l(A)(x) = \bigvee_{\{a/x\} \in M} (\rho_l(\phi(\{a/x\}), A) \circ a).
\]

If \(N_r(A) = H_r(A)\) (resp. \(N_l(A) = H_l(A)\)), then \(A\) is a right (resp. left) definable set. The pair \((N_r(A), H_r(A))\) (resp. \((N_l(A), H_l(A))\)) is called a right (resp. left) rough set.
Theorem 12.  (1) $N_r(\top_X) = \top_X$ and $N_l(\top_X) = \top_X$ where $\top_X(x) = \top$ for all $x \in X$.

(2) $H_r(\bot_X) = \bot_X$ and $H_l(\bot_X) = \bot_X$ where $\bot_X(x) = \bot$ for all $x \in X$.

(3) $\rho_r(A, B) \odot e^r_{LX}(B, C) \leq \rho_r(A, C)$ and $e^l_{LX}(B, C) \odot \rho_l(A, B) \leq \rho_l(A, C)$.

(4) $e^r_{LX}(A, B) \leq e^r_{LX}(N_r(A), N_r(B))$ and $e^l_{LX}(A, B) \leq e^l_{LX}(N_l(A), N_l(B))$.

(5) $e^r_{LX}(A, B) \leq e^r_{LX}(H_r(A), H_r(B))$ and $e^l_{LX}(A, B) \leq e^l_{LX}(H_l(A), H_l(B))$.

Proof. (1) and (2)

$$N_r(\top_X)(x) = \bigvee_{\{a/x\} \in M}(a \odot e^r_{LX}(\phi(\{a/x\}), \top_X))$$
$$= \bigvee_{\{a/x\} \in M}(a \odot \top) = \top.$$

$$H_r(\bot_X)(x) = \bigvee_{\{a/x\} \in M}(a \odot \rho_r(\phi(\{a/x\}), \bot_X))$$
$$= \bigvee_{\{a/x\} \in M}(a \odot \bot) = \bot.$$

Other cases are similarly proved.

(3)

$$\rho_r(A, B) \odot e^r_{LX}(B, C) = \bigvee_{x \in X}(A(x) \odot B(x)) \odot \bigwedge_{y \in Y}(B(y) \Rightarrow C(y))$$
$$\leq \bigvee_{x \in X}(A(x) \odot B(x)) \odot (B(x) \Rightarrow C(x))$$
$$\leq \bigvee_{x \in X}(A(x) \odot C(x)).$$

(4)

$$N_r(A)(x) \odot e^r_{LX}(A, B) = \bigvee_{\{a/x\} \in M}(a \odot e^r_{LX}(\phi(\{a/x\}), A)) \odot e^r_{LX}(A, B)$$
$$\leq \bigvee_{\{a/x\} \in M}(a \odot e^r_{LX}(\phi(\{a/x\}), B))$$
$$\leq N_r(B)(x).$$

So, $e^r_{LX}(A, B) \leq N_r(A)(x) \Rightarrow N_r(B)(x)$.

$$e^l_{LX}(A, B) \odot N_l(A)(x) = e^l_{LX}(A, B) \odot \bigvee_{\{a/x\} \in M}(e^l_{LX}(\phi(\{a/x\}), A) \odot a)$$
$$= \bigvee_{\{a/x\} \in M}((e^l_{LX}(A, B) \odot e^l_{LX}(\phi(\{a/x\}), A)) \odot a)$$
$$= \bigvee_{\{a/x\} \in M}(e^l_{LX}(\phi(\{a/x\}), B)) \odot a$$
$$\leq N_l(B)(x).$$

So, $e^l_{LX}(A, B) \leq N_l(A)(x) \Rightarrow N_l(B)(x)$. 

