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THE GRAPH $\Gamma_2(R)$ OVER A RING R

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Abstract: For a ring R, we define a simple undirected graph $\Gamma_2(R)$ with all the non-zero elements of R as vertices, and two vertices a, b are adjacent if and only if either $ab = 0$ or $ba = 0$ or $a + b$ is a zero-divisor (including 0). We first consider its connectedness. Looking at $\mathbb{Z}_n$, we determine the condition for connectedness of $\Gamma_2(\mathbb{Z}_n)$ and also discuss its structure. We then consider connectedness, 2-connectedness and other properties of $\Gamma_2(R)$ when R is a direct product of rings. Giving particular attention to $\Gamma_2(\mathbb{Z}_n)$, we find out the degree patterns and consider girth, Eulerianity and planarity. Then we look at the non-commutative case of Γ_2 graph over the matrix rings and the infinite case of $\Gamma_2(\mathbb{Z})$ and $\Gamma_2(\mathbb{Z} \times R)$, where R is any ring $\neq \{0\}$.

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1. Introduction

Among graphs associated to a ring, Zero-divisor graph holds a special place due to the fact that it has revealed that the set of Zero-divisors of a ring, which algebraically does not possess a nice structure, has significant and interesting graph-theoretic structures. After the concept of zero-divisor graph was introduced by Beck [3], it was defined by D.F. Anderson and Livingston [2] in the following way : Let R be a commutative ring with 1. Then the zero-divisor graph of R, denoted as $\Gamma(R)$, is the simple undirected graph with all the non-

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zero zero-divisors of R as the vertices, and two vertices x, y are adjacent if and only if $xy = 0$. Though it was at first defined for commutative rings, Redmond [7] modified the definition in order to involve non-commutative rings also.

Significant results obtained from structure of $\Gamma(R)$ motivated us to define newer types of graphs over rings, involving both the operations of a ring, viz. addition and multiplication, in the adjacency conditions (The addition operation involving the zero-divisors has been used to define the "*total graph*" over a commutative ring by Anderson and Badawi [1]). We have earlier defined a graph $\Gamma_1(R)$ over a ring R with unity in the following way :

Definition 1.1. [8] Let R be a ring with unity. Let $G = (V, E)$ be an undirected graph in which $V = R - \{0\}$ and for any $a, b \in V$, $ab \in E$ if and only if $a \neq b$ and either $a.b = 0$ or $b.a = 0$ or $a + b$ is a unit. We denote this graph G by $\Gamma_1(R)$.

Here, a second type of graph $\Gamma_2(R)$ for a ring R is introduced, taking the same vertex set and changing the last part of the adjacency condition of $\Gamma_1(R)$ graphs.

In this paper, the symbol $a \leftrightarrow b$ denotes that vertices a, b are adjacent. For graphs G and H, $G \cup H$ denotes the disjoint union of G and H. $\phi(n)$ is the Euler's phi function. For usual graph-theoretic terms and definitions, one can look at [9].

2. The Graph $\Gamma_2(R)$

Definition 2.1. Let R be a ring . Let $G = (V, E)$ be an undirected graph in which $V = R - \{0\}$ and for any $a, b \in V$, $ab \in E$ if and only if $a \neq b$ and either $a.b = 0$ or $b.a = 0$ or $a + b$ is a zero-divisor (including 0). Clearly G is simple. We denote this simple undirected graph G by $\Gamma_2(R)$.

Remark 2.2. (i) From the definition, it is clear that the Zero-divisor graph $\Gamma(R)$ (or $\overline{\Gamma}(R)$, in the non-commutative case [7]) is a subgraph of $\Gamma_2(R)$ for any ring R.

(ii) Note that the last part of the adjacency condition of $\Gamma_2(R)$ was used as the only adjacency condition for defining The *total graph* over a commutative ring in [1]. In total graph, all the elements of the ring are taken as vertices, i.e., 0 is not excluded, unlike our definition.

