International Journal of Pure and Applied Mathematics
Volume 86 No. 6 2013, 983-992
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v86i6.11
ijpam.eu

SOME INEQUALITIES IN VALUED INNER PRODUCT SPACE

S. Vijayabalaji[1], J. Parthiban[2]

[1]Department of Mathematics
University College of Engineering
(A Constituent College of Anna University, Chennai)
Pantruti, 607 106. Tamilnadu, INDIA

[2]Department of Mathematics
Kurinji College of Engineering and Technology
Tiruchirappalli Dist, 621 307, Tamilnadu, INDIA

Abstract: Algebraic Number theory involves using techniques from Algebra and finite group theory to gain deeper understanding of number fields. Motivated by the theory of valued n-inner product over a valued field that was introduced by us; the main objective of this paper is to derive some interesting inequalities on valued n-inner product space combined with the theory of equations.

AMS Subject Classification: 26D15, 39B62
Key Words: valued inner product, arithmetic mean of vectors

1. Introduction

Functional analysis is a branch of mathematical analysis, the core of which is formed by the study of linear spaces endowed with some kind of limit-related structure (e.g. inner product)[10]. The concept of 2-inner product spaces was first introduced by Diminnie, Gahler and White [4, 5].

A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can

Received: May 9, 2013

be found in the book [2]. Misiak [9] generalize the 2-inner product into $n \geq 2$. Recent results about n-inner product space can be viewed in [3].

In Mathematics, the theory of equations comprises a major part of traditional algebra. An inequality is a relation that holds between two values when they are different [6, 7]. There are many inequalities between means. Now we are trying to discuss some new ones based upon valued inner product space.

2. Preliminaries

Definition 2.1. (see [10]) Let X be a real linear space over R. A real valued function $(\bullet, \bullet)$ on $X \times X$ satisfying the following properties:

(1) $(x, x) \geq 0$.

(2) $(x, y) = (y, x)$

(3) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$, for any $\alpha, \beta \in R$(set of real numbers) is called an inner product on X and the pair $(X, (\bullet, \bullet))$ is called an inner product space.

Example 2.2. (see [10]) $(u, v) = u_1v_1 + u_2v_2$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in R^2.

Theorem 2.3. (Cauchy-Schwarz inequality, see [10]) *For any vectors* $u, v \in X$,$|(u, v)| \leq \|u\| \, \|v\|$.

Theorem 2.4. (Triangle inequality, see [10]) *For any vectors* $u, v \in X$,

$$\|u + v\| \leq \|u\| + \|v\| .$$

Definition 2.5. (see [11]) A valuation $[\bullet]$ on a field K is a function with values in $R \geq 0$ satisfying the following axioms:

(1) $[x] > 0$.

(2) $[x]$=0 if and only if $x = 0$.

(3) $[xy] = [x] \, [y]$.

(4) $[x] \leq 1$ implies $[1 + x] \leq C$ for some constant $C \geq 1$. If (4) is satisfied for $C = 2$,then $[\bullet]$ satisfies the triangle inequality,

(4a) $[x + y] \leq [x] + [y] \, forall x, y \in K$.

Example 2.6. (see [12]) The real valuation on the rational Q is the absolute value on the real numbers, defined by $[x] = x, if x \geq 0$ and $-x, if x < 0$.

3. Valued Inner Product Space

Definition 3.1. Let X be a real linear space over R. A real valued function $[\bullet, \bullet]$ on $X \times X$ satisfying the following properties:

(1) $[x, x] \geq 0$.

(2) $[x, y] = [y, x]$

(3) $[\alpha x + \beta y, z] = [\alpha]\,[x, z] + [\beta]\,[y, z]$, for any $\alpha, \beta \in R$(set of real numbers) is called an valued inner product on X and the pair $[(X, [\bullet, \bullet])$ is called an valued inner product space.

Theorem 3.2. (Arithmetic Mean of Inner Product of Vectors as First Kind) *Let $(X, [\bullet, \bullet])$ be a valued inner product space. Then for any non zero vectors in X whose elements are greater than or equal to 1, then*

$$\frac{[u_1, u_2] + [v_1, v_2]}{2} \geq \frac{[u_1, v_1] + [u_2, v_2]}{2},$$

if $\|u_1\| < \|v_2\|$ and $\|u_2\| < \|v_1\|$ or $\|u_1\| > \|v_2\|$ and $\|u_2\| > \|v_1\|$.