(5)
\[ H_r(A)(x) \odot e_{rX}(A, B) \]
\[ = \bigvee_{a/x \in M} (a \odot \rho_r(\phi(\{a/x\}), A)) \odot e_{rX}(A, B) \]
\[ \leq \bigvee_{a/x \in M} (a \odot \bigvee_{y \in X} (\phi(\{a/x\})(y) \odot A(y)) \odot (A(y) \Rightarrow B(y))) \]
\[ \leq \bigvee_{a/x \in M} (a \odot \bigvee_{y \in X} (\phi(\{a/x\})(y) \odot B(y))). \]
\[ = H_r(B)(x). \]

So, \( e_{rX}(A, B) \leq H_r(A)(x) \Rightarrow H_r(B)(x). \)

(6)
\[ e_{lX}(A, B) \odot H_l(A)(x) \]
\[ = e_{lX}(A, B) \odot \bigvee_{a/x \in M} (\rho_l(\phi(\{a/x\}), A) \odot a) \]
\[ = \bigvee_{a/x \in M} \bigvee_{y \in X} ((A(y) \Rightarrow B(y)) \odot A(y) \odot \phi(\{a/x\})(y)) \odot a \]
\[ = \bigvee_{a/x \in M} (\rho_l(\phi(\{a/x\}), B) \odot a) \]
\[ \leq H_l(B)(x). \]

So, \( e_{lX}(A, B) \leq H_l(A)(x) \rightarrow H_l(B)(x). \)

\[ \square \]

**Theorem 13.** Let \( \phi(\{a/x\}) = \{a/x\} \). Then

(1) \( N_r(A) = A \) and \( N_l(A) = A \).

(2) \( A \leq H_r(A) \) and \( A \leq H_l(A) \).

**Proof.** (1)

\[ N_r(A)(x) = \bigvee_{a/x \in M} (a \odot e_{rX}(\phi(\{a/x\}), A)) \]
\[ = \bigvee_{a/x \in M} (a \odot (a \Rightarrow A(x))) \]
\[ \leq A(x). \]

\[ N_r(A)(x) = \bigvee_{a/x \in M} (a \odot e_{rX}(\phi(\{a/x\}), A)) \]
\[ \geq A(x) \odot e_{rX}(\phi(\{A(x)/x\}), A)) = A(x) \odot \top = A(x). \]

Hence \( N_r(A) = A \). Similarly, \( N_r(A) = A \).

(2)
\[ H_r(A)(x) = \bigvee_{a/x \in M} (a \odot \rho_r(\phi(\{a/x\}), A)) \]
\[ = \bigvee_{a/x \in M} \bigvee_{x \in X} (a \odot (a \odot A(x))) \]
\[ \geq \top \odot (\top \odot A(x)) = A(x). \]

Similarly, \( A \leq H_l(A) \).

\[ \square \]
Theorem 14. Let $\phi(\{a/x\}) = C$ with $C \in L^X$. Then

(1) $N_r(A) = e^r_{LX}(C, A)$ and $N_l(A) = e^l_{LX}(C, A)$.

(2) $H_r(A) = \rho_r(C, A)$ and $H_l(A) = \rho_r(C, A)$.

Proof. (1)

$N_r(A)(x) = \bigvee_{\{a/x\} \in M}(a \land e^r_{LX}(\phi(\{a/x\}), A))$

$= \bigvee_{\{a/x\} \in M}(a \land e^r_{LX}(C, A)) = e^r_{LX}(C, A)$.

(2)

$H_r(A)(x) = \bigvee_{\{a/x\} \in M}(a \land \rho_r(\phi(\{a/x\}), A))$

$= \bigvee_{\{a/x\} \in M}(a \land \rho_r(C, A)) = \rho_r(C, A)$.

Other cases are similarly proved. \hfill \Box

Theorem 15. Let $R \in L^{X \times X}$ and define $\phi(\{a/x\}) = [x]_R$ where $[x]_R(y) = R(x, y)$. Then

(1) $N_r(A)(x) = \bigwedge_{y \in X}(R(x, y) \Rightarrow A(y))$ and $N_l(A)(x) = \bigwedge_{y \in X}(R(x, y) \rightarrow A(y))$.

(2) $H_r(A)(x) = \bigvee_{y \in X}(R(x, y) \land A(y))$ and $H_l(A)(x) = \bigvee_{y \in X}(A(y) \land R(x, y))$.

(3) If $R(x, y) = R(y, x)$, then $H_r(N_r(A)) \leq A$ and $H_l(N_l(A)) \leq A$.

(4) If $R(x, y) = R(y, x)$, then $N_r(H_r(A)) \geq A$ and $N_l(H_l(A)) \geq A$.

