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RIGHT AND LEFT ROUGH SETS INDUCED BY FUNCTIONS

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Abstract: In this paper, we investigate the properties of right and left rough sets induced by functions on a generalized residuated lattice. In particular, we construct right (left) closure (interior) operators.

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1. Introduction

Pawlak [8,9] introduced the rough set as a mathematical tool for dealing information as the uncertainty of data in engineering, physics, computer sciences and many other diverse field. Recently, Bělohlávek [1-3] investigate the properties of fuzzy rough set on a residuated lattice which supports part of foundation of theoretic computer science. Chen and Li [3] introduced fuzzy rough sets induced by functions. Georgescu and Popescue [4.5] introduced non-commutative fuzzy Galois connection in a generalized residuated lattice which is induced by two implications.

In this paper, we investigate the properties of right and left rough sets induced by functions on a generalized residuated lattice. In particular, we construct right (left) closure (interior) operators.

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2. Preliminaries

Definition 1. (see [4,5]) A structure $(L, \vee, \wedge, \odot, \rightarrow, \Rightarrow, \perp, \top)$ is called a *generalized residuated lattice* if it satisfies the following conditions:

(GR1) $(L, \vee, \wedge, \top, \perp)$ is a bounded lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(GR2) $(L, \odot, \top)$ is a monoid;

(GR3) it satisfies a residuation , i.e.

$$a \odot b \leq c \text{ iff } a \leq b \rightarrow c \text{ iff } b \leq a \Rightarrow c.$$

We call that a generalized residuated lattice has the law of double negation if $a = (a^*)^0 = (a^0)^*$ where $a^0 = a \rightarrow \perp$ and $a^* = a \Rightarrow \perp$.

Remark 2. (see [1-7]) (1) A generalized residuated lattice is a residuated lattice ($\rightarrow=\Rightarrow$) iff $\odot$ is commutative.

(2) A left-continuous t-norm $([0,1], \leq, \odot)$ defined by $a \rightarrow b = \bigvee\{c \mid a \odot c \leq b\}$ is a residuated lattice xms

(3) A pseudo MV-algebra is a generalized residuated lattice with the law of double negation.

In this paper, we assume $(L, \wedge, \vee, \odot, \rightarrow, \Rightarrow, \perp, \top)$ is a complete generalized residuated lattice with the law of double negation.

Lemma 3. (see [4,5]) *For each $x, y, z, x_i, y_i \in L$, we have the following properties.*

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$ for $\rightarrow \in \{\rightarrow, \Rightarrow\}$. xms

(2) $x \odot y \leq x \wedge y \leq x \vee y$.

(3) $x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$ for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

(4) $x \rightarrow (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \rightarrow y_i)$, for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

(5) $(\bigwedge_{i\in\Gamma} x_i) \rightarrow y \geq \bigvee_{i\in\Gamma}(x_i \rightarrow y)$, for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

(6) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$ and $(x \odot y) \Rightarrow z = y \Rightarrow (x \Rightarrow z)$.

(7) $x \rightarrow (y \Rightarrow z) = y \Rightarrow (x \rightarrow z)$ and $x \Rightarrow (y \rightarrow z) = y \rightarrow (x \Rightarrow z)$.

(8) $x \odot (x \rightarrow y) \leq y$ and $(x \Rightarrow y) \odot x \leq y$.

(9) $(x \Rightarrow y) \odot (y \Rightarrow z) \leq x \Rightarrow z$ and $(y \rightarrow z) \odot (x \rightarrow y) \leq x \rightarrow z$.

(10) $(x \Rightarrow z) \leq (y \odot x) \Rightarrow (y \odot z)$ *and* $(x \to z) \leq (x \odot y) \to (z \odot y)$.

(11) $(x \Rightarrow y) \leq (y \Rightarrow z) \to (x \Rightarrow z)$ *and* $(y \Rightarrow z) \leq (x \Rightarrow y) \Rightarrow (x \Rightarrow z)$ *xms*
(12) $x_i \to y_i \leq (\bigwedge_{i \in \Gamma} x_i) \to (\bigwedge_{i \in \Gamma} y_i)$ *for* $\to \in \{\to, \Rightarrow\}$.

(13) $x_i \to y_i \leq (\bigvee_{i \in \Gamma} x_i) \to (\bigvee_{i \in \Gamma} y_i)$ *for* $\to \in \{\to, \Rightarrow\}$.

(14) $x \to y = \top$ *iff* $x \leq y$.

(15) $x \to y = y^0 \Rightarrow x^0$ *and* $x \Rightarrow y = y^* \to x^*$.

(16) $\bigwedge_{i \in \Gamma} x_i^* = (\bigvee_{i \in \Gamma} x_i)^*$ *and* $\bigvee_{i \in \Gamma} x_i^* = (\bigwedge_{i \in \Gamma} x_i)^*$.

(17) $\bigwedge_{i \in \Gamma} x_i^0 = (\bigvee_{i \in \Gamma} x_i)^0$ *and* $\bigvee_{i \in \Gamma} x_i^0 = (\bigwedge_{i \in \Gamma} x_i)^0$.

Definition 4. Let X be a set. A function $e_X^r : X \times X \to L$ is called:

(E1) $e_X^r(x, x) = \top$ for all $x \in X$,

(E2) If $e_X^r(x, y) = \top$ and $e_X^r(y, x) = \top$, then $x = y$,

(R) $e_X^r(x, y) \odot e_X^r(y, z) \leq e_X^r(x, z)$, for all $x, y, z \in X$.

Then e_X^r is called a *right partial order*. If e_X^l satisfies (E1), (E2) and
(L) $e_X^l(y, z) \odot e_X^l(x, y) \leq e_X^l(x, z)$, for all $x, y, z \in X$.
Then e_X^l is called a *left partial order*. The triple (X, e_X^r, e_X^l) is called a *bi-partial ordered set*.

