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LINEAR WIRELENGTH OF CIRCULANT NETWORKS

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Abstract: One of the central issues in designing and evaluating an interconnection network is to study how well other existing networks can be embedded into this network. In this paper, we present an algorithm for finding the exact wirelength of circulant networks into a family of grids and prove its correctness using the Congestion lemma and Partition lemma.

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Key Words: wirelength, edge congestion, circulant networks, grids

1. Introduction

Interconnection networks is one of the most basic components of a massively parallel computer system. They are becoming increasingly pervasive in many different applications with the operational costs and characteristics of the networks depending considerably on its applications. As the number of components increases, the number of wires required to connect them also increases. One of the central issues in designing and evaluating an interconnection network is to study how well other existing networks can be embedded into this network. This problem can be modeled by the following graph embedding problem: given a host graph H, which represents the network into which other networks are to be embedded, and a guest graph G, which represents the network to be embedded, the problem is to find a mapping from $V(G)$ to $V(H)$ such that each edge of G can be mapped to a path in H. Two common measures of the

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effectiveness of an embedding are the *dilation*, which measures the slowdown in the new architecture, and *congestion* which is the load factor that gauges the processor utilization [3].

There are several results on the embedding problem of various architectures such as circulant into arbitrary trees, cycles, certain multicyclic graphs and ladders [11], trees on cycles [7], trees on stars [18], hypercubes into grids [3], complete binary tree into grids [15], grids into grids [17], ladders and caterpillars into hypercubes [5], binary trees into hypercubes [8], complete binary trees into hypercubes [2], incomplete hypercube in books [9], enhanced and augmented hypercube into complete binary tree [12] and hypercubes into cylinders, snakes and caterpillars [13].

In this paper, we present an algorithm for finding the exact wirelength of circulant networks into a family of grids and prove its correctness using the Congestion lemma [14] and Partition lemma [14]. This partially solves an open problem posed in [11].

2. Preliminaries

Definition 2.1. [3] Let $G(V, E)$ and $H(V, E)$ be finite graphs with n vertices. An *embedding* f of G into H is defined as follows:

1. f is a bijective map from $V(G) \to V(H)$
2. f is a one-to-one map from $E(G)$ to $\{P_f(f(u), f(v)) : P_f(f(u), f(v))$ is a path in H between $f(u)$ and $f(v)$ for $(u, v) \in E(G)\}$.

The graph G that is being embedded is called a *virtual graph* or a *guest graph* and H is called a *host graph*. Some authors use the name *labelling* instead of embedding [2].

The *edge congestion* of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. In other words,

$$EC_f(G, H) = \max EC_f(G, H(e)),$$

where the maximum is taken over all the edges e of H and then the minimum edge congestion of G into H is defined as

$$EC(G, H) = \min EC_f(G, H)$$

where the minimum is taken over all embeddings f of G into H.

The edge congestion problem of a graph G is to find an embedding of G into H that induces $EC(G, H)$.

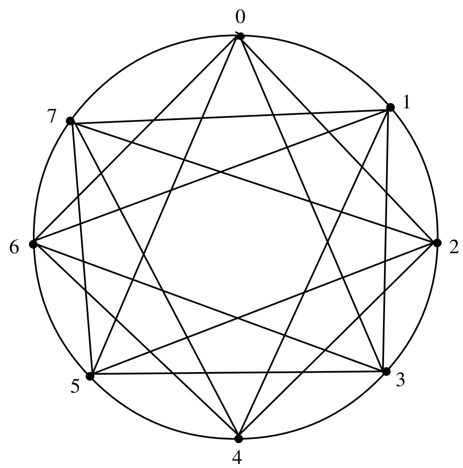

Figure 1: Circulant graph $G(8;\{1,2,3\})$

Definition 2.2. [14] The *wirelength* of an embedding f of G into H is given by

$$WL_f(G,H) = \sum_{(u,v)\in E(G)} d_H(f(u),f(v)) = \sum_{e\in E(H)} EC_f(G,H(e))$$

where $d_H(f(u),f(v))$ denotes the length of the path $P_f(f(u),f(v))$ in H. Then, the *wirelength* of G into H is defined as

$$WL(G,H) = \min WL_f(G,H)$$

where the minimum is taken over all embeddings f of G into H.