Proof. We have

$$\begin{aligned}
&\frac{[u_1, u_2] + [v_1, v_2]}{2} - \frac{[u_1, v_1] + [u_2, v_2]}{2} \\
&\quad \leq \frac{1}{2}\left(\|u_1\|\,\|u_2\| + \|v_1\|\,\|v_2\|\right) - \frac{1}{2}\left(\|u_1\|\,\|v_1\| + \|u_2\|\,\|v_2\|\right) \text{ (by 2.3)} \\
&\quad = \frac{1}{2}\left\{\|u_1\|\left(\|u_2\| - \|v_1\|\right) - \|v_2\|\left(\|u_2\| - \|v_1\|\right)\right\} \\
&\quad = \frac{1}{2}\left\{\left(\|u_1\| - \|v_2\|\right)\left(\|u_2\| - \|v_1\|\right)\right\} \geq 0.
\end{aligned}$$

Therefore $\frac{[u_1,u_2]+[v_1,v_2]}{2} \geq \frac{[u_1,v_1]+[u_2,v_2]}{2}$, if $\|u_1\| < \|v_2\|$ and $\|u_2\| < \|v_1\|$ or $\|u_1\| > \|v_2\|$ and $\|u_2\| > \|v_1\|$.

Theorem 3.3. (Arithmetic Mean of Inner Product of Vectors as Second Kind) *Let $(X, [\bullet, \bullet])$ be a valued inner product space. Then for any non zero vectors in X whose elements are greater than or equal to 1, then*

$$\frac{[u_1, v_1] + [u_2, v_2]}{2} \geq \frac{[u_1, u_2] + [v_1, v_2]}{2},$$

if $\|u_1\| < \|v_2\|$ *and* $\|v_1\| < \|u_2\|$ *or* $\|u_1\| > \|v_2\|$ *and* $\|v_1\| > \|u_2\|$.

Proof.

$$\frac{[u_1,v_1]+[u_2,v_2]}{2} - \frac{[u_1,u_2]+[v_1,v_2]}{2}$$
$$\leq \frac{1}{2}\left(\|u_1\|\,\|v_1\| + \|u_2\|\,\|v_2\|\right) - \frac{1}{2}\left(\|u_1\|\,\|u_2\| + \|v_1\|\,\|v_2\|\right) \text{ (by 2.3)}$$
$$= \frac{1}{2}\left\{\|u_1\|\left(\|v_1\| - \|u_2\|\right) - \|v_2\|\left(\|v_1\| - \|u_2\|\right)\right\}$$
$$= \frac{1}{2}\left\{\left(\|u_1\| - \|v_2\|\right)\left(\|v_1\| - \|u_2\|\right)\right\} \geq 0.$$

Therefore $\frac{[u_1,v_1]+[u_2,v_2]}{2} \geq \frac{[u_1,u_2]+[v_1,v_2]}{2}$,if $\|u_1\| < \|v_2\|$ and $\|v_1\| < \|u_2\|$ or $\|u_1\| > \|v_2\|$ and $\|v_1\| > \|u_2\|$.

Example 3.4. Let $(X, [\bullet,\bullet])$ be a valued inner product space.

$$u_1 = (1,2); u_2 = (3,4).[u_1,u_2] = 1(3) + 2(4) = 11,$$
$$v_1 = (5,6); v_2 = (7,8).[v_1,v_2] = 5(7) + 6(8) = 83,$$
$$\frac{[u_1,u_2]+[v_1,v_2]}{2} = \frac{94}{2} = 47,$$
$$u_1 = (1,2); v_1 = (5,6).[u_1,v_1] = 1(5) + 2(6) = 17,$$
$$u_2 = (3,4); v_2 = (7,8).[u_2,v_2] = 3(7) + 4(8) = 53,$$
$$\frac{[u_1,u_2]+[v_1,v_2]}{2} = \frac{70}{2} = 35.$$

Therefore $\frac{[u_1,u_2]+[v_1,v_2]}{2} \geq \frac{[u_1,v_1]+[u_2,v_2]}{2}$.
Since $\|u_1\| = \sqrt{1^2+2^2} = \sqrt{5}$; $\|u_2\| = \sqrt{3^2+4^2} = \sqrt{25}$ and $\|v_1\| = \sqrt{5^2+6^2} = \sqrt{61}$; $\|v_2\| = \sqrt{7^2+8^2} = \sqrt{113}$, because $\|u_1\| < \|v_2\|$ and $\|u_2\| < \|v_1\|$.

