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EMBEDDING OF CIRCULANT NETWORKS INTO k-ROOTED SIBLING TREES

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Abstract: Graph embedding problems have gained importance in the field of interconnection networks for parallel computer architectures. In this paper, we determine the exact wirelength and dilation of embedding circulant networks into k-rooted sibling trees.

AMS Subject Classification: 05C78, 05C85
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1. Introduction

Graph embedding is an important technique that maps a logical graph into a host graph, usually an interconnection network. Many applications can be modeled as graph embedding [5 - 10]. The quality of an embedding can be measured by certain cost criteria. One of these criteria which is considered very often is the *dilation*. The dilation of an embedding is defined as the maximum

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distance between pairs of vertices of host graph that are images of adjacent vertices of logical graph. It is a measure for the communication time needed when simulating one network on another. Another important cost criteria is the *wirelength* [2, 3]. Graph embeddings have been well studied for a number of networks [5 - 12].

For basic definitions and preliminaries related to embedding problems see [7 - 12].

Definition 1. (see [10]) The undirected circulant graph $G(n;S)$, $S \subseteq \pm\{1,2,\ldots,j\}$, $1 \leq j \leq \lfloor n/2 \rfloor$ is a graph with vertex set $V = \{0,1,\ldots,n-1\}$ and the edge set $E = \{(i,k) : |k-i| \equiv s(mod\ n),\ s \in S\}$.

Lemma 2. (see [8]) *A set of k consecutive vertices of $G(n;\pm 1)$, $1 \leq k \leq n$ induces a maximum subgraph of $G(n;\pm S)$, where $S = \{1,2,\ldots,j\}$, $1 \leq j < \lfloor n/2 \rfloor$, $n \geq 3$.*

Lemma 3. (Congestion Lemma) (see [6]) *Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:*

(i) *For every edge $(a,b) \in G_i$, $i = 1,2, P_f(a,b)$ has no edges in S.*

(ii) *For every edge (a,b) in G with $a \in G_1$ and $b \in G_2$, $P_f(a,b)$ has exactly one edge in S.*

(iii) *G_1 is an optimal set.*

Then $EC_f(S)$ is minimum and $EC_f(S) = \sum_{e \in S} EC_f(e) = r\,|V(G_1)| - 2\,|E(G_1)|$.
□

Lemma 4. (2-Partition Lemma) (see [12]) *Let $f : G \rightarrow H$ be an embedding. Let $[2E(H)]$ denote a collection of edges of H repeated exactly 2 times. Let $\{S_1, S_2, \ldots, S_m\}$ be a partition of $[2E(H)]$ such that each S_i is an edge cut of H. Then*

$$WL_f(G,H) = \frac{1}{2}\sum_{i=1}^{m} EC_f(S_i).$$

For notations in this paper, we refer the readers to [12].

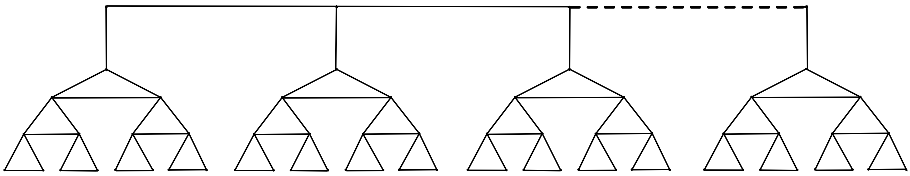

Figure 1: k-rooted Sibling Tree ST_n^k

2. Wirelength of Embedding Circulant Networks into k-rooted Sibling Trees

A tree is a connected graph that contains no cycles. For any non-negative integer n, the complete binary tree of height n, denoted by T_n, is the binary tree where each internal vertex has exactly two children and all the leaves are at the same level. Clearly, a complete binary tree T_n has n levels and contains 2^{i-1} vertices at level i, $1 \leq i \leq n$. Thus T_n has exactly $2^n - 1$ vertices.

The 1-rooted sibling tree ST_n^1 is obtained from the 1-rooted complete binary tree T_n^1 by adding edges (sibling edges) between left and right children of the same parent node. The k-rooted sibling tree ST_n^k is obtained by taking k vertex disjoint 1-rooted sibling tree ST_n^1 on 2^n vertices with roots say $r_1, r_2, ..., r_k$ and adding the edges (r_i, r_{i+1}), $1 \leq i \leq k-1$. See Figure 1. The diameter of ST_n^k is $2n + k - 1$, see [12].

Now, we produce exact wirelength of embedding circulant network

$$G(2^n; \pm\{1, 2, \cdots, j\}), \quad 1 \leq j < \lfloor \frac{n}{2} \rfloor$$

into k-rooted sibling tree $ST_{n_1}^k$, $n \geq 3$ and $k = 2^{n-n_1}$. For brevity, the circulant network $G(2^n; \pm\{1, 2, \cdots, j\})$, $1 \leq j < \lfloor \frac{n}{2} \rfloor$ and k-rooted sibling tree $ST_{n_1}^k$ will be represented by G and H respectively.