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The *edge isoperimetric problem* is NP - complete [10] and is used to solve the wirelength problem when the host graph is a path [4]. When the host graph is a grid, we call the wirelength of the embedding as linear wirelength [16]. The following two versions of the edge isoperimetric problem of a graph $G(V,E)$ have been considered in the literature [4].

Problem 1. Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality. Mathematically, for a given m, if $\theta_G(m) = \min_{A\subseteq V,\ |A|=m} |\theta_G(A)|$ where $\theta_G(A) = \{(u,v)\in E : u\in A, v\notin A\}$, then the problem is to find $A\subseteq V$ such that $|A|=m$ and $\theta_G(m) = |\theta_G(A)|$.

Problem 2. Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m, if $I_G(m) = \max_{A \subseteq V,\ |A|=m} |I_G(A)|$ where $I_G(A) = \{(u,v) \in E : u, v \in A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $I_G(m) = |I_G(A)|$. In the literature, Problem 2 is defined as the *maximum subgraph problem*.

Notation. $EC_f(G, H(e))$ will be represented by $EC_f(e)$. For any set S of edges of H, $EC_f(S) = \sum_{e \in S} EC_f(e)$.

Lemma 2.3. (Congestion Lemma) [14] Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$.

Also S satisfies the following conditions:

(i) For every edge $(a,b) \in G_i$, $i = 1, 2$, $P_f(f(a), f(b))$ has no edges in S.

(ii) For every edge (a,b) in G with $a \in G_1$ and $b \in G_2$, $P_f(f(a), f(b))$ has exactly one edge in S.

(iii) G_1 is an optimal set.

Then $EC_f(S)$ is minimum and $EC_f(S) = r\,|V(G_1)| - 2\,|E(G_1)|$. □

Lemma 2.4. (Partition Lemma) [14] Let $f : G \to H$ be an embedding. Let $\{S_1, S_2, ..., S_p\}$ be a partition of $E(H)$ such that each S_i is an edge cut of H. Then

$$WL_f(G, H) = \sum_{i=1}^{p} EC_f(S_i).$$

3. Circulant Networks

The circulant network is a natural generalization of double loop network and have been used for decades in the design of computer and telecommunication networks due to their optimal fault-tolerance and routing capabilities[6]. It is also used in VLSI design and distributed computation. In addition, these graphs are regular, vertex-symmetric, maximally connected and, after an adequate transformation, they can be represented as mesh-connected topologies[1].

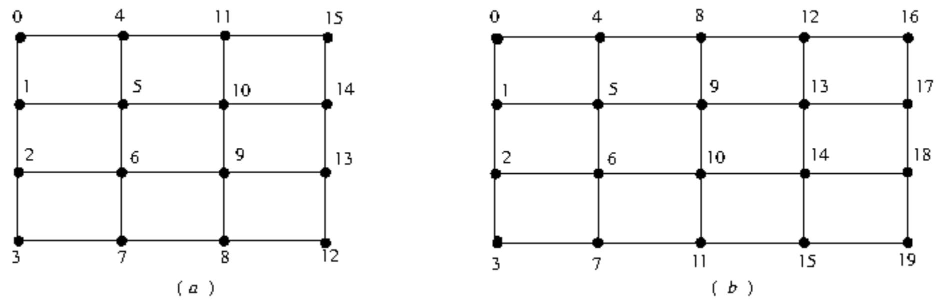

Figure 2: Labeling of $M[4 \times 4]$ and $M[4 \times 5]$

Definition 3.1. [11] A circulant undirected graph, denoted by $G(n; \pm S)$ where $S \subseteq \{1, 2, \cdots, \lfloor n/2 \rfloor\}, n \geq 3$ is defined as a graph consisting of the vertex set $V = \{0, 1, \cdots, n-1\}$ and the edge set $E = \{(i, j) : |j-i| \equiv s(mod n), s \in S\}$. See Figure 1.