Example 5. (1) We define two functions $e_L^r, e_L^l : L \times L \to L$ as $e_L^r(x, y) = x \Rightarrow y$ and $e_L^l(x, y) = x \to y$. Then e_L^r is a right partial order and e_L^l is a left partial order.

(2) We define two functions $e_{L^X}^r, e_{L^X}^l : L^X \times L^X \to L$ as

$$e_{L^X}^r(A, B) = \bigwedge_{x \in X} (A(x) \Rightarrow B(x)),\ e_{L^X}^l(A, B) = \bigwedge_{x \in X} (A(x) \to B(x)).$$

By Lemma 3(9), $e_{L^X}^r$ is a right partial order and $e_{L^X}^l$ is a left partial order.

Definition 6. An operator $I^r : L^X \to L^X$ is called a *right interior operator* on X if it satisfies the following conditions:

(I1) $I^r(A) \leq A$, xms
(I2) $I^r(I^r(A)) \geq I^r(A)$,

(RI) $e_{L^X}^r(A, B) \leq e_{L^X}^r(I^r(A), I^r(B))$.

The pair (X, I^r) is called a right interior space.

An operator $I^l : L^X \to L^X$ is called a *left interior operator* on X if it satisfies the conditions (I1),(I2), and

(LI) $e^l_{L^X}(A,B) \leq e^l_{L^X}(I^l(A), I^l(B))$.

An operator $C^r : L^X \to L^X$ is called a *right closure operator* on X if it satisfies the following conditions:

(C1) $C^r(A) \geq A$,

(C2) $C^r(C^r(A)) \leq C^r(A)$,

(RC) $e^r_{L^X}(A,B) \leq e^r_{L^X}(C^r(A), C^r(B))$.

The pair (X, C^r) is called an r-closure space.

An operator $C^l : L^X \to L^X$ is called a *left closure operator* on X if it satisfies the conditions (C1),(C2) and

(RC) $e^l_{L^X}(A,B) \leq e^l_{L^X}(C^l(A), C^l(B))$.

Definition 7. Let (X, e^r_X, e^l_X) be a bi-partial ordered set and $A \in L^X$.

(1) A point x_0 is called a *right join* of A, denoted by $x_0 = \sqcup_r A$, if it satisfies

(RJ1) $A(x) \leq e^r_X(x, x_0)$,

(RJ2) $\bigwedge_{x\in X}(A(x) \Rightarrow e^r_X(x,y) \leq e^r_X(x_0, y)$.

(2) A point x_0 is called a *left join* of A, denoted by $x_0 = \sqcup_l A$, if it satisfies

(LJ1) $A(x) \leq e^l_X(x, x_0)$,

(LJ2) $\bigwedge_{x\in X}(A(x) \to e^l_X(x,y) \leq e^l_X(x_0, y)$.

An bi-partial ordered set (X, e^r_X, e^l_X) is called a *right (resp. left) join complete lattice* if $\sqcup_r A$ (resp $\sqcup_r A$) exists for $A \in L^X$.

Remark 8. Let (X, e^r_X, e^l_X) be a bi-partial ordered set and $A \in L^X$. If x_0 is a right join of A, then it is unique because $e^r_X(x_0, y) = e^r_X(y_0, y)$ for all $y \in X$, put $y = x_0$ or $y = y_0$, then $e^r_X(x_0, y_0) = e^r_X(y_0, x_0) = \top$ implies $x_0 = y_0$. Similarly, if left join of A exist, then it is unique.

Theorem 9. *Let (X, e^r_X, e^l_X) be a bi-partial ordered set and $A \in L^X$.*

(1) x_0 is a right join of A iff $\bigwedge_{x\in X}(A(x) \Rightarrow e^r_X(x,y)) = e^r_X(x_0, y)$.

(2) x_0 is a left join of A iff $\bigwedge_{x\in X}(A(x) \to e^l_X(x,y) = e^l_X(x_0, y)$.

Proof. (1) ($\Rightarrow$) Let x_0 be a right join of A. Then $A(x) \leq e^r_X(x, x_0)$. Thus, $A(x) \odot e^r_X(x_0, y) \leq e^r_X(x, x_0) \odot e^r_X(x_0, y) \leq e^r_X(x, y)$. Hence $e^r_X(x_0, y) \leq \bigwedge_{x\in X}(A(x) \Rightarrow e^r_X(x,y))$. By (RJ2), the equality holds.

($\Leftarrow$) Since $\bigwedge_{x\in X}(A(x) \Rightarrow e^r_X(x, x_0)) = e^r_X(x_0, x_0) = 1$, then $A(x) \leq e^r_X(x, x_0)$. Hence the result holds. □

Remark 10. Let $(L^X, e^r_{L^X}, e^l_{L^X})$ be a bi-partial ordered set and $A \in L^L$.

(1) Since $\sqcup_r \Phi$ is a right join of Φ iff

$$\bigwedge_{A \in L^X} (\Phi(A) \Rightarrow e^r_{L^X}(A, B)) = e^r_{L^X}(\bigvee_{A \in L^X} (A \odot \Phi(A), B) = e^r_{L^X}(\sqcup_r \Phi, B),$$

then $\sqcup_r \Phi = \bigvee_{A \in L^X} (A \odot \Phi(A))$.

(2) Since $\sqcup_l \Phi$ is a left join of Φ iff

$$\bigwedge_{A \in L^X} (\Phi(A) \rightarrow e^l_{L^X}(A, B)) = e^l_{L^X}(\bigvee_{A \in L^X} (\Phi(A) \odot A), B) = e^l_{L^X}(\sqcup_l \Phi, B),$$

then $\sqcup_l \Phi = \bigvee_{A \in L^X} (\Phi(A) \odot A)$.

