International Journal of Pure and Applied Mathematics
Volume 86 No. 6 2013, 975-981
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v86i6.10

ijpam.eu

CHROMATIC COMPLEMENTARY ACYCLIC DOMINATION IN GRAPHS

M. Valliammal[1], S.P. Subbiah[2], V. Swaminathan[3]

[1]Department of Mathematics
N.M.S. Sermathai Vasan College for Women
Madurai, 12, INDIA

[2]Department of Mathematics
M.T.N. College
Madurai, 04, INDIA

[3]Ramanujan Research Centre
Saraswathi Narayanan College
Madurai, 22, INDIA

Abstract: Let G =(V, E) be a simple graph. A subset D of V(G) is called a complementary acyclic dominating set (c-a dominating set) of G if D is a dominating set and $< V - D >$ is acyclic. D is called a chromatic complementary acyclic dominating set (chromatic c-a dominating set) of G if D is a c-a dominating set and $\chi(< D >) = \chi(G)$.The minimum cardinality of a chromatic c-a dominating set of G is denoted by $\gamma^{\chi}_{c-a}(G)$ and is called chromatic c-a domination number of G.A study of chromatic c-a dominating sets has been made in detail in [5]. In this paper, a study of chromatic c-a dominating sets is initiated.

AMS Subject Classification: 05C35
Key Words: complementary acyclic dominating set, chromatic c-a dominating set, chromatic c-a domination number

Received: May 9, 2013

1. Chromatic Complementary Acyclic Dominating Set

Definition 1.1. Let G =(V, E) be a simple graph. A subset S of V(G) is called a complementary acyclic dominating set (c-a dominating set) of G if S is a dominating set and $< V - S >$ is acyclic.

Definition 1.2. Let $G = (V, E)$ be a simple graph. A subset D of V(G) is called a chromatic complementary acyclic dominating set (chromatic c-a dominating set) of G if D is a complementary acyclic dominating set of G and $\chi(< D >) = \chi(G)$.

Existence. Since $V(G)$ is a c-a dominating set of G and $\chi(< V(G) >) = \chi(G)$, $V(G)$ is a chromatic c-a dominating set of G.

Definition 1.3. The minimum cardinality of a chromatic c-a dominating set of G is called the chromatic c-a domination number of G and is denoted by $\gamma^{\chi}_{c-a}(G)$.

Remark 1.4. $\gamma_{c-a}(G) \leq \gamma^{\chi}_{c-a}(G)$

Theorem 1.5. *Let $G = P_n$. Then*

$$\gamma^{\chi}_{c-a}(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil, & \text{if } n \equiv 1 \ (mod\ 3), \\ \lceil \frac{n}{3} \rceil + 1, & \text{if } n \equiv 0 \text{ or } 2 \ (mod\ 3). \end{cases}$$

Proof. Let D be a γ_{c-a}- set of G.

Case (i), Let $n \equiv 1$ $(mod\ 3)$. Let $V(P_n) = \{u_1, u_2, ..., u_n\}$, $D = \{u_2, u_5, u_8, ..., u_{n-2}, u_{n-1}\}$ is a γ_{c-a}- set of P_n.

Since D is not independent, $\chi(D) = 2 = \chi(P_n)$. D is a chromatic c-a dominating set of G and $\lceil \frac{n}{3} \rceil = \gamma_{c-a}(G) \leq \gamma^{\chi}_{c-a}(G) \leq |D| = \lceil \frac{n}{3} \rceil$.

Therefore $\gamma^{\chi}_{c-a}(P_n) = \lceil \frac{n}{3} \rceil$.

Case (ii) Let $n \equiv 0$ $(mod\ 3)$, where $n = 3k$.

Then $\gamma_{c-a}(P_{3k}) = k$ and it has a unique γ_{c-a}- set. Let $D = \{u_2, u_5, u_8, ..., u_{3k-1}\}$.

Then D is independent. $\chi(< D >) = 1$. But $\chi(P_n) = 2$.

Since $\chi(< D \cup \{u_1\} >) = 2$ and $\gamma^{\chi}_{c-a}(G)$ is an integer, $\gamma^{\chi}_{c-a}(G) > \gamma_{c-a}(G) = |D|$. Therefore $\gamma^{\chi}_{c-a}(G) = |D \cup \{u_1\}| = |D| + 1 = \lceil \frac{n}{3} \rceil + 1$.