Theorem 3.2. [11] *The number of edges in a maximum subgraph on k vertices of $G(n; \pm S), S = \{1, 2, \cdots, j\}, 1 \leq j \leq \lfloor n/2 \rfloor, n \geq 3$ is given by,*

$$\xi = \begin{cases} k(k-1)/2, & k \leq j+1, \\ kj - j(j+1)/2, & j+1 < k \leq n-j, \\ (1/2)\{(n-k)^2 + (4j+1)k - (2j+1)n\}, & n-j < k \leq n. \end{cases}$$

Theorem 3.3. [11] *A set of k consecutive vertices of $G(n; \pm 1), 1 \leq k \leq n$ induces a maximum subgraph of $G(n; \pm S)$ where $S = \{1, 2, \cdots, j\}, 1 \leq j \leq \lfloor n/2 \rfloor, n \geq 3$.*

The proof of the following result is obvious.

Theorem 3.4. *The maximum subgraph on the set of all k vertices of $G(n; \{1, 2, \cdots, j\})$, for $k < j$, is a complete graph on k vertices.*

4. Wirelength of Circulant Networks into Grids

An $l \times m$ grid with l rows and m columns is represented by $M[l \times m]$. In this section, we consider embedding of the circulant network $G(n; \{1, 2, \cdots, \lfloor n/2 \rfloor - 1\}), n \geq 3$ into $M[4 \times m]$, where $m > 2$.

Embedding Algorithm

The embedding of $G(n;\{1,2,\cdots,\lfloor n/2\rfloor-1\})$ with the labeling 0 to $n-1$ of the vertices of $G(n;\pm 1)$ in the clockwise sense into the grid $M[4\times m]$ is an assignment of labels to the nodes of $M[4\times m]$ as follows:

1. For m odd, label the successive vertices of the columns of $M[4\times m]$ consecutively from top to bottom.
2. For m even, label the columns $\{1,2,...,\frac{m}{2}\}$ of $M[4\times m]$ consecutively from top to bottom and label the columns of $M[4\times m]$ consecutively from bottom to top from the column $\frac{m}{2}+1$ to m.

This embedding is denoted by f. See Figure 2.

We note that if we take a subset of k vertices $i_1,i_2,...,i_k$, then $|i_r-i_s|\neq n/2$, for all $1\leq r,s\leq k$. By Theorem 3.4, the subgraph induced by $i_1,i_2,...,i_k$, is a complete graph and hence is a maximum subgraph of G.

From the above observation, we have the following results:

Lemma 4.1. *For m odd,*

$$R_1=\{\ 0,\ \ 1\times 4,\ \ \cdots,\ \ (m-1)\times 4\ \},$$
$$R_2=\{\ 1,\ \ 1\times 4+1,\ \ \cdots,\ \ (m-1)\times 4+1\ \},$$

and for m even, we get,

$$R_1=\left\{\begin{array}{llll} 0, & 1\times 4, & \cdots, & \lceil n/2\rceil-4, \\ & \lceil n/2\rceil+3, & (\lceil n/2\rceil+3)+(4\times 1), & \\ & \cdots, & (\lceil n/2\rceil+3)+(4\times((m/2)-1)) & \end{array}\right\}$$

$$R_2=\left\{\begin{array}{lll} 1, & 1\times 4+1, & \cdots, \\ & (\lceil n/2\rceil-4)+1, & (\lceil n/2\rceil+3)-1, \\ & ((\lceil n/2\rceil+3)+(4\times 1))-1, & \cdots, \\ & (\lceil n/2\rceil+3)+(4\times(m/2-1))-1 & \end{array}\right\}$$

and R_3 which is isomorphic to R_1 in both cases are maximum subgraphs in $G(n;\{1,2,\cdots,\lfloor n/2\rfloor-1\})$.