3. Right and Left Rough Sets Induced by Functions

Definition 11. Let $(L^X, e^r_{L^X}, e^l_{L^X})$ be a bi-partial ordered set and $M = \{\{a/x\} \mid a \in L, a \neq \bot, x \in X\}$ be the set of all singletons. Let $\phi : M \to L^X$

$$\rho_r(A, B) = \bigvee_{x \in X} (A(x) \odot B(x)), \rho_l(A, B) = \bigvee_{x \in X} (B(x) \odot A(x)).$$

$$N_r(A)(x) = \bigvee_{\{a/x\} \in M} (a \odot e^r_{L^X}(\phi(\{a/x\}), A)).$$

$$N_l(A)(x) = \bigvee_{\{a/x\} \in M} (e^l_{L^X}(\phi(\{a/x\}), A) \odot a).$$

$$H_r(A)(x) = \bigvee_{\{a/x\} \in M} (a \odot \rho_r(\phi(\{a/x\}), A)).$$

$$N_l(A)(x) = \bigvee_{\{a/x\} \in M} (\rho_l(\phi(\{a/x\}), A) \odot a).$$

If $N_r(A) = H_r(A)$ (resp. $N_l(A) = H_l(A)$), then A is a *right (resp. left) definable set*. The pair $(N_r(A), H_r(A))$ (resp. $(N_l(A), H_l(A))$) is called a *right (resp. left) rough set*.

Theorem 12. (1) $N_r(\top_X) = \top_X$ *and* $N_l(\top_X) = \top_X$ *where* $\top_X(x) = \top$ *for all* $x \in X$.

(2) $H_r(\bot_X) = \bot_X$ *and* $H_l(\bot_X) = \bot_X$ *where* $\bot_X(x) = \bot$ *for all* $x \in X$.

(3) $\rho_r(A,B) \odot e^r_{L^X}(B,C) \leq \rho_r(A,C)$ *and* $e^l_{L^X}(B,C) \odot \rho_l(A,B) \leq \rho_l(A,C)$.

(4) $e^r_{L^X}(A,B) \leq e^r_{L^X}(N_r(A),N_r(B))$ *and* $e^l_{L^X}(A,B) \leq e^l_{L^X}(N_l(A),N_l(B))$.

(5) $e^r_{L^X}(A,B) \leq e^r_{L^X}(H_r(A),H_r(B))$ *and* $e^l_{L^X}(A,B) \leq e^l_{L^X}(H_l(A),H_l(B))$.

Proof. (1) and (2)

$$\begin{aligned} N_r(\top_X)(x) &= \bigvee_{\{a/x\}\in M}(a \odot e^r_{L^X}(\phi(\{a/x\}),\top_X)) \\ &= \bigvee_{\{a/x\}\in M}(a \odot \top) = \top. \end{aligned}$$

$$\begin{aligned} H_r(\bot_X)(x) &= \bigvee_{\{a/x\}\in M}(a \odot \rho_r(\phi(\{a/x\}),\bot_X)) \\ &= \bigvee_{\{a/x\}\in M}(a \odot \bot) = \bot. \end{aligned}$$

Other cases are simiarly proved.

(3)

$$\begin{aligned} \rho_r(A,B) \odot e^r_{L^X}(B,C) &= \bigvee_{x\in X}(A(x) \odot B(x)) \odot \bigwedge_{y\in Y}(B(y) \Rightarrow C(y)) \\ &\leq \bigvee_{x\in X}(A(x) \odot B(x)) \odot (B(x) \Rightarrow C(x)) \\ &\leq \bigvee_{x\in X}(A(x) \odot C(x)). \end{aligned}$$

(4)

$$\begin{aligned} N_r(A)(x) \odot e^r_{L^X}(A,B) &= \bigvee_{\{a/x\}\in M}(a \odot e^r_{L^X}(\phi(\{a/x\}),A)) \odot e^r_{L^X}(A,B) \\ &\leq \bigvee_{\{a/x\}\in M}(a \odot e^r_{L^X}(\phi(\{a/x\}),B)) \\ &\leq N_r(B)(x). \end{aligned}$$

So, $e^r_{L^X}(A,B) \leq N_r(A)(x) \Rightarrow N_r(B)(x)$.

$$\begin{aligned} e^l_{L^X}(A,B) \odot N_l(A)(x) &= e^l_{L^X}(A,B) \odot \bigvee_{\{a/x\}\in M}(e^l_{L^X}(\phi(\{a/x\}),A) \odot a) \\ &= \bigvee_{\{a/x\}\in M}((e^l_{L^X}(A,B) \odot e^l_{L^X}(\phi(\{a/x\}),A)) \odot a) \\ &= \bigvee_{\{a/x\}\in M}(e^l_{L^X}(\phi(\{a/x\}),B)) \odot a) \\ &\leq N_l(B)(x). \end{aligned}$$

So, $e^l_{L^X}(A,B) \leq N_l(A)(x) \rightarrow N_l(B)(x)$.

(5)

$$\begin{aligned}
&H_r(A)(x) \odot e^r_{L^X}(A, B) \\
&= \bigvee_{\{a/x\}\in M}(a \odot \rho_r(\phi(\{a/x\}), A)) \odot e^r_{L^X}(A, B) \\
&\leq \bigvee_{\{a/x\}\in M}(a \odot \bigvee_{y\in X}(\phi(\{a/x\})(y) \odot A(y)) \odot (A(y) \Rightarrow B(y))) \\
&\leq \bigvee_{\{a/x\}\in M}(a \odot \bigvee_{y\in X}(\phi(\{a/x\})(y) \odot B(y))). \\
&= H_r(B)(x).
\end{aligned}$$

So, $e^r_{L^X}(A, B) \leq H_r(A)(x) \Rightarrow H_r(B)(x)$.

$$\begin{aligned}
&e^l_{L^X}(A, B) \odot H_l(A)(x) \\
&= e^l_{L^X}(A, B) \odot \bigvee_{\{a/x\}\in M}(\rho_l(\phi(\{a/x\}), A) \odot a) \\
&= \bigvee_{\{a/x\}\in M} \bigvee_{y\in X}((A(y) \rightarrow B(y)) \odot A(y) \odot \phi(\{a/x\})(y))) \odot a) \\
&= \bigvee_{\{a/x\}\in M}(\rho_l(\phi(\{a/x\}), B)) \odot a) \\
&\leq H_l(B)(x).
\end{aligned}$$

So, $e^l_{L^X}(A, B) \leq H_l(A)(x) \rightarrow H_l(B)(x)$. □

Theorem 13. *Let $\phi(\{a/x\}) = \{a/x\}$. Then*

(1) $N_r(A) = A$ and $N_l(A) = A$.