Example 3.5. Let $(X, [\bullet,\bullet])$ be a valued inner product space.

$$u_1 = (1,2); u_2 = (5,6).[u_1,u_2] = 1(5) + 2(6) = 17,$$
$$v_1 = (3,4); v_2 = (7,8).[v_1,v_2] = 3(7) + 4(8) = 53,$$
$$\frac{[u_1,u_2]+[v_1,v_2]}{2} = \frac{70}{2} = 35,$$
$$u_1 = (1,2); v_1 = (3,4).[u_1,v_1] = 1(3) + 2(4) = 11,$$
$$u_2 = (5,6); v_2 = (7,8).[u_2,v_2] = 5(7) + 6(8) = 83,$$

$$\frac{[u_1, u_2] + [v_1, v_2]}{2} = \frac{94}{2} = 47.$$

Therefore $\frac{[u_1,v_1]+[u_2,v_2]}{2} \geq \frac{[u_1,u_2]+[v_1,v_2]}{2}$.

Since $\|u_1\| = \sqrt{1^2 + 2^2} = \sqrt{5}; \|u_2\| = \sqrt{5^2 + 6^2} = \sqrt{61}$, and $\|v_1\| = \sqrt{3^2 + 4^2} = \sqrt{25}; \|v_2\| = \sqrt{7^2 + 8^2} = \sqrt{113}$ because $\|u_1\| < \|v_2\|$ and $\|v_1\| < \|u_2\|$.

Theorem 3.6. *Cauchy-Schwarz inequality can be derived from the Arithmetic mean of inner products as first kind.*

Proof. Arithmetic Mean of Inner Product of Vectors as First Kind

$$\frac{[u_1, v_1] + [u_2, v_2]}{2} \leq \frac{[u_1, u_2] + [v_1, v_2]}{2},$$

if $\|u_1\| < \|v_2\|$ and $\|u_2\| < \|v_1\|$ or $\|u_1\| > \|v_2\|$ and $\|u_2\| > \|v_1\|$.

We have *Geometric mean* $\leq$ *Arithmetic mean.*

$\sqrt{[u_1, v_1][u_2, v_2]} \leq \sqrt{[u_1, u_2][v_1, v_2]}$

Squaring and put $u_1 = u_2$ and $v_1 = v_2$.

Then $[u_1, v_1][u_1, v_1] \leq [u_1, u_1][v_1, v_1]$.

Then $[u_1, v_1]^2 \leq \|u_1\|^2 \|v_1\|^2$.

Taking positive square root, we get

$|[u_1, v_1]| \leq \|u_1\| \|v_1\|$.

Theorem 3.7. *Cauchy-Schwarz inequality can be derived from the Arithmetic mean of inner products as second kind.*

Proof. Arithmetic mean of Inner Product of Vectors as Second kind:

$$\frac{[u_1, u_2] + [v_1, v_2]}{2} \leq \frac{[u_1, v_1] + [u_2, v_2]}{2},$$

if $\|u_1\| < \|v_2\|$ and $\|v_1\| < \|u_2\|$ or $\|u_1\| > \|v_2\|$ and $\|v_1\| > \|u_2\|$.

We have *Geometric mean* $\leq$ *Arithmetic mean.*

$\sqrt{[u_1, u_2][v_1, v_2]} \leq \sqrt{[u_1, v_1][u_2, v_2]}$

Squaring and put $u_1 = v_1$ and $u_2 = v_2$.

Then $[u_1, u_2][u_1, u_2] \leq [u_1, u_1][u_2, u_2]$.

