EXISTENCE AND UNIQUENESS OF SELF-SIMILAR
SOLUTIONS OF A NONHOMOGENEOUS EQUATION

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Abstract: In this paper, we shall prove the existence and uniqueness of radial solutions for the nonhomogeneous elliptic equation

$$\text{div} \left( |\nabla u|^{p-2} \nabla u \right) + \beta x \nabla u + \alpha u + |x|^l |u|^{q-1} u = 0, \quad x \in \mathbb{R}^N,$$

These solutions are related to self-similar solutions of the degenerate parabolic equation

$$v_t = \text{div} \left( |\nabla v|^{p-2} \nabla v \right) + |x|^l |v|^{q-1} v, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^N.$$

where \(p > 2, \quad q \geq 1, \quad N \geq 1, \quad -p < l, \quad -N < l\).

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1. Introduction

In this paper, we analyse the self-similar solutions of a nonhomogeneous degenerate...
erate parabolic equation

\[ w_t = \Delta_p w + |x|^l |w|^{q-1} w, \quad \text{in } \mathbb{R}^N \times (0, +\infty) \] (1.1)

where \( p > 2, \quad q \geq 1, \quad N \geq 1, \quad -p < l < 0, \quad -N < l < 0. \)

As usual, \( \nabla U \) denotes the gradient of \( U \), and \( \Delta_p U = \text{div}(|\nabla U|^{p-2} \nabla U) \) is the \( p \)-Laplacian operator.

If we look for self-similar solutions of the form

\[ w(x, t) = t^{-\alpha} U(t^{-\beta} x), \quad \text{in } \mathbb{R}^N \times (0, +\infty), \] (1.2)

then the profile \( U \) must satisfy the nonhomogeneous elliptic equation

\[ \Delta_p U + \beta x \nabla U + |x|^l |U|^{q-1} U = 0, \quad \text{for } x \in \mathbb{R}^N, \] (1.3)

where the scaling factors are given by

\[ \alpha = \frac{p + l}{p(q - 1) + l(p - 2)}, \quad \beta = \frac{q - p + 1}{p(q - 1) + l(p - 2)}, \] (1.4)

with \( l \neq -\frac{p(q - 1)}{p - 2} \).

It is worth mentioning that several recent papers have been devoted to the study of equation (1.3) when \( l = 0 \) and \( p = 2 \), cf. e.g. [5, 6, 7, 8, 10]. Quite recently, in [4] the case \( p = 2 \) and \(-2 < l < 0 \) was treated. Note also when \( l = 0 \) and \( p > 2 \) our equation was investigated in [2] and [9]. The present paper is devoted to the case \( p > 2, \quad -p < l < 0, \quad -N < l < 0 \) and, \( \alpha \) and \( \beta \) are real parameters. More precisely, the main purpose of this paper is to study existence and uniqueness of radially-symmetric solutions; which are functions \( U(y) = u(|y|) = u(r) \) where \( u : \mathbb{R}^+ \to \mathbb{R} \) satisfies the following O.D.E

\[ (|u'|^{p-2} u')' + \frac{N - 1}{r} |u'|^{p-2} u' + \alpha u + \beta ru' + r^l |u|^{q-1} u = 0, \quad r > 0, \] (1.5)

the prime denotes the differentiation with respect to \( r \).

In order to study equation (1.5) we shall use the shooting method, we then consider the following Cauchy problem \((P)\)

\[ (P) \quad \left\{ \begin{array}{l} (|u'|^{p-2} u')' + \frac{N - 1}{r} |u'|^{p-2} u' + \alpha u + \beta ru' + r^l |u|^{q-1} u = 0 \\
 u(0) = a \end{array} \right. \] (1.6)

where \( p > 2, \quad q \geq 1, \quad N \geq 1, \quad 0 > l > -p \) and \( 0 > l > -N \), and \( \alpha, \beta, a \) are real parameters.
2. Existence and Uniqueness of Solutions

In this section we investigate the existence and uniqueness of a solution of the problem (P).

By a solution of (P) on some interval \([0, R]\) we mean a function \(u \in C^1([0, R]) \cap C^0(0, R)\) such that \(|u'|^{p-2} u' \in C^1([0, R])\) and which satisfies (1.6) on the open interval \((0, R)\), with \(\lim_{r \to 0} r^{N-1} |u'|^{p-2} u' = 0\).

To emphasize the dependence of \(u\) on the parameters \(\alpha, \beta\) and \(a\), we sometimes denote a solution of (P) by \(u(., a, \alpha, \beta)\).

We start with a local existence and uniqueness result.

