International Journal of Pure and Applied Mathematics
Volume 86 No. 5 2013, 849-857
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v86i5.8

ijpam.eu

# EXISTENCE AND UNIQUENESS OF SELF-SIMILAR SOLUTIONS OF A NONHOMOGENEOUS EQUATION

Badia Hamri[1], Abdelilah Gmira[2], Youssef Jabri[3 §]

[1,2]Department of Mathematics
Faculty of Science
University Abdelmalek Essaâdi
B.P. 2121, Tétouan, MOROCCO

[3]Department of Mathematics
National School of Applied Sciences
University Mohammed I
B.P. 669, Oujda, MOROCCO

**Abstract:** In this paper, we shall prove the existence and uniqueness of radial solutions for the nonhomogeneous elliptic equation

$$div\left(|\nabla u|^{p-2}\nabla u\right)+\beta x\,\nabla u+\alpha u+|x|^l\,|u|^{q-1}\,u=0,\quad x\in\mathbb{R}^N,$$

These solutions are related to self-similar solutions of the degenerate parabolic equation

$$v_t=div\left(|\nabla v|^{p-2}\nabla v\right)+|x|^l\,|v|^{q-1}\,v,\ \ (t,x)\in(0,+\infty)\times\mathbb{R}^N.$$

where $p>2$, $q\geq 1$, $N\geq 1$, $-p<l$, $-N<l$.

**AMS Subject Classification:** 35K55, 35K65
**Key Words:** elliptic problem, parabolic problem, nonhomogeneous equation, self-similar solution, existence, uniqueness

## 1. Introduction

In this paper, we analyse the self-similar solutions of a nonhomogeneous degen-

Received: May 15, 2013



§Correspondence author

erate parabolic equation

$$w_t = \Delta_p w + |x|^l \, |w|^{q-1} \, w, \text{ in } \mathbb{R}^{\mathbb{N}} \times (0, +\infty) \tag{1.1}$$

where $p > 2$, $q \geq 1$, $N \geq 1$, $-p < l < 0$, $-N < l < 0$.

As usual, $\nabla U$ denotes the gradient of $U$, and $\Delta_p U = div(|\nabla U|^{p-2} \nabla U)$ is the $p$-Laplacian operator.

If we look for self-similar solutions of the form

$$w(x,t) = t^{-\alpha} U(t^{-\beta} x), \quad \text{in } \mathbb{R}^{\mathbb{N}} \times (0, +\infty), \tag{1.2}$$

then the profile $U$ must satisfy the nonhomogeneous elliptic equation

$$\Delta_p U + \beta x \nabla U + \alpha U + |x|^l \, |U|^{q-1} \, U = 0, \text{ for } x \in \mathbb{R}^{\mathbb{N}}, \tag{1.3}$$

where the scaling factors are given by

$$\alpha = \frac{p+l}{p(q-1) + l(p-2)}, \qquad \beta = \frac{q-p+1}{p(q-1) + l(p-2)}, \tag{1.4}$$

with $l \neq -\frac{p(q-1)}{p-2}$.

It is worth mentioning that several recent papers have been devoted to the study of equation (1.3) when $l = 0$ and $p = 2$, cf. e.g; [5, 6, 7, 8, 10]. Quite recently, in [4] the case $p = 2$ and $-2 < l < 0$ was treated. Note also when $l = 0$ and $p > 2$ our equation was investigated in [2] and [9]. The present paper is devoted to the case $p > 2$, $-p < l < 0$, $-N < l < 0$ and, $\alpha$ and $\beta$ are real parameters. More precisely, the main purpose of this paper is to study existence and uniqueness of radially-symmetric solutions; which are functions $U(y) = u(|y|) = u(r)$ where $u : \mathbb{R}^+ \to \mathbb{R}$ satisfies the following O.D.E

$$(|u'|^{p-2} \, u')' + \frac{N-1}{r} \, |u'|^{p-2} \, u' + \alpha u + \beta r u' + r^l \, |u|^{q-1} \, u = 0, r > 0, \tag{1.5}$$

the prime denotes the differentiation with respect to $r$.