Then $[u_1, u_2]^2 \leq \|u_1\|^2 \|u_2\|^2$.

Taking positive square root, we get

$|[u_1, u_2]| \leq \|u_1\| \|u_2\|$.

Theorem 3.8. *The Inequality, Arithmetic mean of inner products as first kind never depends upon the relation between $\|u_2\|$ and $\|v_2\|$.*

Proof. Arithmetic mean of Inner product of vectors as first kind.
$\frac{[u_1,v_1]+[u_2,v_2]}{2} \leq \frac{[u_1,u_2]+[v_1,v_2]}{2}$,if $\|u_1\| < \|v_2\|$ and $\|u_2\| < \|v_1\|$ or $\|u_1\| > \|v_2\|$ and $\|u_2\| > \|v_1\|$.
We have *Geometric mean $\leq$ Arithmetic mean.*
$\sqrt{[u_1,v_1][u_2,v_2]} \leq \frac{[u_1,u_2]+[v_1,v_2]}{2}$
Squaring and put $u_1 = u_2$ and $v_1 = v_2$.
Then $4[u_2,v_2][u_2,v_2] \leq \{[u_2,u_2]+[v_2,v_2]\}^2$.
Then $4[u_2,v_2]^2 \leq \left\{\|u_2\|^2 + \|v_2\|^2\right\}^2$.
Taking positive square root, we get
$2[u_2,v_2] \leq \|u_2\|^2 + \|v_2\|^2$.
$0 \leq \|u_2\|^2 + \|v_2\|^2 - 2[u_2,v_2]$.
$0 \leq \|u_2\|^2 + \|v_2\|^2 - 2\|u_2\|\|v_2\|$.
Thus $(\|u_2\| - \|v_2\|)^2 \geq 0$ or $(\|v_2\| - \|u_2\|)^2 \geq 0$.
Thus $\|u_2\| - \|v_2\| \geq 0$ or $\|v_2\| - \|u_2\| \geq 0$.
Thus $\|u_2\| \geq \|v_2\|$ or $\|u_2\| \leq \|v_2\|$.
We conclude that, Arithmetic mean of inner products as first kind never depends upon the relation between $\|u_2\|$ and $\|v_2\|$.

Theorem 3.9. *The Inequality, Arithmetic mean of inner products as second kind never depends upon the relation between $\|u_1\|$ and $\|u_2\|$.*

Proof. Arithmetic mean of Inner product of vectors as second kind.
$\frac{[u_1,u_2]+[v_1,v_2]}{2} \leq \frac{[u_1,v_1]+[u_2,v_2]}{2}$,if $\|u_1\| < \|v_2\|$ and $\|v_1\| < \|u_2\|$ or $\|u_1\| > \|v_2\|$ and $\|v_1\| > \|u_2\|$.
We have *Geometric mean $\leq$ Arithmetic mean.*
$\sqrt{[u_1,u_2][v_1,v_2]} \leq \frac{[u_1,v_1]+[u_2,v_2]}{2}$
Squaring and put $u_1 = v_1$ and $u_2 = v_2$.
Then $4[u_1,u_2][u_1,u_2] \leq \{[u_1,u_1]+[u_2,u_2]\}^2$.
Then $4[u_1,u_2]^2 \leq \left\{\|u_1\|^2 + \|u_2\|^2\right\}^2$.
Taking positive square root, we get
$2[u_1,u_2] \leq \|u_1\|^2 + \|u_2\|^2$.
$0 \leq \|u_1\|^2 + \|u_2\|^2 - 2[u_1,u_2]$.
$0 \leq \|u_1\|^2 + \|u_2\|^2 - 2\|u_1\|\|u_2\|$.
Thus $(\|u_1\| - \|u_2\|)^2 \geq 0$ or $(\|u_2\| - \|u_1\|)^2 \geq 0$.
Thus $\|u_1\| - \|u_2\| \geq 0$ or $\|u_2\| - \|u_1\| \geq 0$.

Thus $\|u_1\| \geq \|u_2\|$ or $\|u_1\| \leq \|u_2\|$.
We conclude that, Arithmetic mean of inner products as first kind never depends upon the relation between $\|u_1\|$ and $\|u_2\|$.