**Proposition 2.1.** Assume \(p > 2, q \geq 1, N \geq 1, -p < l < 0, -N < l\) and \(\alpha, \beta \in \mathbb{R}^*\). Then for any \(a \in \mathbb{R}^*\), there exists a constant \(R(a) > 0\) such that the problem (P) has a unique maximal solution \(u\) defined in \([0, R(a)]\).

**Remark 2.1.** Note that equation (1.6) can be written as in the following system

\[
(S) \begin{cases}
  u' = |v|^{-\frac{p-2}{p-1}} v, \\
  v' = -\frac{N-1}{r} v - \beta r |v|^{-\frac{p-2}{p-1}} v - \alpha u - r^l |u|^q u.
\end{cases}
\]

Let \(r_0 > 0, \mu \in \mathbb{R}\) and \(\eta \in \mathbb{R}^*\) such that

\[
\begin{cases}
  u(r_0) = \mu, \\
  v(r_0) = \eta.
\end{cases}
\]

As \((r, u, v) \mapsto (r, |v|^{-\frac{p-2}{p-1}} v, -\frac{N-1}{r} v - \beta r |v|^{-\frac{p-2}{p-1}} v - \alpha u - r^l |u|^q u)\) is locally Lipschitz continuous function in the set \(\{(r, u, v) \in \mathbb{R}^*_+ \times \mathbb{R} \times \mathbb{R}^*\}\), we deduce (from the theory of O.D.E [1]) the local existence and uniqueness solution of (2.1)–(2.2).

In spite of the degenerescence of (P) at \(r = 0\), the Banach fixed point theorem ensures the local existence and uniqueness. We prove that this solution is global indeed.

Let \(u\) be a solution of problem (P) in some interval \([0, R]\); then if we multiply (1.6) by \(r^{N-1}\) and integrate twice from 0 to \(r\) \((0 < r < R)\) we get

\[
u(r) = a - \int_0^r \Psi \{f_u(s)\} ds,
\]

where

\[f_u(s) = \beta su(s)\]
\[ + s^{1-N} \left[ (\alpha - \beta N) \int_0^s \sigma^{N-1} u(\sigma) d\sigma + \int_0^s \sigma^{l+N-1} |u|^{q-1} u(\sigma) d\sigma \right] \] (2.4)

and

\[ \Psi(s) = |s|^{-r/p} s \quad \text{for any } s \in \mathbb{R}^*. \] (2.5)

Let us introduce some notations. For any \( 0 < M < a \), we denote

\[ E_{M,a} = \left\{ u \in C^0([0, r_a]); \|u - a\|_{E_{M,a}} \leq M \right\}, \] (2.6)

where \( C^0([0, r_a]) \) is the Banach space of continuous functions on \([0, r_a]\) where

\[ r_a = \min \{r_i, i = 1, 2, 3\}. \] (2.7)

The reals \( r_i \) are given explicitly by

\[ r_1 = k_2^{-1/p}\left( \frac{p + l}{p - 1} M \right)^{\frac{p-1}{p+l}}, \] (2.8)

\[ r_2 = K_1^{\frac{p-2}{p+l}} \left( \frac{p + l}{2q} \frac{l + N}{(M + a)^{q-1}} \right)^{\frac{p-1}{p+l}}, \] (2.9)

\[ r_3 = k_1^{\frac{p-2}{p(l+2)}} \left[ \frac{p - l(p - 2)}{2} \left( |\beta| + \frac{|\alpha - N\beta|}{N} \right)^{-1} \right]^{\frac{p-1}{p-(p-2)}}, \] (2.10)

with

\[ K_1 = \frac{(a - M)^q}{2(l + N)} \quad \text{and} \quad K_2 = 2 \frac{(a + M)^q}{(l + N)}. \] (2.11)

The following result is the keystone of the proof of the existence.

**Lemma 2.1.** Let \( a \in \mathbb{R}_+^* \) and \( 0 < M < a \). For each \( v \in E_{M,a} \), the function \( f_v \) given by (2.4) satisfies

\[ K_1 s^{l+1} < f_v(s) \leq K_2 s^{l+1}, \quad \forall 0 \leq s \leq r_0, \] (2.12)

where \( r_a \) as in the formula (2.7) and \( K_1 \) and \( K_2 \) are given by (2.11).