(iii) Since in a finite ring, every nonzero element is either a zero-divisor or a unit, it is clear that for a finite field F, $\overline{\Gamma_1(F)} \cong \Gamma_2(F)$.

Regarding the connectedness of the $\Gamma_2(R)$ graph, we start with the case when $R = \mathbb{Z}_n = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{n-1}\}$. For convenience, we write m instead of $\overline{m}$ for elements in $\mathbb{Z}_n$.

Theorem 2.3. $\Gamma_2(\mathbb{Z}_n)$ *is never a complete graph unless* $n = 2$ *or* 3.

Proof. $\Gamma_2(\mathbb{Z}_3)$ is complete graph (K_2). $\Gamma_2(\mathbb{Z}_2)$ has only one vertex. For all $n > 3$, $\mathbb{Z}_n$ has two units 1 and -1. Now in these cases 1 is not adjacent to -2. So $\Gamma_2(\mathbb{Z}_n)$ cannot be complete. □

Now we try to find out the values of n for which $\Gamma_2(\mathbb{Z}_n)$ is connected. We have the following result:

Theorem 2.4. $\Gamma_2(\mathbb{Z}_n)$ *is connected if and only if either* $n \leq 3$ *or the prime-power factorization of* n *has more than one prime factor and in the latter case* $diam(\Gamma_2(\mathbb{Z}_n)) = 2$.

Proof. (i) $\Gamma_2(\mathbb{Z}_2)$ and $\Gamma_2(\mathbb{Z}_3)$ being respectively a singleton graph and a K_2, are connected. Now suppose $n > 3$ and let n have only one prime factor p. Here the zero-divisors are precisely the multiples of p and hence sum of a unit and a zero-divisor cannot be a multiple of p, i.e., a zero-divisor (note that it is clearly $\neq 0$). So there is no edge between a zero-divisor and a unit and hence the subgraphs induced respectively by zero-divisors and units are mutually disjoint in this case. So $\Gamma_2(\mathbb{Z}_n)$ is not connected in this case.

(ii) Now consider the case when $n = p_1^{q_1} p_2^{q_2} \ldots p_r^{q_r}$ where $r > 1$.

Case I : Let z_1, z_2 be two zero-divisors. If they are not adjacent, then they cannot have a common prime factor p_i among the prime factors of n. So we have the path $z_1 - (n/z_1) - z_2$.

Case II : Let z be a zero-divisor and u be a unit. Now suppose they are not adjacent . Let z be not a multiple of some p_i among the prime factors of n. Then there exist integers x, y such that $xz + yp_i = 1$. So $uxz + uyp_i = u \implies u - uxz = uyp_i$. So u is adjacent to $(-uxz)$. So we have the path $u - (-uxz) - z$. Again, if z is a multiple of all the prime factors of n, then considering p_1 and p_2, there exist integers x, y such that $xp_1 + yp_2 = 1 \implies u - uxp_1 = uyp_2$. So we have the path $u - (-uxp_1) - z$.

Case III : Let u_1, u_2 be two units. Suppose that they are not adjacent. If $u_1 - u_2$ is a zero divisor, then we have the path $u_1 - (-u_2) - u_2$. Now let $u_1 - u_2 = v$ be a unit. Now there exist integers x, y such that $xp_1 + yp_2 = 1$. So $v - vxp_1 = vyp_2 \implies u_1 - u_2 - vxp_1 = vyp_2$. So we have the path $u_1 - (-u_2 - vxp_1) - u_2$.

From the above three cases it is clear that between any two vertices there is a path of length at most 2 . From Theorem 2.3, it is clear that diameter is 1 if and only if $n = 3$ and hence when n has more than one prime factor, $\Gamma_2(\mathbb{Z}_n)$ is connected with diameter 2. □

Remark 2.5. Proof of Theorem 2.4 shows that the subgraph of $\Gamma_2(\mathbb{Z}_n)$ induced by the zero-divisors is also of diameter ≤ 2.