Theorem 3.10. *From the Arithmetic mean of Inner product of vectors, we have*
$4[u_1, v_1][u_2, v_2][v_1, v_2] \leq \left(\|u_1\|^4 + \|u_2\|^4\right)\left(\|u_1\|^2 + \|u_2\|^2\right).$

Proof. We have *Geometric mean* $\leq$ *Arithmetic mean.*
From 3.8 and 3.9, wehave
$\sqrt{[u_1, v_1][u_2, v_2]} \leq \frac{[u_1,u_2]+[v_1,v_2]}{2}$
$\sqrt{[u_1, u_2][v_1, v_2]} \leq \frac{[u_1,v_1]+[u_2,v_2]}{2}$
Multiplying together both equations, we get
$\sqrt{[u_1, v_1][u_2, v_2][u_1, u_2][v_1, v_2]} \leq \frac{[u_1,u_2]+[v_1,v_2]}{2}\frac{[u_1,v_1]+[u_2,v_2]}{2}.$
Apply the tools of second kind $u_1 = v_1$ and $u_2 = v_2$ at right hand side, and squaring
$16[u_1, v_1][u_2, v_2][u_1, u_2][v_1, v_2] \leq ([u_1, u_2] + [u_1, u_2])^2 ([u_1, u_1] + [u_2, u_2])^2.$
$16[u_1, v_1][u_2, v_2][u_1, u_2][v_1, v_2] \leq (2[u_1, u_2])^2 \left(\|u_1\|^2 + \|u_2\|^2\right)^2.$
$4[u_1, v_1][u_2, v_2][v_1, v_2] \leq [u_1, u_2]\left(\|u_1\|^2 + \|u_2\|^2\right)^2.$
$4[u_1, v_1][u_2, v_2][v_1, v_2] \leq (\|u_1\| \|u_2\|)\left(\|u_1\|^2 + \|u_2\|^2\right)\left(\|u_1\|^2 + \|u_2\|^2\right).$
$4[u_1, v_1][u_2, v_2][v_1, v_2] \leq \left(\|u_1\|^3 \|u_2\| + \|u_1\| \|u_2\|^3\right)(\|u_1\|^2 + \|u_2\|^2)$
$4[u_1, v_1][u_2, v_2][v_1, v_2] < \left(\|u_1\|^4 + \|u_2\|^4\right)\left(\|u_1\|^2 + \|u_2\|^2\right).$
(By the result $a^3b + ab^3 < a^4 + b^4$)

Theorem 3.11. (Inner Product of Arithmetic Mean of Vectors as First Kind) *Let* $(X, [\bullet, \bullet])$ *be a valued inner product space. Then for any non zero vectors in* X *whose elements are greater than or equal to* 1*, then*
$\frac{[u_1,u_2]+[v_1,v_2]}{2} \geq \left[\frac{u_1+v_1}{2}, \frac{u_2+v_2}{2}\right]$ *,if* $\|u_1\| < \|v_1\|$ *and* $\|u_2\| < \|v_2\|$ *or* $\|u_1\| > \|v_1\|$ *and* $\|u_2\| > \|v_2\|$.

Proof. $\left[\frac{u_1+v_1}{2}, \frac{u_2+v_2}{2}\right] - \frac{[u_1,u_2]+[v_1,v_2]}{2}$
$\leq \frac{1}{4}(\|u_1 + v_1\| \|u_2 + v_2\|) - \frac{1}{2}(\|u_1\| \|u_2\| + \|v_1\| \|v_2\|)(by 2.3).$
$\leq \frac{1}{4}(\|u_1\| + \|v_1\|)(\|u_2\| + \|v_2\|) - \frac{1}{2}(\|u_1\| \|u_2\| + \|v_1\| \|v_2\|)(by 2.4).$
$= \frac{1}{4}(\|u_1\| \|u_2\| + \|u_1\| \|v_2\| + \|u_2\| \|v_1\| + \|v_1\| \|v_2\|) - \frac{1}{2}(\|u_1\| \|u_2\| + \|v_1\| \|v_2\|).$
$= \frac{1}{4}(-\|u_1\| \|u_2\| + \|u_1\| \|v_2\| + \|u_2\| \|v_1\| - \|v_1\| \|v_2\|).$
$= \frac{1}{4}(\|u_1\| (\|v_2\| - \|u_2\|) - \|v_1\| (\|v_2\| - \|u_2\|)).$