**Proof.** The idea is to limit the function \( f_v(r) \) with two expressions having the same sign. These expressions depend strongly on the sign of \( \beta \) and \( \alpha - N\beta \). For this purpose we assume \( \beta \geq 0 \) and \( \alpha - N\beta \geq 0 \), (the reasoning in the others three cases is similar). Using the definition of \( E_{M,a} \), the following estimates hold for any \( v \in E_{M,a} \):
\[
\left\{ \left( \frac{\alpha}{N} - \beta \right) + \frac{(a - M)^{q-1}}{l + N} s^l \right\} (a - M)s \leq g(s) \\
\leq \left\{ \left( \frac{\alpha}{N} - \beta \right) + \frac{(a + M)^{q-1}}{l + N} s^l \right\} (a + M)s, \quad (2.13)
\]
where

\[
g(s) = s^{1-N} \left[ (\alpha - \beta N) \int_0^s \sigma^{N-1} v(\sigma)d\sigma + \int_0^s \sigma^{l+N-1} |v|^{q-1} v(\sigma)d\sigma \right]. \quad (2.14)
\]

Hence if we chose \( s \) small enough we get

\[
\frac{(a - M)^q}{2(l + N)} s^{l+1} \leq f_v(s) \leq 2 \frac{(a + M)^q}{(l + N)} s^{l+1} \quad (2.15)
\]

Then the lemma follows. \( \Box \)

Now we are able to prove the proposition.

Proof. The proof is divided into two steps.

Step 1: Local Existence and Uniqueness. For the set \( E_{M,a} \) given by (2.6), consider the mapping \( T \) defined on \( E_{M,a} \) by

\[
T(v)(r) = a - \int_0^r \Psi(f_v)(s)ds \quad (2.16)
\]

where \( \Psi \) and \( f_v \) are given by (2.4) and (2.5). Using Lemma 2.1 we deduce

\[
|T(v)(r) - a| \leq \frac{p - 1}{l + p} K_{\frac{1}{l+p}} \frac{1}{r^{\frac{1}{l+p}}} r^{\frac{l+p}{l+p}}. \quad (2.17)
\]

From the choice of (2.7) we deduce that \( T \) maps \( E_{M,a} \) into itself.

Now we assert that \( T \) is a contraction. In fact let \( v, w \in E_{M,a} \) and \( r \in [0, r_a] \), then

\[
|T(v)(r) - T(w)(r)| \\
\leq \int_0^r |\Psi(f_v(s)) - \Psi(f_w(s))| ds, \text{ for any } r \in [0, r_a]. \quad (2.18)
\]

Set

\[
\Phi(s) = \min(|f_v(s)|, |f_w(s)|) \quad \text{for any } s \in [0, r_a], \quad (2.19)
\]

then

\[
|T(v)(r) - T(w)(r)| \leq \int_0^r \frac{\Psi(\Phi(s))}{\Phi(s)} |(f_v - f_w)(s)| ds. \quad (2.20)
\]
Since the estimate (2.12) holds for \( f_v \) and also for \( f_w \), we get
\[
|\Psi(f_v(s)) - \Psi(f_w(s))| \leq \frac{1}{p-1} \left| K_1 s^{l+1} \right| \frac{p-2}{p-1} |f_v(s) - f_w(s)|
\]
for any \( s \in [0, r_a] \) \hspace{1cm} (2.21)

But (2.11) and the definition of the space \( E_{M,a} \), imply that the following estimate holds true
\[
|f_v(s) - f_w(s)| \leq \left[ (|\beta| + \frac{|\alpha - \beta N|}{N}) s + q \frac{(M + a)^{q-1}}{l + N} s^{l+1} \right] \|v - w\|_0 , \hspace{1cm} (2.22)
\]
for any \( s \in [0, r_a] \). Putting together (2.20), (2.21) and (2.22) we get
\[
|T(v)(r) - T(w)(r)| \leq \frac{1}{l+p} K_1 \frac{p-2}{p-1} \left[ q \frac{(M + |a|)^{q-1}}{l + N} r^{l+p} \right] + \frac{1}{p-l(p-2)} (|\beta| + \frac{|\alpha - \beta N|}{N}) r^{p-l(p-2)} \|v - w\|_{E_{M,a}} , \hspace{1cm} (2.23)
\]
for any \( 0 < r \leq r_a \).

From the choice of \( r_a \), we conclude that \( T \) is a contraction; and then the Banach contraction theorem implies that there exists a unique function \( u \) solving the problem \((P)\) in \([0, r_a]\).

Note that as \( u \) is continuous, as given by (2.3). We deduce easily that \( u \in C^1([0, r_a]) \) and also \( |u'|^{p-2} u' \in C^1([0, r_a]) \).

**Step 2.** \( \lim_{r \to 0} r^{N-1} |u'|^{p-2} u' = 0. \)

Using again the implicit formula given by (2.7) we deduce
\[
|u'|^{p-2} u'(r) = \beta r u(r) + r^{1-N} \left[ (\alpha - \beta N) \int_0^r s^{N-1} u(s) ds \right.
\]
\[
+ \left. \int_0^r s^{l+N-1} |u|^{q-1} u(s) ds \right] \hspace{1cm} (2.24)
\]
for any \( 0 < r \leq r_a \). As \( l > -N \) we obtain \( \lim_{r \to 0} r^{N-1} |u'|^{p-2} u' = 0 \), which completes the proof.