In order to study equation (1.5) we shall use the shooting method, we then consider the following Cauchy problem $(P)$

$$(P) \quad \begin{cases} (|u'|^{p-2} \, u')' + \frac{N-1}{r} \, |u'|^{p-2} \, u' + \alpha u + \beta r u' + r^l \, |u|^{q-1} \, u = 0 \\ u(0) = a \end{cases} \tag{1.6}$$

where $p > 2$, $q \geq 1$, $N \geq 1, 0 > l > -p$ and $0 > l > -N$, and $\alpha, \beta, a$ are real parameters.

## 2. Existence and Uniqueness of Solutions

In this section we investigate the existence and uniqueness of a solution $u$ of the problem $(P)$.

By a solution of $(P)$ on some interval $[0,R[$ we mean a function $u \in C^0([0,R[) \cap C^1(]0,R[)$ such that $|u'|^{p-2}u' \in C^1(]0,R[)$ and which satisfies (1.6) on the open interval $]0,R[$, with $\lim_{r\to 0} r^{N-1}|u'|^{p-2}u' = 0$.

To emphasize the dependence of $u$ on the parameters $\alpha$, $\beta$ and $a$, we sometimes denote a solution of $(P)$ by $u(.,a,\alpha,\beta)$.

We start with a local existence and uniqueness result.

**Proposition 2.1.** *Assume $p > 2, q \geq 1, N \geq 1, -p < l < 0, -N < l$ and $\alpha, \beta \in \mathbb{R}^*$. Then for any $a \in \mathbb{R}^*$, there exists a constant $R(a) > 0$ such that the problem $(P)$ has a unique maximal solution $u$ defined in $[0, R(a)[$.*

**Remark 2.1.** Note that equation (1.6) can be written as in the following system

$$(S)\begin{cases} u' = |v|^{-\frac{p-2}{p-1}}v, \\ v' = -\frac{N-1}{r}v - \beta r|v|^{-\frac{p-2}{p-1}}v - \alpha u - r^l|u|^{q-1}u. \end{cases} \tag{2.1}$$

Let $r_0 > 0$, $\mu \in \mathbb{R}$ and $\eta \in \mathbb{R}^*$ such that

$$\begin{cases} u(r_0) = \mu, \\ v(r_0) = \eta. \end{cases} \tag{2.2}$$

As $(r,u,v) \mapsto \left(r, |v|^{-\frac{p-2}{p-1}}v, -\frac{N-1}{r}v - \beta r|v|^{-\frac{p-2}{p-1}}v - \alpha u - r^l|u|^{q-1}u\right)$ is locally Lipschitz continuous function in the set $\left\{(r,u,v) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}^*\right\}$, we deduce (from the theory of O.D.E [1]) the local existence and uniqueness solution of (2.1)–(2.2).

In spite of the degenerescence of $(P)$ at $r = 0$, the Banach fixed point theorem ensures the local existence and uniqueness. We prove that this solution is global indeed.

Let $u$ be a solution of problem $(P)$ in some interval $[0,R[$; then if we multiply (1.6) by $r^{N-1}$ and integrate twice from 0 to $r$ $(0 < r < R)$ we get

$$u(r) = a - \int_0^r \Psi\{f_u(s)\}\,ds, \tag{2.3}$$

where

$$f_u(s) = \beta s u(s)$$

$$+ s^{1-N}\left[(\alpha-\beta N)\int_0^s \sigma^{N-1}u(\sigma)d\sigma+\int_0^s \sigma^{l+N-1}\left|u\right|^{q-1}u(\sigma)d\sigma\right] \quad (2.4)$$

and

$$\Psi(s)=|s|^{-(p-2)/(p-1)}\,s \qquad \text{for any } s\in\mathbb{R}^*. \quad (2.5)$$

Let us introduce some notations. For any $0<M<a$, we denote

$$E_{M,a}=\left\{u\in C^0([0,r_a]);\ \|u-a\|_{E_{M,a}}\le M\right\}, \quad (2.6)$$

where $C^0([0,r_a])$ is the Banach space of continuous functions on $[0,r_a]$ where

$$r_a=\min\{r_i,\ i=1,2,3\}. \quad (2.7)$$

The reals $r_i$ are given explicitly by

$$r_1=k_2^{-\frac{1}{p+l}}\left[\frac{p+l}{p-1}M\right]^{\frac{p-1}{p+l}}, \quad (2.8)$$

$$r_2=K_1^{\frac{p-2}{l+p}}\left[\frac{p+l}{2q}\frac{l+N}{(M+a)^{q-1}}\right]^{\frac{p-1}{p+l}}, \quad (2.9)$$

$$r_3=k_1^{\frac{p-2}{p-l(p-2)}}\left[\frac{p-l(p-2)}{2}(|\beta|+\frac{|\alpha-N\beta|}{N})^{-1}\right]^{\frac{p-1}{p-l(p-2)}}, \quad (2.10)$$

with

$$K_1=\frac{(a-M)^q}{2(l+N)} \text{ and } K_2=2\frac{(a+M)^q}{(l+N)}. \quad (2.11)$$

The following result is the keystone of the proof of the existence.