Proof. (1)

$N_r(A)(x) = \bigvee_{\{a/x\} \in M}(a \land e^r_{LX}(\phi(\{a/x\}), A))$

$= \bigvee_{\{a/x\} \in M}(a \land e^r_{LX}([x]_R, A)) = e^r_{LX}([x]_R, A)$

$= \bigwedge_{y \in X}(R(x, y) \Rightarrow A(y))$.

$N_l(A)(x) = \bigvee_{\{a/x\} \in M}(e^l_{LX}(\phi(\{a/x\}), A) \land a)$

$= \bigvee_{\{a/x\} \in M}(e^l_{LX}([x]_R, A) \land a) = e^l_{LX}([x]_R, A)$

$= \bigwedge_{y \in X}(R(x, y) \rightarrow A(y))$.

(2)

$H_r(A)(x) = \bigvee_{\{a/x\} \in M}(a \land \rho_r(\phi(\{a/x\}), A))$

$= \bigvee_{\{a/x\} \in M}(a \land \rho_r([x]_R, A)) = \rho_r([x]_R, A)$

$= \bigvee_{y \in X}(R(x, y) \land A(y))$. 
Similarly, $H_l(N_l(A)) \leq A$.

(4)

\[
H_r(N_r(A))(x) = \bigwedge_{y \in X} (R(x, y) \Rightarrow H_r(A)(y))
\]
\[
= \bigwedge_{y \in X} (R(x, y) \Rightarrow \bigvee_{w \in X} (R(y, w) \Rightarrow A(w)))
\]
\[
\geq \bigwedge_{y \in X} (R(x, y) \Rightarrow R(y, x) \Rightarrow A(x))
\]
\[
\geq A(x).
\]

Similarly, $N_l(H_l(A)) \geq A$. 

Example 16. Let $K = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$ be a set and we define an operation $\otimes : K \times K \to K$ as follows:

\[
(x_1, y_1) \otimes (x_2, y_2) = (x_1 x_2, x_1 y_2 + y_1).
\]

Then $(K, \otimes)$ is a group with $e = (1, 0)$, $(x, y)^{-1} = (\frac{1}{x}, -\frac{y}{x})$.

We have a positive cone $P = \{(a, b) \in \mathbb{R}^2 \mid a = 1, b \geq 0$, or $a > 1\}$ because $P \cap P^{-1} = \{(1, 0)\}$, $P \circ P \subset P$, $(a, b)^{-1} \circ P \circ (a, b) = P$ and $P \cup P^{-1} = K$. For $(x_1, y_1), (x_2, y_2) \in K$, we define

\[
(x_1, y_1) \leq (x_2, y_2) \iff (x_1, y_1)^{-1} \circ (x_2, y_2) \in P, (x_2, y_2) \circ (x_1, y_1)^{-1} \in P
\]
\[
\iff x_1 < x_2 \text{ or } x_1 = x_2, y_1 \leq y_2.
\]

Then $(K, \leq \otimes)$ is a lattice-group.

The structure $(L, \odot, \rightarrow, \rightarrow, (\frac{1}{2}, 1), (1, 0))$ is a generalized residuated lattice with strong negation where $\bot = (\frac{1}{2}, 1)$ is the least element and $\top = (1, 0)$ is the greatest element if we define:

\[
(x_1, y_1) \odot (x_2, y_2) = (x_1, y_1) \odot (x_2, y_2) \vee \left(\frac{1}{2}, 1\right) = (x_1 x_2, x_1 y_2 + y_1) \vee \left(\frac{1}{2}, 1\right),
\]
\[
(x_1, y_1) \Rightarrow (x_2, y_2) = ((x_1, y_1)^{-1} \odot (x_2, y_2)) \wedge (1, 0) = \left(\frac{x_2}{x_1}, \frac{y_2-y_1}{x_1}\right) \wedge (1, 0),
\]
\[
(x_1, y_1) \rightarrow (x_2, y_2) = ((x_2, y_2) \odot (x_1, y_1)^{-1}) \wedge (1, 0) = \left(\frac{x_2}{x_1}, -\frac{x_2 y_1}{x_1} + y_2\right) \wedge (1, 0).
\]
We have \((x, y) = (x, y)^* = (x, y)^{op}\) from:

\[
(x, y)^* = (x, y) \Rightarrow \left( \frac{1}{2}, 1 \right) = \left( \frac{1}{2x}, \frac{1-y}{x} \right),
\]

\[
(x, y)^{op} = \left( \frac{1}{2x}, \frac{1-y}{x} \right) \Rightarrow \left( \frac{1}{2}, 1 \right) = (x, y).
\]