(2) $A \leq H_r(A)$ and $A \leq H_l(A)$.

Proof. (1)

$$\begin{aligned}
N_r(A)(x) &= \bigvee_{\{a/x\}\in M}(a \odot e^r_{L^X}(\phi(\{a/x\}), A)) \\
&= \bigvee_{\{a/x\}\in M}(a \odot (a \Rightarrow A(x))) \\
&\leq A(x).
\end{aligned}$$

$$\begin{aligned}
N_r(A)(x) &= \bigvee_{\{a/x\}\in M}(a \odot e^r_{L^X}(\phi(\{a/x\}), A)) \\
&\geq A(x) \odot e^r_{L^X}(\phi(\{A(x)/x\}), A)) = A(x) \odot \top = A(x).
\end{aligned}$$

Hence $N_r(A) = A$. Similarly, $N_r(A) = A$.

(2)

$$\begin{aligned}
H_r(A)(x) &= \bigvee_{\{a/x\}\in M}(a \odot \rho_r(\phi(\{a/x\}), A)) \\
&= \bigvee_{\{a/x\}\in M} \bigvee_{x\in X}(a \odot (a \odot A(x))) \\
&\geq \top \odot (\top \odot A(x)) = A(x).
\end{aligned}$$

Similarly, $A \leq H_l(A)$. □

Theorem 14. *Let $\phi(\{a/x\}) = C$ with $C \in L^X$. Then*

(1) $N_r(A) = e^r_{L^X}(C, A)$ *and* $N_l(A) = e^l_{L^X}(C, A)$.

(2) $H_r(A) = \rho_r(C, A)$ *and* $H_l(A) = \rho_r(C, A)$.

Proof. (1)

$$\begin{aligned} N_r(A)(x) &= \bigvee_{\{a/x\}\in M}(a \odot e^r_{L^X}(\phi(\{a/x\}), A)) \\ &= \bigvee_{\{a/x\}\in M}(a \odot e^r_{L^X}(C, A)) = e^r_{L^X}(C, A). \end{aligned}$$

(2)

$$\begin{aligned} H_r(A)(x) &= \bigvee_{\{a/x\}\in M}(a \odot \rho_r(\phi(\{a/x\}), A)) \\ &= \bigvee_{\{a/x\}\in M}(a \odot \rho_r(C, A)) = \rho_r(C, A). \end{aligned}$$

Other cases are similarly proved. □

Theorem 15. *Let $R \in L^{X\times X}$ and define $\phi(\{a/x\}) = [x]_R$ where $[x]_R(y) = R(x, y)$. Then*

(1) $N_r(A)(x) = \bigwedge_{y\in X}(R(x, y) \Rightarrow A(y))$ *and* $N_l(A)(x) = \bigwedge_{y\in X}(R(x, y) \to A(y))$.

(2) $H_r(A)(x) = \bigvee_{y\in X}(R(x, y) \odot A(y))$ *and* $H_l(A)(x) = \bigvee_{y\in X}(A(y) \odot R(x, y))$.

(3) *If $R(x, y) = R(y, x)$, then $H_r(N_r(A)) \leq A$ and $H_l(N_l(A)) \leq A$.*

(4) *If $R(x, y) = R(y, x)$, then $N_r(H_r(A)) \geq A$ and $N_l(H_l(A)) \geq A$.*

Proof. (1)

$$\begin{aligned} N_r(A)(x) &= \bigvee_{\{a/x\}\in M}(a \odot e^r_{L^X}(\phi(\{a/x\}), A)) \\ &= \bigvee_{\{a/x\}\in M}(a \odot e^r_{L^X}([x]_R, A)) = e^r_{L^X}([x]_R, A) \\ &= \bigwedge_{y\in X}(R(x, y) \Rightarrow A(y)). \end{aligned}$$

$$\begin{aligned} N_l(A)(x) &= \bigvee_{\{a/x\}\in M}(e^l_{L^X}(\phi(\{a/x\}), A) \odot a) \\ &= \bigvee_{\{a/x\}\in M}(e^l_{L^X}([x]_R, A) \odot a) = e^l_{L^X}([x]_R, A) \\ &= \bigwedge_{y\in X}(R(x, y) \to A(y)). \end{aligned}$$

(2)

$$\begin{aligned} H_r(A)(x) &= \bigvee_{\{a/x\}\in M}(a \odot \rho_r(\phi(\{a/x\}), A)) \\ &= \bigvee_{\{a/x\}\in M}(a \odot \rho_r([x]_R, A)) = \rho_r([x]_R, A) \\ &= \bigvee_{y\in X}(R(x, y) \odot A(y)). \end{aligned}$$

$$\begin{aligned} H_l(A)(x) &= \bigvee_{\{a/x\}\in M}(\rho_l(\phi(\{a/x\}),A)\odot a) \\ &= \bigvee_{\{a/x\}\in M}(\rho_l([x]_R,A)\odot a) = \rho_l([x]_R,A) \\ &= \bigvee_{y\in X}(A(y)\odot R(x,y)). \end{aligned}$$

(3)

$$\begin{aligned} H_r(N_r(A))(x) &= \bigvee_{y\in X}(R(x,y)\odot \bigwedge_{w\in X}(R(y,w)\Rightarrow A(w))) \\ &\le \bigvee_{y\in X}(R(x,y)\odot (R(y,x)\Rightarrow A(x))) \\ &= \bigvee_{y\in X}(R(x,y)\odot (R(x,y)\Rightarrow A(x))) \\ &\le A(x). \end{aligned}$$

Similarly, $H_l(N_l(A)) \le A$.