Case (iii) Let $n \equiv 2$ $(mod\ 3)$. Let $n = 3k + 2$. Then $\gamma_{c-a}(P_{3k+2}) = k + 1$.

Let D be a γ_{c-a}- set. Suppose D is not independent. Let u_i, $u_{i+1} \in D$.

Subcase(i) Let $i = 3l$.

Consider the path $P_1 :\ u_1, u_2, ..., u_{3l-2}$ and the path $P_2 :\ u_{3l+3}, ..., u_{3k+2}$.

Then $\gamma_{c-a}(P_1) = \left\lceil \frac{3l-2}{3} \right\rceil = l.$ and $\gamma_{c-a}(P_2) = \left\lceil \frac{3(k-l)}{3} \right\rceil = k - l.$

Therefore $\gamma_{c-a}(P_n) = \gamma_{c-a}(P_1) + 2 + \gamma_{c-a}(P_2)$.

(i.e) $\gamma_{c-a}(P_n) = l + 2 + k - l. = k + 2$, a contradiction. Therefore D is independent for every γ_{c-a}- set of P_n.

Since $\chi(P_n) = 2$, $\gamma_{c-a}(G) < \gamma^{\chi}_{c-a}(G)$. Let $u \in D$ and $v \in N(u)$.

Then $D \cup \{v\}$ is a $c-a$ dominating set, $\chi(D \cup \{v\}) = 2$, $D \cup \{v\}$ is a chromatic c-a dominating set. Therefore $\gamma^{\chi}_{c-a}(G) \leq |D \cup \{v\}| = |D| + 1 = \gamma_{c-a}(G) + 1$.

Since $\gamma_{c-a}(G) < \gamma^{\chi}_{c-a}(G)$, $\gamma^{\chi}_{c-a}(G) = \gamma_{c-a}(G) + 1 = \left\lceil \frac{n}{3} \right\rceil + 1$.

Therefore $\gamma^{\chi}_{c-a}(P_n) = \left\lceil \frac{n}{3} \right\rceil + 1$.

Theorem 1.6. $\gamma^{\chi}_{c-a}(K_{m,n}) = min\{m, n\}, \quad m, n \geq 2$

Proof. Let $G = K_{m,n}$, $m \leq n$, $m, n \geq 2$.

Le V_1 and V_2 be the bipartite sets of G. Let $|V_1| = m$ and $|V_2| = n$.

Subcase (i) $m = 2$. Then $\gamma^{\chi}_{c-a}(G) = 2$.

Choose a vertex u from V_1 and a vertex v from V_2. Then $\{u, v\}$ is a chromatic $c - a$ dominating set of G. Therefore $\gamma^{\chi}_{c-a}(K_{m,n}) = 2 = min\{m, n\}$.

Subcase (ii) $m \geq 3$. Let $D = (V_1 - \{u\}) \cup \{v\}$ where $u \in V_1$ and $v \in V_2$. Then D is a chromatic $c - a$ dominating set of G. Therefore $\gamma^{\chi}_{c-a}(G) \leq m$.

Let D_1 be a minimum chromating $c-a$ dominating set of G. If D_1 contains $m-2$ or less points from V_1, the remaining vertices of V_1 will form a cycle with any set of vertices of V_2 of cardinality greater than 2. Therefore D_1 contains at least $m - 1$ vertices from V_1. For chromatic $c - a$ domination, D_1 contains at least one vertex from V_2. Therefore $|D_1| \geq m = min\{m, n\}$.

Therefore $\gamma^{\chi}_{c-a}(K_{m,n}) = m = min\{m, n\}$.

Theorem 1.7. *Let $G = K_n$. Then $\gamma^{\chi}_{c-a}(K_n) = n$ for all $n \geq 1$.*

Theorem 1.8. *Let $G = W_n$.Then* $\gamma^{\chi}_{c-a}(W_n) = \begin{cases} n & if\ n\ is\ even \\ 3 & if\ n\ is\ odd \end{cases}$

Theorem 1.9. *Let $D_{r,s}$ be the double star obtained by joining the central vertices of two stars $K_{1,r}$ and $K_{1,s}$. Then $\gamma^{\chi}_{c-a}(D_{r,s}) = 2$, for all $2 \leq r \leq s$.*

Theorem 1.10. *Let $G = F_n$. Then $\gamma^{\chi}_{c-a}(F_n) = 3$, where F_n denotes a Fan on n vertices, $n \geq 3$.*

Theorem 1.11. $\gamma^{\chi}_{c-a}(G) = 1$ *if and only if* $G = K_1$.