Let \(X = \{a, b, c\}\) be a set. Define \((R_1(a, b)), (R_2(a, b)) \in L^X \times X\) as

\[
R_1 = \begin{pmatrix}
(1, 0) & (\frac{5}{8}, 2) & (\frac{5}{6}, \frac{5}{3}) \\
(\frac{5}{7}, \frac{30}{7}) & (1, 0) & (\frac{5}{7}, \frac{5}{3}) \\
(1, -2) & (\frac{5}{7}, \frac{10}{3}) & (1, 0)
\end{pmatrix}

R_2 = \begin{pmatrix}
(1, 0) & (\frac{2}{3}, 5) & (\frac{5}{6}, 1) \\
(\frac{2}{7}, 5) & (1, 0) & (\frac{3}{7}, 4) \\
(\frac{3}{6}, 1) & (\frac{3}{7}, 4) & (1, -2)
\end{pmatrix}
\]

(1) For \(\phi(\{a/x\}) = [x]_{R_1}\) where \([x]_{R_1}(y) = R_1(x, y)\) and

\[
A = ((\frac{3}{4}, 1), (\frac{5}{6}, 2), (\frac{3}{5}, 0))^t,
\]

\[
N_r(A)(a) = \bigvee_{\{k/x\} \in M}(k \odot e_{L,X}^{r}([x]_{R_1}, A))
\]

\[
N_r(A)(b) = \left(\frac{5}{6}, 2\right), \quad N_r(A)(c) = \left(\frac{3}{5}, 0\right)
\]

\[
H_r(N_r(A)) = \left(\frac{18}{25}, -2\right), \left(\frac{6}{5}, 2\right), \left(\frac{18}{25}, -4\right))^t \not\subseteq A.
\]

\[
N_l(A)(a) = \bigvee_{\{k/x\} \in M}(e_{L,X}^{l}(\{k/x\}, A) \odot k)
\]

\[
N_l(A)(b) = \left(\frac{5}{6}, 2\right), \quad N_l(A)(c) = \left(\frac{3}{5}, 0\right)
\]

\[
H_l(N_l(A)) = \left(\frac{18}{25}, -\frac{6}{5}\right), \left(\frac{5}{6}, 2\right), \left(\frac{18}{25}, \frac{86}{25}\right))^t \not\subseteq A.
\]

\[
H_r(A) = \left(\frac{3}{4}, 1\right), \left(\frac{5}{6}, 2\right), \left(\frac{3}{4}, -\frac{7}{5}\right))^t
\]

\[
H_l(A) = \left(\frac{3}{4}, 1\right), \left(\frac{5}{6}, 2\right), \left(\frac{3}{4}, -\frac{1}{2}\right))^t
\]

\[
N_r(H_r(A)) = H_r(A), \quad N_l(H_l(A)) = H_l(A).
\]

(2) For \(\phi(\{a/x\}) = [x]_{R_2}\) where \([x]_{R_2}(y) = R_2(x, y)\) and

\[
A = ((\frac{3}{4}, 1), (\frac{5}{6}, 2), (\frac{3}{5}, 0))^t,
\]

\[
N_r(A) = ((\frac{3}{4}, 1), (\frac{5}{6}, 2), (\frac{3}{5}, 2))^t
\]

Y.C. Kim
$$H_r(N_r(A)) = ((\frac{3}{4}, 1), (\frac{5}{6}, 2), (\frac{3}{5}, 0))^t = A,$$
$$N_l(A) = ((\frac{3}{4}, 1), (\frac{5}{6}, 2), (\frac{3}{5}, \frac{6}{5}))^t,$$
$$H_l(N_l(A)) = ((\frac{3}{4}, 1), (\frac{5}{6}, 2), (\frac{3}{5}, 0))^t = A,$$
$$H_r(A) = ((\frac{3}{4}, 1), (\frac{5}{6}, 2), (\frac{3}{5}, -2))^t,$$
$$N_r(H_r(A)) = ((\frac{3}{4}, 1), (\frac{5}{6}, 2), (\frac{3}{5}, 0))^t = A,$$
$$H_l(A) = ((\frac{3}{4}, 1), (\frac{5}{6}, 2), (\frac{3}{5}, -\frac{6}{5}))^t,$$
$$N_l(H_l(A)) = ((\frac{3}{4}, 1), (\frac{5}{6}, 2), (\frac{3}{5}, 0))^t = A.$$