Lemma 4.2. *For $1\leq j\leq m-1$, $C_j=\{0,1,2,\cdots,4j-1\}$, is maximum in $G(n;\{1,2,\cdots,\lfloor n/2\rfloor-1\})$.*

The proof follows from Theorem 3.3.

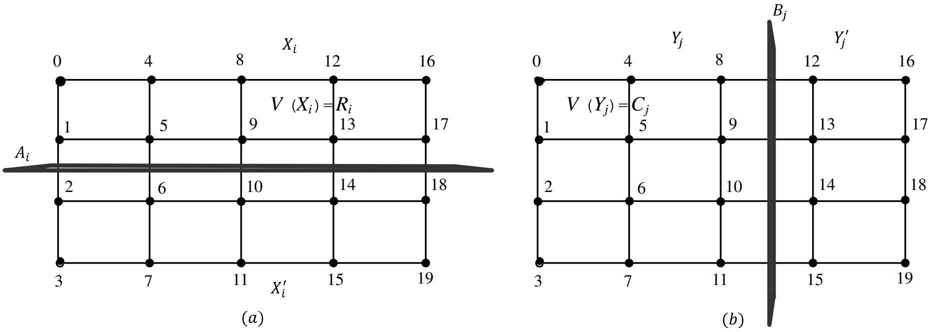

Figure 3: (a) Each A_i is an edge cut on $M[4 \times 5]$ which disconnects $M[4 \times 5]$ into two components X_i and $X_i^{'}$ where $V(X_i)$ is R_i. (b) Each B_j is an edge cut on $M[4 \times 5]$ which disconnects $M[4 \times 5]$ into two components Y_j and $Y_j^{'}$ where $V(Y_j)$ is C_j.

Theorem 4.3. *The embedding f of the circulant network $G(n; \{1, 2, \cdots, \lfloor n/2 \rfloor - 1\})$ into the grid $M[4 \times m]$, where $m > 2$ induces a minimum wirelength $WL(G(n; \{1, 2, \cdots, \lfloor n/2 \rfloor - 1\}), M[4 \times m])$.*

Proof. Let A_i be an edge cut of the grid $M[4 \times m]$ such that A_i disconnects $M[4 \times m]$ into two components X_i and $X_i^{'}$ where $V(X_i)$ is $R_i, i = 1, 2$. See Figure 3(a). Let B_j be an edge cut of the grid $M[4 \times m]$ such that B_j disconnects $M[4 \times m]$ into two components Y_j and $Y_j^{'}$ where $V(Y_j)$ is $C_j, j = 1, 2, ..., m-1$. See Figure 3(b). Let G_i and $G_i^{'}$ be the inverse images of X_i and $X_i^{'}$ under f respectively. The edge cut A_i satisfies conditions (i) and (ii) of the Congestion Lemma. Further by Lemma 4.1, the subgraph G_i induced by the vertices of R_i is maximum. Thus by the Congestion Lemma, $EC_f(A_i)$ is minimum for $i = 1, 2$. Let G_j and $G_j^{'}$ be the inverse images of Y_j and $Y_j^{'}$ under f respectively. The edge cut B_j satisfies conditions (i) and (ii) of the Congestion Lemma. Further by the Lemma 4.2, the subgraph G_j induced by the vertices of C_j is maximum. Thus by the Congestion Lemma, $EC_f(B_j)$ is minimum for $j = 1, 2, ..., m-1$. The Partition Lemma implies that the wirelength is minimum. ◻

Theorem 4.4. *The exact wirelength of $G(4n; \{1, 2, \cdots, \lfloor(2n-1)\})$ into the grid $M[4 \times n]$, where $n \geq 2$ is given by*

$$WL(G(4n; \{1, 2, \cdots, (2n-1)\})), M[4 \times n]) = 2(5n^2 + 6n - 10) + \sum_{j=2}^{n-1} [8j(2n-1) - 16(n-j)^2 - 4j(8n-3) + 4n(4n-1)].$$

5. Conclusion

In this paper, we have partially solved an open problem posed in [11]. The general problem still remains open. It would be interesting to find a strategy to solve this problem.

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