**Lemma 2.1.** *Let $a\in\mathbb{R}_*^+$ and $0<M<a$. For each $v\in E_{M,a}$,the function $f_v$ given by (2.4) satisfies*

$$K_1 s^{l+1}<f_v(s)\le K_2 s^{l+1},\ \forall 0\le s\le r_0, \quad (2.12)$$

*where $r_a$ as in the formula (2.7) and $K_1$ and $K_2$ are given by (2.11).*

*Proof.* The idea is to limit the function $f_v(r)$ with two expressions having the same sign. These expressions depend strongly on the sign of $\beta$ and $\alpha-N\beta$. For this purpose we assume $\beta\ge 0$ and $\alpha-N\beta\ge 0$, (the reasoning in the others three cases is similar). Using the definition of $E_{M,a}$, the following estimates hold for any $v\in E_{M,a}$:

$$\left\{\left(\frac{\alpha}{N}-\beta\right)+\frac{(a-M)^{q-1}}{l+N}s^{l}\right\}(a-M)s \leq g(s)$$
$$\leq \left\{\left(\frac{\alpha}{N}-\beta\right)+\frac{(a+M)^{q-1}}{l+N}s^{l}\right\}(a+M)s, \quad (2.13)$$

where

$$g(s)=s^{1-N}\left[(\alpha-\beta N)\int_0^s \sigma^{N-1}v(\sigma)d\sigma+\int_0^s \sigma^{l+N-1}\left|v\right|^{q-1}v(\sigma)d\sigma\right]. \quad (2.14)$$

Hence if we chose $s$ small enough we get

$$\frac{(a-M)^q}{2(l+N)}s^{l+1} \leq f_v(s) \leq 2\frac{(a+M)^q}{(l+N)}s^{l+1} \quad (2.15)$$

Then the lemma follows. □

Now we are able to prove the proposition.

*Proof.* The proof is divided into two steps.

*Step 1: Local Existence and Uniqueness.* For the set $E_{M,a}$ given by (2.6), consider the mapping $T$ defined on $E_{M,a}$ by

$$T(v)(r)=a-\int_0^r \Psi(f_v)(s)ds \quad (2.16)$$

where $\Psi$ and $f_v$ are given by (2.4) and (2.5). Using Lemma 2.1 we deduce

$$|T(v)(r)-a| \leq \frac{p-1}{l+p}K_2^{\frac{1}{p-1}}r^{\frac{l+p}{p-1}}. \quad (2.17)$$

From the choice of (2.7) we deduce that $T$ maps $E_{M,a}$ into itself.

Now we assert that $T$ is a contraction. In fact let $v,w \in E_{M,a}$ and $r \in [0,r_a]$, then

$$|T(v)(r)-T(w)(r)|$$
$$\leq \int_0^r \left|\Psi(f_v(s))-\Psi(f_w(s))\right| ds\,, \text{ for any } r \in [0,r_a]\,. \quad (2.18)$$

Set

$$\Phi(s)=\min\left(\left|f_v(s)\right|,\left|f_w(s)\right|\right) \text{ for any } s \in [0,r_a]\,, \quad (2.19)$$

then

$$|T(v)(r)-T(w)(r)| \leq \int_0^r \frac{\Psi(\Phi(s))}{\Phi(s)}\left|(f_v-f_w)(s)\right| ds. \quad (2.20)$$

Since the estimate (2.12) holds for $f_v$ and also for $f_w$, we get

$$|\Psi(f_v(s)) - \Psi(f_w(s))| \leq \frac{1}{p-1}\left|K_1 s^{l+1}\right|^{-\frac{p-2}{p-1}} |f_v(s) - f_w(s)| \quad \text{for any } s \in [0, r_a] \quad (2.21)$$