**Theorem 17.** Let \((L^X, e^r_{L^X}, e^l_{L^X})\) be a bi-partial ordered set.

1. \(R = \{H_r(A) \mid A \in L^X\}\) is a right join complete lattice.
2. \(S = \{H_l(A) \mid A \in L^X\}\) is a left join complete lattice.
3. \(\rho_r(A, \sqcap_r \Phi) = \bigvee_{B \in X} \rho_r(A, B) \circ \Phi(B)\).
4. \(\rho_l(A, \sqcap_l \Phi) = \bigvee_{B \in X} \Phi(B) \circ \rho_l(A, B)\).

**Proof.**

1. For \(\Phi \in L^{L^X}\),

\[
\sqcap_r \Phi(x) = \bigvee_{A \in L^X} (H_r(A)(x) \circ \Phi(H_r(A))
= \bigvee_{A \in L^X} (\bigvee_{\{a/x\} \in M} (a \circ \rho_r(\phi(\{a/x\}), A)) \circ \Phi(H_r(A))
= \bigvee_{A \in L^X} (\bigvee_{\{a/x\} \in M} (a \circ \bigvee_{y \in X} (\phi(\{a/x\})(y) \circ A(y))) \circ \Phi(H_r(A))
= \bigvee_{A \in L^X} (\bigvee_{\{a/x\} \in M} (a \circ \bigvee_{y \in X} (\phi(\{a/x\})(y) \circ A(y)) \circ \Phi(H_r(A))
= \bigvee_{A \in L^X} (\bigvee_{\{a/x\} \in M} (a \circ \bigvee_{y \in X} (\phi(\{a/x\})(y) \circ E(y))) = H_r(E)(x).
\]

where \(E = A \circ \Phi(H_r(A))\). Hence \(\sqcap_r \Phi \in R\).

2. For \(\Phi \in L^{L^X}\),

\[
\sqcap_l \Phi(x) = \bigvee_{A \in L^X} (\Phi(H_l(A) \circ H_l(A)(x))
= \bigvee_{A \in L^X} (\Phi(H_l(A) \circ \bigvee_{\{a/x\} \in M} (\rho_l(\phi(\{a/x\}), A) \circ a))
= \bigvee_{A \in L^X} \bigvee_{\{a/x\} \in M} (\Phi(H_l(A)) \circ \bigvee_{y \in X} (A(y) \circ \phi(\{a/x\})(y)) \circ a)
= \bigvee_{\{a/x\} \in M} \bigvee_{y \in X} (\bigvee_{A \in L^X} (\Phi(H_l(A) \circ A(y)) \circ \phi(\{a/x\})(y)) \circ a)
= \bigvee_{\{a/x\} \in M} \bigvee_{y \in X} (D(y) \circ \phi(\{a/x\})(y)) \circ a)
= \bigvee_{\{a/x\} \in M} (\rho_l(\phi(\{a/x\}), D) \circ a)
= H_l(D)(x).
\]
where $D = \Phi(H_l(A)) \odot A$. Hence $\sqcup_l \Phi \in S$.

(3) 
\[
\rho_r(A, \sqcup_r \Phi) = \bigvee_{x \in X} (A(x) \odot \bigvee_{B \in X} (B(x) \odot \Phi(B)) \\
= \bigvee_{B \in X} (\bigvee_{x \in X} (A(x) \odot B(x)) \odot \Phi(B)) \\
= \bigvee_{B \in X} \rho_r(A, B) \odot \Phi(B).
\]

(4) 
\[
\rho_l(A, \sqcup_l \Phi) = \bigvee_{x \in X} (\sqcup_l \Phi(x) \odot A(x)) \\
= \bigvee_{x \in X} (\bigvee_{B \in X} (\Phi(B) \odot B(x)) \odot A(x)) \\
= \bigvee_{B \in X} \Phi(B) \odot \rho_l(A, B).
\]

\[\square\]

**Theorem 18.** (1) If $\{a/x\} \preceq \phi(\{a/x\})$, then $N_r(A) \leq A \leq H_r(A)$ and $N_l(A) \leq A \leq H_l(A)$.