Now we consider the the number of components of $\Gamma_2(\mathbb{Z}_n)$ when n has only one prime factor. We know that a natural number can be expressed as sum of two natural numbers in several ways. For an odd prime p, the number of ways is $(p-1)/2$ viz. $\{1, p-1\}, \{2, p-2\}, \ldots\ldots, \{(p-1)/2, (p+1)/2\}$. With repect to the operation addition modulo p, these can be represented as $[p \pm 1], [p \pm 2], [p \pm 3] \ldots., [p \pm (p-1)/2]$ repectively. It can be shown that in the subgraph of $(\Gamma_2(\mathbb{Z}_{p^r}))$ induced by the units, there are $(p-1)/2$ components, each corresponding to those $(p-1)/2$ modulo classes. We give the following result whose proof in fact follows from Theorem 2.2 of [1]:

Theorem 2.6. *Let $n = p^r$, where p is an odd prime and $r \in \mathbb{N}$. Then $\Gamma_2(\mathbb{Z}_n)$ has $(p+1)/2$ components, one $K_{p^{r-1}-1}$ consisting of the zero-divisors, and $(p-1)/2$ copies of $K_{p^{r-1},p^{r-1}}$ for the units.*

Theorem 2.7. *$\Gamma_2(\mathbb{Z}_{2^r})$, where $r \in \mathbb{N}$, has two components consisting of zero-divisors and units of $\mathbb{Z}_{2^r}$ respectively. The first is a $K_{2^{r-1}-1}$ and the other is a $K_{2^{r-1}}$.*

Proof. Here zero-divisors are precisely the even numbers and units are precisely the odd numbers. So the result follows since sum of two odd or two even is a even number, but sum of an odd and an even is odd. □

So $\Gamma_2(R)$ is not always a connected graph. However, it is connected when R is a direct product of rings.

Theorem 2.8. *Let $R = R_1 \times R_2 \times \ldots\ldots. \times R_n$, where R_i's are rings with unity and $n > 1, n \in \mathbb{N}$. Then $\Gamma_2(R)$ is connected with $diam(\Gamma_2(R)) \leq 2$ and $girth(\Gamma_2(R)) = 3$.*

Proof. (i) From Theorem 2.3 in [2], there is a path between any two vertices a, b, corresponding to two zero-divisors of R. So if we can show that each vertex which corresponds to an element which is not a zero-divisor of R, is adjacent to at least one zero-divisor, then it is assured that there will be a path between any two vertices. Now consider a vertex $u = (a_1, a_2, \ldots, a_n)$ such that u is not a zero-divisor in R. So $a_i \neq 0$ for $i = 1, 2, .., n$. Now u is adjacent to $(-a_1, 0, 0, \ldots, 0)$, the latter being a zero-divisor. So any vertex u which is not a zero-divisor in R, has a zero-divisor adjacent with it. So as we argued, $\Gamma_2(R)$ is connected.

(ii) Consider two vertices $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$. If they are not adjacent, then we must have $a_i + b_i \neq 0$ for $i = 1, 2, ..., n$. Now if at least one of a_1 and b_n is non-zero, we have $(a_1, a_2, \ldots, a_n) \leftrightarrow (-a_1, 0, 0, ..., 0, -b_n) \leftrightarrow (b_1, b_2, \ldots, b_n)$.. Note that $(-a_1, 0, 0, ..., 0, -b_n)$ is equal to neither $(a_1, a_2, \ldots, a_n)$ nor $(b_1, b_2, \ldots, b_n)$ since $a_1 + b_1 \neq 0$ and $a_n + b_n \neq 0$. Again, let $a_1 = b_n = 0$. Then if $n > 2$, we have $(a_1, a_2, \ldots, a_n) \leftrightarrow (0, 1, 0, ..., 0) \leftrightarrow (b_1, b_2, \ldots, b_n)$ and if $n = 2$, then $(a_1, a_2) \leftrightarrow (b_1, b_2)$. So diameter is ≤ 2.