Proof. Suppose $\gamma^{\chi}_{c-a}(G) = 1$. Let D be a chromatic c-a dominating set of G such that $\chi(< D >) = \chi(G)$.

Since $\gamma^{\chi}_{c-a}(G) = 1$, $|D| = 1$. Therefore $\chi(< D >) = 1 = \chi(G)$.

That is $G = \overline{K_n}$. If $n \geq 2$, then D is not a dominating set of G.

Therefore $G = K_1$. The converse is obvious.

Theorem 1.12. *$\gamma^{\chi}_{c-a}(G) = 2$ if and only if G is either $\overline{K_2}$ or bipartite such that if V_1 and V_2 are bipartite sets, each is dominated by a single vertex in the other and the removal of these two vertices results in an acyclic graph.*

Proof. Let $\gamma^{\chi}_{c-a}(G) = 2$. Let $D = \{u_1, u_2\}$ be a γ^{χ}_{c-a}- set of G.

Case (i) D is independent. Therefore $\chi(G) = \chi(< D >) = 1$. Hence $G = \overline{K_n}$. If $n \geq 3$, then $\gamma(G) \geq 3$ and hence $\gamma^{\chi}_{c-a}(G) \geq 3$, a contradiction. Therefore $n \leq 2$. If $n = 1$, then $\gamma^{\chi}_{c-a}(G) = 1$, a contradiction. Therefore $n = 2$. Hence $G = \overline{K_2}$.

Case (ii) D is not independent. Therefore $\chi(< D >) = 2 = \chi(G)$.
Hence G is bipartite. Further if V_1 and V_2 are partite sets of G, then there exists a vertex $v_1 \in V$ which is adjacent with every vertex of V_2 and there exists a vertex $v_2 \in V_2$ such that v_2 is adjacent with every vertex of V_1. Also $< V(G) - \{v_1, v_2\} >$ is acyclic. The converse is obvious.

Theorem 1.13. *Let D be any chromatic c-a dominating set of G.*
Then $|V - D| = \sum_{u \in D} deg(u)$ if and only if $G = \overline{K_n}$.

Proof. If $G = \overline{K_n}$, then $D = V$ and hence the result follows.
Suppose $|V - D| = \sum_{u \in D} deg(u) = k$ (say). Let $k \geq 1$. Then G has an edge and hence $\chi(G) \geq 2$. Let $V - D = \{u_1, u_2, ..., u_k\}$. Since D is a chromatic c-a dominating set of G, each u_i is adjacent to a vertex of D, $1 \leq i \leq k$ and hence contributes at least one degree to every vertex of D. Since $\chi(< D >) \geq 2$, D contains atleast one edge which contributes 2 to degree sum of vertices of D. Hence $\sum_{u \in D} deg(u) \geq k + 2$. Therefore $|V - D| = \sum_{u \in D} deg(u) = k \geq k + 2$, a contradiction. Therefore $k = 0$. Therefore D is independent and $V - D$ is empty. Therefore $V(G) = D$. Hence $G = \overline{K_n}$.

Corollary 1.14. *For any non trivial connected graph with a chromatic c-a dominating set D, $\sum_{u \in D} deg(u) \geq |V - D| + 2$*

Proof. If G is χ-critical, then $V = D$ and $\sum_{u \in D} deg(u) = 2m \geq 2 = |V - D| + 2$.
Suppose G is not χ-critical. Since G is not trivial, $\chi(G) \geq 2$.
By similar argument in the Theorem 2.13, $\sum_{u \in D} deg(u) \geq |V - D| + 2$.

Theorem 1.15. *For any graph G,* $\left\lfloor \frac{n}{\Delta(G)+1} \right\rfloor \leq \gamma_{c-a}^{\chi}(G)$ *and the equality holds if and only if* $G = \overline{K_n}$.