(2) If $\phi(\{a/x\}) \preceq \{a/x\}$, then $N_r(A) \leq A \leq H_r(A)$ and $N_l(A) \leq A \leq H_l(A)$.

(3) If $a \circ \phi(\{a/x\})(y) \odot b \preceq \phi(\{b/y\})(x)$, then $H_r(N_r(A)) \leq A$ for $A \in L^X$.

(4) If $b \circ \phi(\{a/x\})(y) \odot a \preceq \phi(\{b/y\})(x)$, then $H_l(N_l(A)) \leq A$ for $A \in L^X$.

(5) If $\phi(\{a/x\})(y) \preceq b \circ \phi(\{b/y\})(x)$, then $N_r(H_r(A)) \geq A$ for $A \in L^X$.

(6) If $\phi(\{a/x\})(y) \preceq \phi(\{b/y\})(x) \circ b$, then $N_l(H_l(A)) \geq A$ for $A \in L^X$.

**Proof.** (1) 
\[
N_r(A)(x) = \bigvee_{\{a/x\} \in M} (a \circ e^r_{L^X}(\phi(\{a/x\}), A)) \\
\leq \bigvee_{\{a/x\} \in M} (a \circ e^r_{L^X}(\{a/x\}, A)) \\
= \bigvee_{\{a/x\} \in M} (a \circ (a \Rightarrow A(x))) \\
\leq A(x).
\]

\[
H_r(A)(x) = \bigvee_{\{a/x\} \in M} (a \circ \rho_r(\phi(\{a/x\}), A)) \\
\geq A(x) \circ e^r_{L^X}(\{A(x)/x\}, A) = A(x) \circ \top = A(x).
\]

Similarly, $N_l(A) \leq A \leq H_l(A)$.

(2) 
\[
H_r(A)(x) = \bigvee_{\{a/x\} \in M} (a \circ \rho_r(\phi(\{a/x\}), A)) \\
\leq \bigvee_{\{a/x\} \in M} (a \circ \rho_r(\{a/x\}, A)) = \top \circ A(x) = A(x).
\]
Similarly, \( H_l(A) \leq A \leq N_l(A) \).

\[
N_l(A)(x) = V_{\left\{ a/x \right\} \in M}(e_{L,X}^r(\phi(\{a/x\}), A)) \\
\geq V_{\left\{ a/x \right\} \in M}(e_{L,X}^r(\{a/x\}, A)) \\
= V_{\left\{ a/x \right\} \in M}(a \circ (\Rightarrow A(x))) \\
\geq A(x) \circ (A(x) \Rightarrow A(x)) = A(x).
\]

(4)

\[
H_l(A)(x) = V_{\left\{ a/x \right\} \in M}(\rho_l(\phi(\{a/x\}), A) \circ a) \\
\geq A(x) \circ \{T/x\}(x) \circ T = A(x) \circ T = A(x).
\]

(3)

\[
H_r(N_r(A))(x) \\
= V_{\left\{ a/x \right\} \in M}(a \circ \rho_r(\phi(\{a/x\}), N_r(A))) \\
= V_{\left\{ a/x \right\} \in M}(a \circ V_{y \in X}(\phi(\{a/x\})(y) \circ V_{\left\{ b/y \right\} \in M}(b \circ e_{L,X}^r(\phi(\{b/y\}), A)))) \\
= V_{\left\{ a/x \right\} \in M} V_{y \in X} V_{\left\{ b/y \right\} \in M}(a \circ (\phi(\{a/x\})(y) \circ (b \circ e_{L,X}^r(\phi(\{b/y\}), A)))) \\
\leq V_{\left\{ a/x \right\} \in M} V_{y \in X} V_{\left\{ b/y \right\} \in M}(\phi(\{b/y\})(x) \circ e_{L,X}^r(\phi(\{b/y\}), A)) \\
\leq V_{\left\{ a/x \right\} \in M} V_{y \in X} V_{\left\{ b/y \right\} \in M}(\phi(\{b/y\})(x) \circ (\phi(\{b/y\})(x) \Rightarrow A(x))) \\
\leq A(x).
\]