$= \frac{1}{4}(\|u_1\| - \|v_1\|)(\|v_2\| - \|u_2\|)$.
$= -\frac{1}{4}(\|u_1\| - \|v_1\|)(\|u_2\| - \|v_2\|) \leq 0$.
Hence $\left[\frac{u_1+v_1}{2}, \frac{u_2+v_2}{2}\right] \leq \frac{[u_1,u_2]+[v_1,v_2]}{2}$,
if $\|u_1\| < \|v_1\|$ and $\|u_2\| < \|v_2\|$ or $\|u_1\| > \|v_1\|$ and $\|u_2\| > \|v_2\|$.

Theorem 3.12. (Inner Product of Arithmetic Mean of Vectors as Second Kind) *Let $(X, [\bullet, \bullet])$ be a valued inner product space. Then for any non zero vectors in X whose elements are greater than or equal to 1, then* $\frac{[u_1,u_2]+[v_1,v_2]}{2} \leq \left[\frac{u_1+v_1}{2}, \frac{u_2+v_2}{2}\right]$ *,if $\|u_1\| < \|v_1\|$ and $\|v_2\| < \|u_2\|$ or $\|u_1\| > \|v_1\|$ and $\|v_2\| > \|u_2\|$.*

Proof. $\frac{[u_1,u_2]+[v_1,v_2]}{2} - \left[\frac{u_1+v_1}{2}, \frac{u_2+v_2}{2}\right]$
$\leq \frac{1}{2}\left(\|u_1\|\|u_2\| + \|v_1\|\|v_2\|\right) - \frac{1}{4}\left(\|u_1 + v_1\|\|u_2 + v_2\|\right) (by 2.3)$.
$\leq \frac{1}{2}\left(\|u_1\|\|u_2\| + \|v_1\|\|v_2\|\right) - \frac{1}{4}\left(\|u_1\| + \|v_1\|\right)\left(\|u_2\| + \|v_2\|\right) (by 2.4)$.
$= \frac{1}{2}(\|u_1\|\|u_2\| + \|v_1\|\|v_2\|) - \frac{1}{4}(\|u_1\|\|u_2\| + \|u_1\|\|v_2\| + \|u_2\|\|v_1\| + \|v_1\|\|v_2\|$.
$= \frac{1}{4}(\|u_1\|\|u_2\| - \|u_1\|\|v_2\| - \|u_2\|\|v_1\| + \|v_1\|\|v_2\|)$.
$= \frac{1}{4}(\|u_1\|(\|u_2\| - \|v_2\|) - \|v_1\|(\|u_2\| - \|v_2\|))$.
$= \frac{1}{4}(\|u_1\| - \|v_1\|)(\|u_2\| - \|v_2\|)$.
$= -\frac{1}{4}(\|u_1\| - \|v_1\|)(\|v_2\| - \|u_2\|) \leq 0$.
Hence $\frac{[u_1,u_2]+[v_1,v_2]}{2} \leq \left[\frac{u_1+v_1}{2}, \frac{u_2+v_2}{2}\right]$,
if $\|u_1\| < \|v_1\|$ and $\|v_2\| < \|u_2\|$ or $\|u_1\| > \|v_1\|$ and $\|v_2\| > \|u_2\|$.

Example 3.13. Let $(X, [\bullet, \bullet])$ be a valued inner product space.
$u_1 = (1, 2); u_2 = (3, 4).[u_1, u_2] = 1(3) + 2(4) = 11$.
$v_1 = (5, 6); v_2 = (7, 8).[v_1, v_2] = 5(7) + 6(8) = 83$.
$\frac{[u_1,u_2]+[v_1,v_2]}{2} = \frac{94}{2} = 47$.