Wirelength Algorithm

Input: The circulant network G and the k-rooted sibling tree H.

Algorithm: Label the consecutive vertices of $G(2^n; \pm 1)$ in G as $0, 1, 2, \ldots,$ $2^n - 1$ in the clockwise sense and label the vertices of ST_n^l, $1 \leq l \leq k$ in H using postorder traversal, see [14] and Figure 2.

Output:

An embedding f of G into H given by $f(x) = x$ with minimum wirelength.

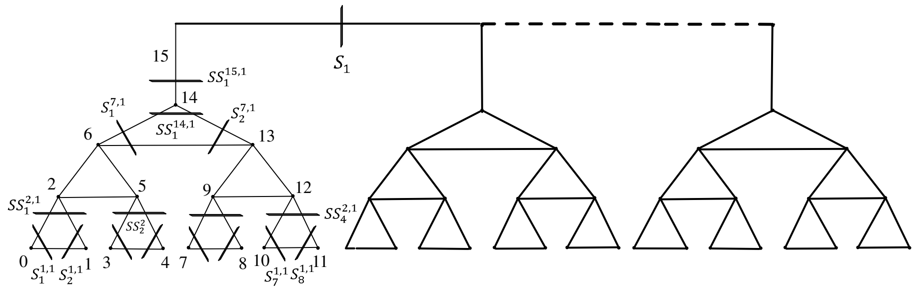

Figure 2: Edge cut of ST_4^k

Proof of Correctness

For $l = 1, 2, ..., k,\ j = 1, 2, ..., n$ and $i = 1, 2, ..., 2^{n-j}$, let $S_i^{2^j-1,l}$ be an edge cut of H consisting of edges induced by the $\lceil i/2 \rceil^{\text{th}}$ parent vertex from left to right in level $n - j$ with its left child if i is odd and its right child if i is even together with the corresponding sibling edge which is the same edge in either case, such that $S_i^{2^j-1,l}$ disconnects H into two components $H_i^{2^j-1,l}$ and $\overline{H}_i^{2^j-1,l}$ where $V(H_i^{2^j-1,l})$ is consecutively labeled (see [5]). See Figure 2. Let $G_i^{2^j-1,l}$ and $\overline{G}_i^{2^j-1,l}$ be the inverse images of $H_i^{2^j-1,l}$ and $\overline{H}_i^{2^j-1,l}$ under f respectively. By Lemma 2, $G_i^{2^j-1.l}$ is an optimal set in G. Thus $EC_f(S_i^{2^j-1,l})$ is minimum for $l = 1, 2, ..., k,\ j = 1, 2, ..., n$ and $i = 1, 2, ..., 2^{n-j}$, by Congestion Lemma.

For $l = 1, 2, ..., k,\ j = 1, 2, ..., n-1$ and $i = 1, 2, ..., 2^{n-j-1}$, let $SS_i^{2(2^j-1),l}$ be an edge cut of H consisting of the edges induced by the i^{th} parent vertex from left to right in level $n-j$ and its two children, such that $SS_i^{2(2^j-1),l}$ disconnects H into two components
$H_i^{2(2^j-1),l}$ and $\overline{H}_i^{2(2^j-1),l}$ where $V(H_i^{2(2^j-1)})$ is consecutively labeled, see [5]. Let $G_i^{2(2^j-1),l}$ and $\overline{G}_i^{2(2^j-1),l}$ be the inverse images of $H_i^{2(2^j-1),l}$ and $\overline{H}_i^{2(2^j-1),l}$ under f respectively. By Lemma 2, $G_i^{2(2^j-1),l}$ is an optimal set in G. Therefore $EC_f(SS_i^{2(2^j-1),l})$ is minimum for $l = 1, 2, ..., k,\ j = 1, 2, ..., n-1$ and $i = 1, 2, ..., 2^{n-j-1}$, by Congestion Lemma.

For $l = 1, 2, ..., k,\ SS_1^{2^{n_1}-1,l}$ is identified with $S_1^{2^{n_1}-1,l}$, where $SS_1^{2^{n_1}-1,l}$ consists of the edge joining vertices at level 0 and 1 in ST_n^1. For $l = 1, 2, ..., k-1$, let $S_l = S'_l = (r_l, r_{l+1})$. By Congestion Lemma, $EC_f(SS_1^{2^{n_1}-1,l})$ is minimum for $l = 1, 2, ..., k$. In the same manner, $EC_f(S_l)$ is minimum for $l = 1, 2, ..., k-1$.