But (2.11) and the definition of the space $E_{M,a}$, imply that the following estimate holds true

$$|f_v(s) - f_w(s)| \leq \left[(|\beta| + \frac{|\alpha - \beta N|}{N})s + q\frac{(M+a)^{q-1}}{l+N}s^{l+1}\right] \|v - w\|_0 , \quad (2.22)$$

for any $s \in [0, r_a]$. Putting together (2.20), (2.21) and (2.22) we get

$$|T(v)(r) - T(w)(r)| \leq \frac{1}{l+p} K_1^{-\frac{p-2}{p-1}} \left[q\frac{(M+|a|)^{q-1}}{l+N} r^{\frac{l+p}{p-1}} + \frac{1}{p-l(p-2)}(|\beta| + \frac{|\alpha - \beta N|}{N})r^{\frac{p-l(p-2)}{p-1}}\right] \|v - w\|_{E_{M,a}} , \quad (2.23)$$

for any $0 < r \leq r_a$.

From the choice of $r_a$, we conclude that $T$ is a contraction; and then the Banach contraction theorem implies that there exists a unique function $u$ solving the problem $(P)$ in $[0, r_a]$.

Note that as $u$ is continuous, as given by (2.3). We deduce easily that $u \in C^1(]0, r_a[)$ and also $|u'|^{p-2} u' \in C^1(]0, r_a[)$.

*Step 2.* $\lim_{r \to 0} r^{N-1} |u'|^{p-2} u' = 0$.

Using again the implicit formula given by (2.7) we deduce

$$\left|u'\right|^{p-2} u'(r) = \beta r u(r) + r^{1-N}\left[(\alpha - \beta N)\int_0^r s^{N-1}u(s)ds + \int_0^r s^{l+N-1}|u|^{q-1} u(s)ds\right] \quad (2.24)$$

for any $0 < r \leq r_a$. As $l > -N$ we obtain $\lim_{r \to 0} r^{N-1} |u'|^{p-2} u' = 0$, which completes the proof. □

**Remark 2.2.** The following holds true:

i) It is easy to see that the function $f_v$ has the following behavior

$$f_v(r) \simeq a\left\{\frac{\alpha}{N} + \frac{1}{l+N}a^{q-1}r^l\right\} r, \quad (2.25)$$

when $r$ tends towards 0. We deduce that if $-N < l < 0$ and $-p < l < 0$ then for any $a > 0$ the solution $u$ starts decreasing independently of sign of $\alpha$ and $\beta$. Moreover, (1.6) implies that if $\alpha > 0$ and $\beta > 0$ the solution $u$ is strictly decreasing until it reaches the axis of $x$.

ii) Proposition 2.1 is also valid for $l \geq 0$, if we replace the power $l+1$ in formula (2.15) by 1.

iii) Note that $u(., a, \alpha, \beta) = -u(., -a, \alpha, \beta)$.

**Proposition 2.2.** *Let $\alpha$, $\beta \in \mathbb{R}$, $q > 1$, $p > 2$, $-p < l < 0$ and $-N < l$. Then for any $a > 0$, the solution $u$ of the problem $(P)$ is global.*

*Proof.* Let $[0, R(a)[$ be the maximum interval of existence. Integrating equation (1.6) on $]r_0, r[$ for some $0 < r_0 < r < R(a)$, we get

$$\begin{aligned} &\frac{p-1}{p}\left|u'\right|^p(r) - \frac{p-1}{p}\left|u'\right|^p(r_0) - \frac{\alpha}{2}u^2(r_0) - \frac{r_0^l u^{q+1}(r_0)}{q+1} = \\ &+u^2(r)\left[\frac{\alpha}{2} + \frac{r^l\left|u(r)\right|^{q-1}}{q+1}\right] - (N-1)\int_{r_0}^{r}\frac{1}{s}\left|u'(s)\right|^p ds \\ &-\beta\int_{r_0}^{r} s(u')^2(s)\,ds + \frac{l}{q+1}\int_{r_0}^{r} s^{l-1}\left|u(s)\right|^{q+1} ds \end{aligned} \tag{2.26}$$