(4)

$$\begin{aligned} N_r(H_r(A))(x) &= \bigwedge_{y\in X}(R(x,y)\Rightarrow H_r(A)(y))) \\ &= \bigwedge_{y\in X}(R(x,y)\Rightarrow \bigvee_{w\in X}(R(y,w)\odot A(w)))) \\ &\ge \bigwedge_{y\in X}(R(x,y)\Rightarrow R(y,x)\odot A(x)) \\ &\ge A(x). \end{aligned}$$

Similarly, $N_l(H_l(A)) \ge A$. □

Example 16. Let $K = \{(x,y)\in R^2 \mid x>0\}$ be a set and we define an operation $\otimes : K\times K\to K$ as follows:

$$(x_1,y_1)\otimes(x_2,y_2) = (x_1x_2, x_1y_2+y_1).$$

Then $(K,\otimes)$ is a group with $e=(1,0)$, $(x,y)^{-1} = (\frac{1}{x}, -\frac{y}{x})$.

We have a positive cone $P = \{(a,b)\in R^2 \mid a=1, b\ge 0 \text{ ,or } a>1\}$ because $P\cap P^{-1} = \{(1,0)\}$, $P\odot P\subset P$, $(a,b)^{-1}\odot P\odot (a,b) = P$ and $P\cup P^{-1} = K$. For $(x_1,y_1),(x_2,y_2)\in K$, we define

$$\begin{aligned} (x_1,y_1)\le(x_2,y_2) &\Leftrightarrow (x_1,y_1)^{-1}\odot(x_2,y_2)\in P,\ (x_2,y_2)\odot(x_1,y_1)^{-1}\in P \\ &\Leftrightarrow x_1<x_2 \text{ or } x_1=x_2, y_1\le y_2. \end{aligned}$$

Then $(K,\le\otimes)$ is a lattice-group.

The structure $(L,\odot,\Rightarrow,\rightarrow,(\frac{1}{2},1),(1,0))$ is a generalized residuated lattice with strong negation where $\bot = (\frac{1}{2},1)$ is the least element and $\top=(1,0)$ is the greatest element if we define:

$$\begin{aligned} (x_1,y_1)\odot(x_2,y_2) &= (x_1,y_1)\otimes(x_2,y_2)\vee(\tfrac{1}{2},1) = (x_1x_2,x_1y_2+y_1)\vee(\tfrac{1}{2},1), \\ (x_1,y_1)\Rightarrow(x_2,y_2) &= ((x_1,y_1)^{-1}\otimes(x_2,y_2))\wedge(1,0) = (\tfrac{x_2}{x_1},\tfrac{y_2-y_1}{x_1})\wedge(1,0), \\ (x_1,y_1)\rightarrow(x_2,y_2) &= ((x_2,y_2)\otimes(x_1,y_1)^{-1})\wedge(1,0) = (\tfrac{x_2}{x_1},-\tfrac{x_2y_1}{x_1}+y_2)\wedge(1,0). \end{aligned}$$

We have $(x,y) = (x,y)^{*\circ} = (x,y)^{\circ *}$ from:

$$(x,y)^* = (x,y) \Rightarrow (\frac{1}{2},1) = (\frac{1}{2x}, \frac{1-y}{x}),$$

$$(x,y)^{*\circ} = (\frac{1}{2x}, \frac{1-y}{x}) \to (\frac{1}{2},1) = (x,y).$$

Let $X = \{a,b,c\}$ be a set. Define $(R_1(a,b)), (R_2(a,b)) \in L^{X\times X}$ as

$$R_1 = \begin{pmatrix} (1,0) & (\frac{5}{8},\frac{5}{2}) & (\frac{5}{6},\frac{5}{3}) \\ (\frac{5}{7},\frac{30}{7}) & (1,0) & (\frac{5}{8},-\frac{5}{4}) \\ (1,-2) & (\frac{5}{7},\frac{10}{3}) & (1,0) \end{pmatrix} R_2 = \begin{pmatrix} (1,0) & (\frac{2}{3},5) & (\frac{5}{6},1) \\ (\frac{2}{3},5) & (1,0) & (\frac{6}{7},4) \\ (\frac{5}{6},1) & (\frac{6}{7},4) & (1,-2) \end{pmatrix}$$

(1) For $\phi(\{a/x\}) = [x]_{R_1}$ where $[x]_{R_1}(y) = R_1(x,y)$ and

$$A = ((\frac{3}{4},1),(\frac{5}{6},2),(\frac{3}{5},0))^t,$$

$$\begin{aligned} N_r(A)(a) &= \bigvee_{\{k/x\}\in M}(k \odot e^r_{L^X}([x]_{R_1}, A)) \\ &= e^r_{L^X}([x]_{R_1}, A) = (\tfrac{18}{25},-2), \\ N_r(A)(b) &= (\tfrac{5}{6},2), \quad N_r(A)(c) = (\tfrac{3}{5},0) \end{aligned}$$

$$H_r(N_r(A)) = ((\frac{18}{25},-2),(\frac{5}{6},2),(\frac{18}{25},-4))^t \not\leq A.$$

$$\begin{aligned} N_l(A)(a) &= \bigvee_{\{k/x\}\in M}(e^l_{L^X}([x]_{R_1}, A) \odot k) \\ &= e^l_{L^X}([x]_{R_1}, A) = (\tfrac{18}{25},-\tfrac{6}{5}), \\ N_l(A)(b) &= (\tfrac{5}{6},2), \quad N_l(A)(c) = (\tfrac{3}{5},0) \end{aligned}$$