(4)

\[
H_l(N_l(A))(x) \\
= V_{\left\{ a/x \right\} \in M}(\rho_l(\phi(\{a/x\}), N_l(A)) \circ a) \\
= V_{\left\{ a/x \right\} \in M}(V_{y \in X}(V_{\left\{ b/y \right\} \in M}(e_{L,X}^r(\phi(\{b/y\}), A) \circ b) \circ \phi(\{a/x\})(y) \circ a)) \\
= V_{\left\{ a/x \right\} \in M} V_{y \in X} V_{\left\{ b/y \right\} \in M}(e_{L,X}^r(\phi(\{b/y\}, A) \circ b) \circ \phi(\{a/x\})(y) \circ a) \\
\leq V_{\left\{ a/x \right\} \in M} V_{y \in X} V_{\left\{ b/y \right\} \in M}(e_{L,X}^r(\phi(\{b/y\}, A) \circ \phi(\{a/x\})(y))) \\
\leq A(x).
\]

(5)

\[
N_r(H_r(A))(x) \\
= V_{\left\{ a/x \right\} \in M}(a \circ e_{L,X}^r(\phi(\{a/x\}), H_r(A))) \\
= V_{\left\{ a/x \right\} \in M}(a \circ V_{y \in X}(\phi(\{a/x\})(y) \Rightarrow V_{\left\{ b/y \right\} \in M}(b \circ \rho_r(\phi(\{b/y\}), A)))) \\
\geq V_{\left\{ a/x \right\} \in M} V_{y \in X} V_{\left\{ b/y \right\} \in M}(a \circ (\phi(\{a/x\})(y) \Rightarrow (b \circ \phi(\{b/y\})(x) \circ A(x)))) \\
\geq V_{\left\{ a/x \right\} \in M} V_{y \in X} V_{\left\{ b/y \right\} \in M}(a \circ (\phi(\{a/x\})(y) \Rightarrow (\phi(\{a/x\})(y) \circ A(x)))) \\
\geq V_{\left\{ a/x \right\} \in M}(a \circ A(x)) = A(x).
\]
By Theorem 18(1), $H$ is a left closure operator.

So, $H(N)$ is a right closure operator.

By Theorem 18(3), then $H \circ N : L^X \to L^X$ is a right closure operator.

Thus, $H : L^X \to L^X$ is a right closure operator.

(1) (I1) By Theorem 18(1), $H_r(N_r(A)) \leq A$.

(I2) By (I1), since $N_r$ is an increasing function, $N_r(H_r(N_r(A))) \leq N_r(A)$. By Theorem 18(3), $N_r(H_r(N_r(A))) \geq N_r(A)$. Hence $N_r(H_r(N_r(A))) = N_r(A)$.

So, $H_r(N_r(H_r(N_r(A)))) = H_r(N_r(A))$.

(RI) $e_{L^X}^r(A, B) \leq e_{L^X}^r(N_r(A), N_r(B)) \leq e_{L^X}^r(H_r(N_r(A)), H_r(N_r(A)))$.

Thus $H_r \circ N_r : L^X \to L^X$ is a right interior operator.

(C1) By Theorem 18(1), $N_r(H_r(A)) \geq A$.

(C2) By (C1), since $H_r$ is an increasing function, $H_r(N_r(H_r(A))) \geq H_r(A)$. By Theorem 18(1), $H_r(N_r(H_r(A))) \leq H_r(A)$. Hence $H_r(N_r(H_r(A))) = H_r(A)$.

So, $N_r(H_r(N_r(H_r(A)))) = N_r(H_r(A))$.

(RC) $e_{L^X}^r(A, B) \leq e_{L^X}^r(H_r(A), H_r(B)) \leq e_{L^X}^r(N_r(H_r(A)), N_r(H_r(A)))$.

Thus, $N_r \circ H_r : L^X \to L^X$ is a right closure operator.

(2) It is similarly proved as (1). \qed

From Theorems 15 and 19, we obtain the following corollary.

**Corollary 20.** Let $R \in L^X \times X$ and define $\phi(\{a/x\}) = [x]_R$ where $[x]_R(y) = R(x, y)$. If $R(x, y) = R(y, x)$, then $H_r \circ N_r$ (resp. $H_l \circ N_l$) is a right (resp. left) interior operator and $N_r \circ H_r$ (resp. $N_l \circ H_l$) is a right (resp. left) closure operator.
References