We note that the set $\{S_i^{2^j-1,l} : 1 \leq l \leq k, 1 \leq j \leq n, 1 \leq i \leq 2^{n-j}\} \cup$

$\{SS_i^{2(2^j-1),l} : 1 \leq l \leq k, 1 \leq j \leq n-1, 1 \leq i \leq 2^{n-j-1}\} \cup \{SS_1^{2^{n_1}-1,l} : 1 \leq l \leq k\} \cup \{S_l : 1 \leq l \leq k-1\}$ forms a partition of $E^2(H)$. The 2-Partition Lemma implies that $WL_f(G,H)$ is minimum.

Theorem 5. *The exact wirelength of embedding G into H is given by*

$$\begin{aligned} WL(G,H) &= \frac{k}{2}\sum_{j=1}^{n}(2^{n-j})\theta_G(2^j-1) + \frac{k}{2}\sum_{j=1}^{n-1}(2^{n-j-1})\theta_G(2(2^j-1)) \\ &\quad +k\ \theta_G(2^{n_1}-1) + (k-1)\ \theta_G(2^{n_1}). \end{aligned}$$

Proof. By Congestion Lemma,

(i) $EC_f(S_i^{2^j-1,l}) = \theta_G(2^j-1)$, $1 \leq l \leq k, 1 \leq j \leq n, 1 \leq i \leq 2^{n-j}$

(ii) $EC_f(SS_i^{2(2^j-1),l}) = \theta_G(2(2^j-1))$, $1 \leq l \leq k, 1 \leq j \leq n-1, 1 \leq i \leq 2^{n-j-1}$

(iii) $EC_f(SS_1^{2^{n_1}-1,l}) = EC_f(S_1^{2^{n_1}-1,l}) = \theta_G(2^{n_1}-1)$, $1 \leq l \leq k$ and

(iv) $EC_f(S_l) = EC_f(S'_l) = \theta_G(2^{n_1})$, $1 \leq l \leq k-1$.

Then by 2-Partition Lemma,

$$\begin{aligned} WL(G,H) &= \frac{1}{2}\,[\sum_{l=1}^{k}\sum_{j=1}^{n}\sum_{i=1}^{2^{n-j}}\theta_G(2^j-1) + \sum_{l=1}^{k}\sum_{j=1}^{n-1}\sum_{i=1}^{2^{n-j-1}}\theta_G(2(2^j-1)) \\ &\quad +2\sum_{l=1}^{k}\theta_G(2^{n_1}-1) + 2\sum_{l=1}^{k-1}\theta_G(2^{n_1})] \\ &= \frac{k}{2}\sum_{j=1}^{n}(2^{n-j})\theta_G(2^j-1) + \frac{k}{2}\sum_{j=1}^{n-1}(2^{n-j-1})\theta_G(2(2^j-1)) \\ &\quad +k\ \theta_G(2^{n_1}-1) + (k-1)\ \theta_G(2^{n_1}). \qquad \square \end{aligned}$$

3. Dilation of Embedding Circulant Networks into k-rooted Sibling Trees

The dilation problem and the wirelength problem are different in the sense that an embedding that gives minimum dilation need not give minimum wirelength and vice-versa. In the literature there is no efficient method to compute exact dilation of graph embeddings, see [4, 7, 10]. The Dilation Lemma [13] formulated by Sundara Rajan, Paul Manuel and Indra Rajasingh is a powerful tool to compute exact dilation of an embedding.

Lemma 6. (Dilation Lemma, see [13]) *Let G be an r-regular graph of order n. Let H be a graph on n vertices such that for $u \in V(H)$, $D_\delta(u) \neq \phi$, where $D_\delta(u)$ denotes the set of all diametrically opposite vertices of u in G. If $|D_\delta(u)| + |D_{\delta-1}(u)| + \cdots + |D_{\delta-k(u)}(u)| \geq n - r$, then the dilation of embedding G onto H is at least $\delta - k$, where $k = \min\limits_{u \in V(H)} k(u)$ and δ is the diameter of H.*

For brevity, the circulant network $G(2^n; \pm\{1, 2, \cdots, 2^{n-1} - 2\})$ and k-rooted sibling tree $ST_{n_1}^k$ will be represented by G and H respectively.

Theorem 7. *The exact dilation of embedding G into H is given by $dil(G, H) = 2n_1 + k - 1$.*

Proof. G is vertex-transitive and hence without loss of generality select any vertex u in G. By the definition of circulant graph, there are $n - (2j + 1) = 2^n - [2(2^{n-1} - 2) + 1] = 3$ vertices which are not adjacent to u in G. Since $|D_\delta(f(u))| \geq 4$, by Dilation Lemma $dil(G, H) \geq 2n_1 + k - 1$. It is clear that $dil(G, H) \leq 2n_1 + k - 1$, as the diameter of H is $2n_1 + k - 1$. Hence the theorem. □

4. Conclusion

In this paper, we compute the minimum wirelength and dilation of embedding circulant networks into k-rooted sibling trees. Finding the wirelength and dilation of embedding circulant networks into certain trees such as X-trees, Christmas tree and Brother tree are under investigation.

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