$$H_l(N_l(A)) = ((\frac{18}{25},-\frac{6}{5}),(\frac{5}{6},2),(\frac{18}{25},-\frac{86}{25}))^t \not\leq A.$$

$$H_r(A)) = ((\frac{3}{4},1),(\frac{5}{6},2),(\frac{3}{4},-\frac{7}{5}))^t,$$

$$H_l(A)) = ((\frac{3}{4},1),(\frac{5}{6},2),(\frac{3}{4},-\frac{1}{2}))^t,$$

$$N_r(H_r(A)) = H_r(A), \; N_l(H_l(A)) = H_l(A).$$

(2) For $\phi(\{a/x\}) = [x]_{R_2}$ where $[x]_{R_2}(y) = R_2(x,y)$ and

$$A = ((\frac{3}{4},1),(\frac{5}{6},2),(\frac{3}{5},0))^t,$$

$$N_r(A) = ((\frac{3}{4},1),(\frac{5}{6},2),(\frac{3}{5},2))^t,$$

$$H_r(N_r(A)) = ((\frac{3}{4},1),(\frac{5}{6},2),(\frac{3}{5},0))^t = A,$$
$$N_l(A) = ((\frac{3}{4},1),(\frac{5}{6},2),(\frac{3}{5},\frac{6}{5}))^t,$$
$$H_l(N_l(A)) = ((\frac{3}{4},1),(\frac{5}{6},2),(\frac{3}{5},0))^t = A,$$
$$H_r(A) = ((\frac{3}{4},1),(\frac{5}{6},2),(\frac{3}{5},-2))^t,$$
$$N_r(H_r(A)) = ((\frac{3}{4},1),(\frac{5}{6},2),(\frac{3}{5},0))^t = A,$$
$$H_l(A) = ((\frac{3}{4},1),(\frac{5}{6},2),(\frac{3}{5},-\frac{6}{5}))^t,$$
$$N_l(H_l(A)) = ((\frac{3}{4},1),(\frac{5}{6},2),(\frac{3}{5},0))^t = A.$$

Theorem 17. *Let* $(L^X, e^r_{L^X}, e^l_{L^X})$ *be a bi-partial ordered set.*

(1) $R = \{H_r(A) \mid A \in L^X\}$ *is a right join complete lattice.*

(2) $S = \{H_l(A) \mid A \in L^X\}$ *is a left join complete lattice.*

(3) $\rho_r(A, \sqcup_r\Phi) = \bigvee_{B\in X} \rho_r(A,B) \odot \Phi(B)$.

(4) $\rho_l(A, \sqcup_l\Phi) = \bigvee_{B\in X} \Phi(B) \odot \rho_l(A,B)$.

Proof. (1) For $\Phi \in L^{L^X}$,

$$\begin{aligned}
\sqcup_r\Phi(x) = &\bigvee_{A\in L^X}(H_r(A)(x) \odot \Phi(H_r(A))\\
&= \bigvee_{A\in L^X}(\bigvee_{\{a/x\}\in M}(a \odot \rho_r(\phi(\{a/x\}),A)) \odot \Phi(H_r(A))\\
&= \bigvee_{A\in L^X}(\bigvee_{\{a/x\}\in M}(a \odot \bigvee_{y\in X}(\phi(\{a/x\})(y) \odot A(y))) \odot \Phi(H_r(A))\\
&= \bigvee_{A\in L^X}(\bigvee_{\{a/x\}\in M}(a \odot \bigvee_{y\in X}(\phi(\{a/x\})(y) \odot A(y))) \odot \Phi(H_r(A))\\
&= \bigvee_{A\in L^X}(\bigvee_{\{a/x\}\in M}(a \odot \bigvee_{y\in X}(\phi(\{a/x\})(y) \odot A(y) \odot \Phi(H_r(A))))\\
&= \bigvee_{A\in L^X}(\bigvee_{\{a/x\}\in M}(a \odot \bigvee_{y\in X}(\phi(\{a/x\})(y) \odot E(y))) = H_r(E)(x).
\end{aligned}$$

where $E = A \odot \Phi(H_r(A))$. Hence $\sqcup_r\Phi \in R$.

(2) For $\Phi \in L^{L^X}$,

$$\begin{aligned}
\sqcup_l\Phi(x) &= \bigvee_{A\in L^X}(\Phi(H_l(A) \odot H_l(A)(x))\\
&= \bigvee_{A\in L^X}(\Phi(H_l(A) \odot \bigvee_{\{a/x\}\in M}(\rho_l(\phi(\{a/x\}),A) \odot a))\\
&= \bigvee_{A\in L^X}\bigvee_{\{a/x\}\in M}(\Phi(H_l)(A) \odot \bigvee_{y\in X}(A(y) \odot \phi(\{a/x\})(y)) \odot a)\\
&= \bigvee_{\{a/x\}\in M}\bigvee_{y\in X}(\bigvee_{A\in L^X}(\Phi(H_l)(A) \odot A(y)) \odot \phi(\{a/x\})(y)) \odot a)\\
&= \bigvee_{\{a/x\}\in M}\bigvee_{y\in X}(D(y) \odot \phi(\{a/x\})(y)) \odot a)\\
&= \bigvee_{\{a/x\}\in M}(\rho_l(\phi(\{a/x\}),D) \odot a)\\
&= H_l(D)(x).
\end{aligned}$$

where $D = \Phi(H_l(A)) \odot A$. Hence $\sqcup_l \Phi \in S$.