**Remark 2.2.** The following holds true:

i) It is easy to see that the function \( f_v \) has the following behavior
\[
f_v(r) \simeq a \left\{ \frac{\alpha}{N} + \frac{1}{l + N} q^{-1} r^l \right\} r , \hspace{1cm} (2.25)
\]
when \( r \) tends towards 0. We deduce that if \(-N < l < 0\) and \(-p < l < 0\) then for any \( a > 0 \) the solution \( u \) starts decreasing independently of sign of \( \alpha \) and \( \beta \). Moreover, (1.6) implies that if \( \alpha > 0 \) and \( \beta > 0 \) the solution \( u \) is strictly decreasing until it reaches the axis of \( x \).

ii) Proposition 2.1 is also valid for \( l \geq 0 \), if we replace the power \( l + 1 \) in formula (2.15) by 1.

iii) Note that \( u(., a, \alpha, \beta) = -u(., -a, \alpha, \beta) \).

**Proposition 2.2.** Let \( \alpha, \beta \in \mathbb{R}, \ q > 1, \ p > 2, \ -p < l < 0 \) and \(-N < l \). Then for any \( a > 0 \), the solution \( u \) of the problem \( (P) \) is global.

**Proof.** Let \([0, R(a)]\) be the maximum interval of existence. Integrating equation (1.6) on \([r_0, r]\) for some \( 0 < r_0 < r < R(a) \), we get

\[
\frac{p-1}{p} u'^p(r) - \frac{p-1}{p} u'^p(r_0) - \frac{\alpha}{2} u^2(r_0) - \frac{r_0 u^{q+1}(r_0)}{q+1} = \tag{2.26}
\]

\[
+ u^2(r) \left[ \frac{\alpha}{2} + \frac{p}{q+1} r^l u(r)^{q-1} \right] - (N-1) \int_{r_0}^r \frac{1}{s} |u'(s)|^p \, ds
\]

\[
- \beta \int_{r_0}^r s(u')^2(s) \, ds + \frac{l}{q+1} \int_{r_0}^r s^{l-1} |u(s)|^{q+1} \, ds
\]

Assume \( R(a) \) is finite. Then, the functions \( u(r) \) and \( u'(r) \) go to \( \infty \) when \( r \) goes to \( R(a) \). As \( l < 0 \) we claim that the right hand side of (2.26) is strictly negative for \( r \) close to \( R(a) \). This is obvious for \( \beta \geq 0 \). On the other hand, when \( \beta < 0 \) we use the following estimate

\[
\int_{r_0}^r s(u')^2(s) \, ds \leq \left[ \frac{r^2}{2} - \frac{r_0^2}{2} \right] \frac{p-2}{p} \left[ \int_{r_0}^r \frac{1}{s} |u'(s)|^p \, ds \right]^{\frac{2}{p}} \tag{2.27}
\]

to deduce that

\[
-(N-1) \int_{r_0}^r \frac{1}{s} |u'(s)|^p \, ds - \beta \int_{r_0}^r s(u')^2(s) \, ds \leq \frac{2}{s} \left[ \int_{r_0}^r \frac{1}{s} |u'(s)|^p ds \right]^{\frac{2}{p}} \left\{ \beta \left( \frac{r^2}{2} - \frac{r_0^2}{2} \right) - \frac{p-2}{p} - (N-1) \left[ \int_{r_0}^r \frac{|u'(s)|^p}{s} ds \right]^{\frac{2}{p}} \right\}, \tag{2.28}
\]

which implies that the right hand side of (2.26) is strictly negative, for \( r \) close to \( R(a) \), while the expression \( \frac{2}{2} + \frac{p |u(r)|^{q-1}}{q+1} \) goes to \( +\infty \) when \( r \) tends to \( R(a) \), and so the left side of (2.26) is strictly positive. This is a contradiction. Thereby \([0, R(a)] = [0, \infty]\). \( \square \)
**Remark 2.3.** Note that if $\alpha > 0$, $\beta > 0$ and $-p < l < 0$ and $-N < l$ the global existence can be shown easily. In fact consider the energy function defined by

$$E(r) = \frac{p-1}{p} |u'|^p + \frac{\alpha}{2} u^2 + r^l |u|^{q+1}$$

According to equation (1.6), $E$ satisfies

$$E'(r) = -\left\{ \frac{N-1}{r} |u'|^p + \beta ru^2 - \frac{l}{q+1} r^{l-1} |u|^{q+1} \right\}.$$  

As $l < 0$, the energy is strictly decreasing and then the maximal existence interval is $[0, \infty[$.

**References**