Proof. Since $\left\lfloor \frac{n}{\Delta(G)+1} \right\rfloor \leq \gamma(G) \leq \gamma_{c-a}^{\chi}(G)$, the lower bound is attained. If $G = \overline{K_n}$, then the result is obvious. Suppose $\left\lfloor \frac{n}{\Delta(G)+1} \right\rfloor = \gamma_{c-a}^{\chi}(G) = k$ and D is a γ_{c-a}^{χ}-set of G. Suppose $G \neq \overline{K_n}$, then $\gamma_{c-a}^{\chi}(G) \geq 2$.

Then by corollary 1.14 , $|V - D| < \sum_{u \in D} deg(u)$.

Thus $n - k < \sum_{u \in D} deg(u) \leq k\Delta(G)$ and hence $\frac{n}{\Delta(G)+1} < k$.

But $k > \frac{n}{\Delta(G)+1} \geq \left\lfloor \frac{n}{\Delta(G)+1} \right\rfloor = k$, a contradiction. Thus $G = \overline{K_n}$.

Theorem 1.16. *If* G *is a perfect graph, then* $\gamma_{c-a}^{\chi}(G) \leq \gamma_{c-a}(G) + \omega(G)$.

Proof. Let S be a maximum clique in G and D a γ_{c-a}-set of G. Since G is perfect, $\omega(G) = \chi(G)$ and hence $\chi(G) = \omega(G) = |S| = \chi(< S >)$.

Therefore $\chi(< S \cup D >) = \chi(G)$.Thus $S \cup D$ is a chromatic $c-a$ dominating set of G which implies that $\gamma_{c-a}^{\chi}(G) \leq |S \cup D| \leq |S| + |D| = \omega(G) + \gamma_{c-a}(G)$.

Remark 1.17. If G is a graph with $\Delta(G) = n-1$, then $\gamma_{c-a}^{\chi}(G)$ need not be equal to $\chi(G)$.

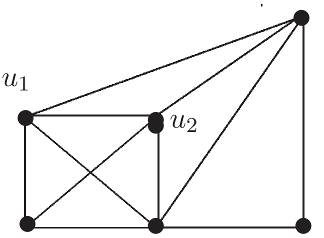
Example 1.18. $\gamma_{c-a}^{\chi}(W_6) = 6$, $\chi(W_6) = 4$ and hence$\gamma_{c-a}^{\chi}(W_6) \neq \chi(W_6)$.

Theorem 1.19. *If* G *is a perfect graph with a full degree vertex and if there exists a maximum clique whose complement is acyclic, then* $\gamma_{c-a}^{\chi}(G) = \chi(G)$.

Proof. Let u be a vertex in $V(G)$ of degree $n-1$. Let S be a maximum clique in G such that $< V - S >$ is acyclic. Clearly S contains u.

Otherwise, $S \cup \{u\}$ is a clique, a contradiction. Therefore S is a chromatic c-a dominating set of G. Therefore $\gamma_{c-a}^{\chi}(G) \leq |S| = \chi(G)$, since G is a perfect graph. But $\chi(G) \leq \gamma_{c-a}^{\chi}(G)$. Therefore $\gamma_{c-a}^{\chi}(G) = \chi(G)$.

Illustration 1.20. Consider the graph G:



Here $\omega(G) = 4 = \chi(G)$. $S = \{u_1, u_2, u_3, u_4\}$ induces a maximum clique and S contains a full degree vertex. $< V - S >$ is acyclic. Hence $\gamma^{\chi}_{c-a}(G) = \chi(G) = 4$.

Theorem 1.21. *Given a positive integer $k \geq 2$, there exists a connected graph G such that $\gamma^{\chi}_{c-a}(G) = k$.*

Proof. (i) $\gamma^{\chi}_{c-a}(K_{k+1,k}) = min\{k + 1, k\} = k$.
(ii) $\gamma^{\chi}_{c-a}(K_k) = k$.

Theorem 1.22. *Given a positive integer $k \geq 1$, there exists a graph G such that $\gamma^{\chi}_{c-a}(G) - \gamma(G) = k$.*

Proof. Let $G = K_{k+1}$. Then $\gamma(G) = 1$ and $\gamma^{\chi}_{c-a}(G) = k + 1$.
Therefore $\gamma^{\chi}_{c-a}(G) - \gamma(G) = k + 1 - 1 = k$.