(3)

$$\begin{aligned}\rho_r(A, \sqcup_r \Phi) &= \bigvee_{x\in X}(A(x) \odot \bigvee_{B\in X}(B(x) \odot \Phi(B)) \\ &= \bigvee_{B\in X}(\bigvee_{x\in X}(A(x) \odot B(x)) \odot \Phi(B)) \\ &= \bigvee_{B\in X} \rho_r(A, B) \odot \Phi(B).\end{aligned}$$

(4)

$$\begin{aligned}\rho_l(A, \sqcup_l \Phi) &= \bigvee_{x\in X}(\sqcup_l \Phi(x) \odot A(x)) \\ &= \bigvee_{x\in X}(\bigvee_{B\in X}(\Phi(B) \odot B(x)) \odot A(x)) \\ &= \bigvee_{B\in X}(\Phi(B) \odot \bigvee_{x\in X}(B(x) \odot A(x))) \\ &= \bigvee_{B\in X} \Phi(B) \odot \rho_l(A, B).\end{aligned}$$

□

Theorem 18. (1) *If* $\{a/x\} \leq \phi(\{a/x\})$, *then* $N_r(A) \leq A \leq H_r(A)$ *and* $N_l(A) \leq A \leq H_l(A)$.

(2) *If* $\phi(\{a/x\}) \leq \{a/x\}$, *then* $N_r(A) \leq A \leq H_r(A)$ *and* $N_l(A) \leq A \leq H_l(A)$.

(3) *If* $a \odot \phi(\{a/x\})(y) \odot b \leq \phi(\{b/y\})(x)$, *then* $H_r(N_r(A)) \leq A$ *for* $A \in L^X$.

(4) *If* $b \odot \phi(\{a/x\})(y) \odot a \leq \phi(\{b/y\})(x)$, *then* $H_l(N_l(A)) \leq A$ *for* $A \in L^X$.

(5) *If* $\phi(\{a/x\})(y) \leq b \odot \phi(\{b/y\})(x)$, *then* $N_r(H_r(A)) \geq A$ *for* $A \in L^X$.

(6) *If* $\phi(\{a/x\})(y) \leq \phi(\{b/y\})(x) \odot b$, *then* $N_l(H_l(A)) \geq A$ *for* $A \in L^X$.

Proof. (1)

$$\begin{aligned}N_r(A)(x) &= \bigvee_{\{a/x\}\in M}(a \odot e^r_{L^X}(\phi(\{a/x\}), A)) \\ &\leq \bigvee_{\{a/x\}\in M}(a \odot e^r_{L^X}(\{a/x\}, A)) \\ &= \bigvee_{\{a/x\}\in M}(a \odot (a \Rightarrow A(x))) \\ &\leq A(x).\end{aligned}$$

$$\begin{aligned}H_r(A)(x) &= \bigvee_{\{a/x\}\in M}(a \odot \rho_r(\phi(\{a/x\}), A)) \\ &\geq A(x) \odot e^r_{L^X}(\{A(x)/x\}, A)) = A(x) \odot \top = A(x).\end{aligned}$$

Similarly, $N_l(A) \leq A \leq H_l(A)$.

(2)

$$\begin{aligned}H_r(A)(x) &= \bigvee_{\{a/x\}\in M}(a \odot \rho_r(\phi(\{a/x\}), A)) \\ &\leq \bigvee_{\{a/x\}\in M}(a \odot \rho_r(\{a/x\}, A)) = \top \odot A(x) = A(x).\end{aligned}$$

$$\begin{aligned}
N_r(A)(x) &= \bigvee_{\{a/x\}\in M}(a \odot e^r_{L^X}(\phi(\{a/x\}), A))\\
&\geq \bigvee_{\{a/x\}\in M}(a \odot e^r_{L^X}(\{a/x\}, A))\\
&= \bigvee_{\{a/x\}\in M}(a \odot (a \Rightarrow A(x)))\\
&\geq A(x) \odot (A(x) \Rightarrow A(x)) = A(x).
\end{aligned}$$

Similarly, $H_l(A) \leq A \leq N_l(A)$.

(4)

$$\begin{aligned}
N_l(A)(x) &= \bigvee_{\{a/x\}\in M}(e^l_{L^X}(\phi(\{a/x\}), A) \odot a)\\
&\leq \bigvee_{\{a/x\}\in M}(e^l_{L^X}(\{a/x\}, A) \odot a)\\
&= \bigvee_{\{a/x\}\in M}((a \to A(x)) \odot a)\\
&\leq A(x).
\end{aligned}$$

$$\begin{aligned}
H_l(A)(x) &= \bigvee_{\{a/x\}\in M}(\rho_l(\phi(\{a/x\}), A) \odot a)\\
&\geq A(x) \odot \{\top/x\}(x) \odot \top = A(x) \odot \top = A(x).
\end{aligned}$$

(3)

$$\begin{aligned}
&H_r(N_r(A))(x)\\
&= \bigvee_{\{a/x\}\in M}(a \odot \rho_r(\phi(\{a/x\}), N_r(A)))\\
&= \bigvee_{\{a/x\}\in M}(a \odot \bigvee_{y\in X}(\phi(\{a/x\})(y) \odot \bigvee_{\{b/y\}\in M}(b \odot e^r_{L^X}(\phi(\{b/y\}), A))))\\
&= \bigvee_{\{a/x\}\in M}\bigvee_{y\in X}\bigvee_{\{b/y\}\in M}(a \odot (\phi(\{a/x\})(y) \odot (b \odot e^r_{L^X}(\phi(\{b/y\}), A))))\\
&\leq \bigvee_{\{a/x\}\in M}\bigvee_{y\in X}\bigvee_{\{b/y\}\in M}(\phi(\{b/y\})(x) \odot e^r_{L^X}(\phi(\{b/y\}), A))\\
&\leq \bigvee_{\{a/x\}\in M}\bigvee_{y\in X}\bigvee_{\{b/y\}\in M}(\phi(\{b/y\})(x) \odot (\phi(\{b/y\})(x) \Rightarrow A(x)))\\
&\leq A(x).
\end{aligned}$$