$u_1 = (1, 2); v_1 = (5, 6). \frac{u_1+v_1}{2} = \frac{(1,2)+(5,6)}{2} = (3, 4)$.
$u_2 = (3, 4); v_2 = (7, 8). \frac{u_2+v_2}{2} = \frac{(3,4)+(7,8)}{2} = (5, 6)$.
$[\frac{u_1+v_1}{2}, \frac{u_2+v_2}{2}] = 3(5) + 4(6) = 39$.

Therefore $\left[\frac{u_1+v_1}{2}, \frac{u_2+v_2}{2}\right] \leq \frac{[u_1,u_2]+[v_1,v_2]}{2}$.

Since $\|u_1\| = \sqrt{1^2 + 2^2} = \sqrt{5}; \|u_2\| = \sqrt{3^2 + 4^2} = \sqrt{25}$.
and $\|v_1\| = \sqrt{5^2 + 6^2} = \sqrt{61}; \|v_2\| = \sqrt{7^2 + 8^2} = \sqrt{113}$.
Since $\|u_1\| < \|v_1\|$ and $\|u_2\| < \|v_2\|$.

Example 3.14. Let $(X, [\bullet, \bullet])$ be a valued inner product space.
$u_1 = (1,2); u_2 = (7,8).[u_1, u_2] = 1(7) + 2(8) = 23.$
$v_1 = (5,6); v_2 = (3,4).[v_1, v_2] = 5(3) + 6(4) = 39.$
$\frac{[u_1,u_2]+[v_1,v_2]}{2} = \frac{62}{2} = 31.$

$u_1 = (1,2); v_1 = (5,6).\frac{u_1+v_1}{2} = \frac{(1,2)+(5,6)}{2} = (3,4).$
$u_2 = (7,8); v_2 = (3,4).\frac{u_2+v_2}{2} = \frac{(7,8)+(3,4)}{2} = (5,6).$
$[\frac{u_1+v_1}{2}, \frac{u_2+v_2}{2}] = 3(5) + 4(6) = 39.$
Therefore $\left[\frac{u_1+v_1}{2}, \frac{u_2+v_2}{2}\right] \geq \frac{[u_1,u_2]+[v_1,v_2]}{2}$.
Since $\|u_1\| = \sqrt{1^2 + 2^2} = \sqrt{5}; \|u_2\| = \sqrt{7^2 + 8^2} = \sqrt{113}.$
and $\|v_1\| = \sqrt{5^2 + 6^2} = \sqrt{61}; \|v_2\| = \sqrt{3^2 + 4^2} = \sqrt{25}.$
Since $\|u_1\| < \|v_1\|$ and $\|u_2\| > \|v_2\|$.

References

[1] Tom M. Apostal, *An Introduction to Analytic Number Theory*, Springer,Newyork (1976).

[2] Y.J. Cho, P.C.S. Lin, S.S. Kim, A. Misiak, *Theory of 2-Inner Product Spaces*, Nova Science Publishers, New York (2001).

[3] Y.J. Cho, M. Matic, J. Pecaric, Inequalities of Hlawka's type in n-inner product spaces, *Commun. Korean Math. Soc.*, No. 4 (2002), 583-592.

[4] C. Diminnie, S. Gahler, A. White, 2-inner product spaces, *Demonstratio Math.* (1973), 523-536.

[5] C. Diminnie, S. Gahler, A. White, 2-inner product spaces II, *Demonstratio Math.* (1977), 169-188.

[6] G.H. Hardy, E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford Univ. Press, Newyork (1)980.

[7] G.H. Hardy, Wood J.E. Little, G. Polya, *Inequalities,* Cambridge University Press (1999).

[8] J.S. Milne, *Algebraic Number Theory*, University of Michigan (2011), 101-110.

[9] A. Misiak, n-inner product spaces, *Math. Nachr.* (1989), 299-319.

[10] W. Rudin, *Functional Analysis,* McGraw-Hill Science (1991).

[11] Lang Serge, *Algebraic Number Theory*, Chapman and Hall, CRC Press (1987).

[12] I.N. Stewart, Tall Do, *Algebraic Number Theory*, Chapman and Hall/CRC Press (1987).

[13] S. Vijayabalaji, N. Thillaigovindan, Fuzzy n-inner product space, *Bulletin of Korean Mathematical Society*, **43**, No. 3 (2007), 447-459.