Theorem 1.23. *Let G be a bipartite graph containing C_4 as an induced subgraph. Then $\gamma_{c-a}(G) = \gamma^{\chi}_{c-a}(G)$.*

Proof. Let u_1, u_2, u_3, u_4 be the vertices of C_4 such that u_1 and u_3 are adjacent. Let D be a γ_{c-a}-set of G such that D contains u_1 and u_3. Then $\chi(< D >) = 2 = \chi(G)$. Therefore D is a chromatic $c - a$ dominating set of G.
Therefore $\gamma^{\chi}_{c-a}(G) \leq |D| = \gamma_{c-a}(G) \leq \gamma^{\chi}_{c-a}(G)$. Hence $\gamma_{c-a}(G) = \gamma^{\chi}_{c-a}(G)$.

Theorem 1.24. *Let G be a χ-critical graph.Then $\gamma^{\chi}_{c-a}(G) = n$.*

Proof. Let D be a γ^{χ}_{c-a}-set of G. Then $< D >$ is a subgraph of G with $\chi(< D >) = \chi(G)$. Since G is χ-critical, $D = V(G)$.
Therefore $\gamma^{\chi}_{c-a}(G) = |D| = n$.

Remark 1.25. There exists a non χ-critical graph G such that $\gamma_{c-a}(G) = n$.

For example, let $G = C_{2n+1} \cup tK_1$, $t \geq 1$.Then G contains C_{2n+1} with $\chi(G) = \chi(C_{2n+1})$.Therefore G is non -critical. Clearly $\gamma^{\chi}_{c-a}(G) = |V(G)|$.

Theorem 1.26. *Let $G = H_1 \cup H_2$, where H_1 is χ-critical, and $\gamma(H_2) = |V(H_2)|$.Then $\gamma^{\chi}_{c-a}(G) = |V(G)|$.*

Proof. Let D be a γ^{χ}_{c-a}-set of G.Since $\gamma(G) = \gamma(H_1) + \gamma(H_2)$, $|D| \geq \gamma(H_1) + \gamma(H_2) = \gamma(H_1) + |V(H_2)|$. Since H_2 is totally disconnected, $\chi(< D >) = \chi(G) = \chi(H_1)$. If $D \cap V(H_1) \neq V(H_1)$,then $\chi(< D \cap V(H_1) >) < \chi(H_1)$.

Therefore $\chi(< D >) = \chi(< D \cap V(H_1) >) < \chi(H_1) = \chi(G)$, a contradiction. Therefore$D \supseteq V(H_1)$. Also $D \supseteq V(H_2)$. Therefore$D \supseteq V(G)$.

But $D \subseteq V(G)$.Therefore $D = V(G)$.

Theorem 1.27. *Let G be a graph without isolates. If $\gamma^{\chi}_{c-a}(G) = n$, then G is χ-critical.*

Proof. Suppose $\gamma^{\chi}_{c-a}(G) =$ n. Let D be a γ^{χ}_{c-a} - set of G.Then D = V(G). Let D_1 be a subset of V(G) with $|D_1| = n - 1$. Since G has no isolates, D_1 is a c-a dominating set of G. Since $\gamma^{\chi}_{c-a}(G) = n$, $\chi(< D_1 >) \neq \chi(G)$. Therefore any subset S of V(G) of cardinality n-1 is such that $\chi(< S >) < \chi(G)$. Let T be any proper subgraph of G. Then $|V(T)| \leq n - 1$.

Therefore $\chi(T) < \chi(G)$(Since V(T) is contained in a subset S_1 of V(G) of cardinality n-1 and $\chi(T) \leq \chi(< S_1 >) < \chi(G)$). Therefore G is χ-critical.

References

[1] Frank Harary, *Graph Theory*, Narosa Publishing House, Reprint (1997).

[2] Gary Chartrand, Ping Zhang, *Chromatic Graph Theory*, CRC Press (2008).

[3] Teresa W. Haynes, Stephen T. Hedetniemi, Peter J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York (1998).

[4] S.M. Hedetniemi, S.T. Hedetniemi, D.F Rall, Acyclic domination, *Discrete Mathematics*, **222** (2000).

[5] M. Poopalaranjani, *On Some Coloring and Domination Parameters in Graphs*, Ph.D. Thesis, Bharathidasan University, India (2006).