(4)

$$\begin{aligned}
&H_l(N_l(A))(x)\\
&= \bigvee_{\{a/x\}\in M}(\rho_l(\phi(\{a/x\}), N_l(A)) \odot a)\\
&= \bigvee_{\{a/x\}\in M}(\bigvee_{y\in X}(\bigvee_{\{b/y\}\in M}(e^r_{L^X}(\phi(\{b/y\}), A) \odot b) \odot \phi(\{a/x\})(y)) \odot a))\\
&= \bigvee_{\{a/x\}\in M}\bigvee_{y\in X}\bigvee_{\{b/y\}\in M}(e^r_{L^X}(\phi(\{b/y\}), A) \odot b) \odot \phi(\{a/x\})(y)) \odot a))\\
&\leq \bigvee_{\{a/x\}\in M}\bigvee_{y\in X}\bigvee_{\{b/y\}\in M}(e^r_{L^X}(\phi(\{b/y\}), A) \odot \phi(\{a/x\})(y))\\
&\leq A(x).
\end{aligned}$$

(5)

$$\begin{aligned}
&N_r(H_r(A))(x)\\
&= \bigvee_{\{a/x\}\in M}(a \odot e^r_{L^X}(\phi(\{a/x\}), H_r(A)))\\
&= \bigvee_{\{a/x\}\in M}(a \odot \bigvee_{y\in X}(\phi(\{a/x\})(y) \Rightarrow \bigvee_{\{b/y\}\in M}(b \odot \rho_r(\phi(\{b/y\}), A))))\\
&\geq \bigvee_{\{a/x\}\in M}\bigvee_{y\in X}\bigvee_{\{b/y\}\in M}(a \odot (\phi(\{a/x\})(y) \Rightarrow (b \odot \phi(\{b/y\})(x) \odot A(x))))\\
&\geq \bigvee_{\{a/x\}\in M}\bigvee_{y\in X}\bigvee_{\{b/y\}\in M}(a \odot (\phi(\{a/x\})(y) \Rightarrow (\phi(\{a/x\})(y) \odot A(x))))\\
&\geq \bigvee_{\{a/x\}\in M}(a \odot A(x)) = A(x).
\end{aligned}$$

(6)

$$
\begin{aligned}
&N_l(H_l(A))(x)\\
&= \bigvee_{\{a/x\}\in M}(e^l_{L^X}(\phi(\{a/x\}), H_l(A)) \odot a)\\
&= \bigvee_{\{a/x\}\in M}(\bigvee_{y\in X}(\bigvee_{\{b/y\}\in M}((\phi(\{a/x\})(y) \to A(x) \odot \phi(\{b/y\})(x) \odot b) \odot a)\\
&\geq \bigvee_{\{a/x\}\in M}(\bigvee_{y\in X}(\bigvee_{\{b/y\}\in M}((\phi(\{a/x\})(y) \to A(x) \odot \phi(\{a/x\})(y)) \odot a)\\
&\geq \bigvee_{\{a/x\}\in M}(A(x) \odot a) = A(x).
\end{aligned}
$$

□

Theorem 19. *(1) If $a\odot\phi(\{a/x\})(y)\odot b \leq \phi(\{b/y\})(x)$ and $\phi(\{a/x\})(y) \leq b \odot \phi(\{b/y\})(x)$, then $H_r \circ N_r : L^X \to L^X$ is a right interior operator and $N_r \circ H_r : L^X \to L^X$ is a right closure operator.*

(2) If $b\odot\phi(\{a/x\})(y)\odot a \leq \phi(\{b/y\})(x)$ and $\phi(\{a/x\})(y) \leq \phi(\{b/y\})(x)\odot b$, then $H_l \circ N_l : L^X \to L^X$ is a left interior operator and $N_l \circ H_l : L^X \to L^X$ is a left closure operator.

Proof. (1) (I1) By Theorem 18(1), $H_r(N_r(A)) \leq A$.

(I2) By (I1), since N_r is an increasing function, $N_r(H_r(N_r(A))) \leq N_r(A)$. By Theorem 18(3), $N_r(H_r(N_r(A))) \geq N_r(A)$. Hence $N_r(H_r(N_r(A))) = N_r(A)$. So, $H_r(N_r(H_r(N_r(A)))) = H_r(N_r(A))$.

(RI) $e^r_{L^X}(A, B) \leq e^r_{L^X}(N_r(A), N_r(B)) \leq e^r_{L^X}(H_r(N_r(A)), H_r(N_r(A)))$.
Thus $H_r \circ N_r : L^X \to L^X$ is a right interior operator.

(C1) By Theorem 18(1), $N_r(H_r(A)) \geq A$.

(C2) By (C1), since H_r is an increasing function, $H_r(N_r(H_r(A))) \geq H_r(A)$. By Theorem 18(1), $H_r(N_r(H_r(A))) \leq H_r(A)$. Hence $H_r(N_r(H_r(A))) = H_r(A)$. So, $N_r(H_r(N_r(H_r(A)))) = N_r(H_r(A))$.

(RC) $e^r_{L^X}(A, B) \leq e^r_{L^X}(H_r(A), H_r(B)) \leq e^r_{L^X}(N_r(H_r(A)), N_r(H_r(A)))$.

Thus, $N_r \circ H_r : L^X \to L^X$ is a right closure operator.

(2) It is similarly proved as (1). □

From Theorems 15 and 19, we obtain the following corollary.

Corollary 20. *Let $R \in L^{X\times X}$ and define $\phi(\{a/x\}) = [x]_R$ where $[x]_R(y) = R(x, y)$. If $R(x, y) = R(y, x)$, then $H_r \circ N_r$(resp. $H_l \circ N_l$) is a right (resp. left) interior operator and $N_r \circ H_r$(resp. $N_l \circ H_l$) is a right (resp. left) closure operator.*

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