Assume $R(a)$ is finite. Then, the functions $u(r)$ and $u^{'}(r)$ go to $\infty$ when $r$ goes to $R(a)$. As $l < 0$ we claim that the right hand side of (2.26) is strictly negative for $r$ close to $R(a)$. This is obvious for $\beta \geq 0$. On the other hand, when $\beta < 0$ we use the following estimate

$$\int_{r_0}^{r} s(u')^2(s)\,ds \leq \left[\frac{r^2}{2} - \frac{r_0^2}{2}\right]^{\frac{p-2}{p}}\left[\int_{r_0}^{r}\frac{1}{s}\left|u'(s)\right|^p ds\right]^{\frac{2}{p}} \tag{2.27}$$

to deduce that

$$\begin{aligned} &-(N-1)\int_{r_0}^{r}\frac{1}{s}\left|u'(s)\right|^p ds - \beta\int_{r_0}^{r} s(u')^2(s)\,ds \leq \\ &\left[\int_{r_0}^{r}\frac{\left|u'(s)\right|^p}{s}ds\right]^{\frac{2}{p}}\left\{|\beta|\,(\frac{r^2}{2} - \frac{r_0^2}{2})^{\frac{p-2}{p}} - (N-1)\left[\int_{r_0}^{r}\frac{\left|u'(s)\right|^p}{s}ds\right]^{\frac{p-2}{p}}\right\}, \end{aligned} \tag{2.28}$$

which implies that the right hand side of (2.26) is strictly negative, for $r$ close to $R(a)$, while the expression $\frac{\alpha}{2} + \frac{r^l|u(r)|^{q-1}}{q+1}$ goes to $+\infty$ when $r$ tends to $R(a)$, and so the left side of (2.26) is strictly positive. This is a contradiction. Thereby $[0, R(a)[ = [0, \infty[$. □

**Remark 2.3.** Note that if $\alpha > 0$, $\beta > 0$ and $-p < l < 0$ and $-N < l$ the global existence can be shown easily. In fact consider the energy function defined by

$$E(r) = \frac{p-1}{p}\left|u'\right|^p + \frac{\alpha}{2}u^2 + r^l\frac{|u|^{q+1}}{q+1} \tag{2.29}$$

According to equation (1.6), $E$ satisfies

$$E'(r) = -\left\{\frac{N-1}{r}\left|u'\right|^p + \beta r u'^2 - \frac{l}{q+1}r^{l-1}|u|^{q+1}\right\}. \tag{2.30}$$

As $l < 0$, the energy is strictly decreasing and then the maximal existence interval is $[0,\infty[$.

## References


[1] H. Amann, *Ordinary Differential Equations*, Walter de Gruyter Inc., Germany, 1996.

[2] B. Benyounes, A. Gmira, On the radial solutions of degenerate quasilinear elliptic equation, *Annales de la Faculté des Sciences de Toulouse*, **8**, No. 3 (1999), 411-438.

[3] M.F. Bidaut-Véron , S. Pohozaev, Nonexistence results and estimates for some nonlinear elliptic problems, *J. Anal. Math.*, **84** (2001), 1-49.

[4] S. Filippas, A. Tertikas, On similarity solutions of a heat equation with a nonhomogeneous nonlilearity, *J. Diff. Equ.*, **165** (2000), 468-492.

[5] A. Haraux, F.B. Weissler, Non-uniqueness for a semilinear initial value problem, *Indiana Univ. Math. J.*, **31** (1982) 167-189.

[6] M. Hirose, E. Yanagida, Global structure of self-similar solutions for semilinear heat equations, *J. Diff. Equ.*, **163** (2000), 407-428.

[7] L.A. Peletier, D. Terman, F.B. Weissler, On the equation $\triangle u + \frac{1}{2}x.\nabla u + f(u) = 0$, *Archive for Rational Mechanics and Analysis* , **94** (1986), 83-99.

[8] Y. Naito, T. Suzuki, Radial symmetry self- similar solutions for semilinear heat equations, *J. Diff. Equ.*, **163** (2000), 407-428.

[9] Y.W. Qi, The global existence and non uniqueness of a non linear degenerate equation, *Nonlinear Analysis: Theory, Methods and Applications - Journal*, **31**, No. 9 (1998), 117-136.

[10] E. Yanagida, Uniqueness of rapidly decaying solutions to the Haraux weissler equation, *J. Diff. Equ.*, **127** (1996), 561-570.