(iii) First, let $R_i \ncong \mathbb{Z}_2$ for at least one i. Then that R_i has an element, say x, apart from $0, 1$. Let ${e_i}^x$ be the n-tuple whose i-th component is x and others are 0. Then for any $j \neq i$, we have the 3-cycle ${e_i}^x - {e_j}^1 - {e_i}^1 - {e_i}^x$. Again, let $R_i \cong \mathbb{Z}_2$ for $i = 1, 2, \ldots, n$. Then we have the 3-cycle ${e_1}^1 - {e_n}^1 - (1, 1, \ldots, 1) - {e_1}^1$. So girth of $\Gamma_2(R)$ is 3. Hence the result. □

Example 2.9. The graph $\Gamma_2(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is a 5-regular graph. The two subgraphs induced by the zero-divisors and the units are both K_4.(i.e., the complete graph with 4 vertices).

Remark 2.10. Regarding $diam(\Gamma_2(R_1 \times R_2 \cdots \times R_n))$ where R_i's are finite rings with unity, if at least one R_i has characteristic $\neq 2$, then that R_i has at least one unit $x \neq 1$ (cf. Corollary 4 of [6]). Here $(1, 1, \ldots, 1)$ is not adjacent to ${e_i}^{x-1}$. So diameter cannot be 1 in this case. In particular, for direct product of fields, $diam(\Gamma_2(F_1 \times F_2 \cdots \times F_n))$ has diameter 1 (i.e., a complete graph) if and only if $F_i \cong \mathbb{Z}_2$ for $i = 1, 2, .., n$.

We now a give a result regarding the 2-connectedness of $\Gamma_2(R)$.

Theorem 2.11. *Let R be a commutative ring such that $diam(\Gamma_2(R)) = 2$. Then the graph $\Gamma_2(R)$ is 2-connected.*

Proof. Let diameter of $\Gamma_2(R)$ be 2. We show that if we delete any one vertex from the graph, it still remains connected. Suppose u is the vertex deleted. If u is a unit, then the connected subgraph induced by the zero-divisors remain unchanged. So since every unit has an adjacent zero-divisor, the graph remains connected. So let u be a zero-divisor. Consider any two vertices x, y. If there is a path between x, y without involving u, then it is unaffected by the deletion. So suppose the only path between x, y is $x - u - y$. (Note that since diameter is 2, there will be no other vertex in that path). So we have three possibilities as following:

Case I : Let $xu = 0$ and $uy = 0$. In this case $(x + y)u = 0$, so $x + y$ is a zero-divisor (may be 0). So this case does not arise.

Case II : (Without loss of generality) Let $xu = 0$ and $yu \neq 0$ but $y + u$ is a zero-divisor. If $y + u = 0$, then $xy = -xu = 0$, i.e., $x \leftrightarrow y$. So $y + u = z$ for some non-zero zero-divisor. Let $zt = 0$. If $-z = u$, then $2u + y = 0$. So we have the path $x - 2u - y$. (Note that $2u \neq 0$ as that would imply $y = 0$). Now let $-z \neq u$. If $ut \neq u$, we have the path $x - ut - (-z) - y$. If $ut = u$, then $uz = utz = 0$, so $(x - z)u = 0$. Hence $x \leftrightarrow -z$. So we have the path $x - (-z) - y$.

Case III : Let $xu \neq 0$ and $uy \neq 0$, but $x+u = z_1$ and $u+y = z_2$ where z_1, z_2 are zero-divisors. Now both of z_1, z_2 cannot be 0 together. If, without loss of generality, $z_1 = 0$, then $x - y = -z_2$. So we have the path $x - (-y) - y$. Note that $-y \neq u$ since $z_2 \neq 0$. Now let both z_1, z_2 be non-zero. $x - y = z_1 - z_2 = v$. If v is a zero-divisor, then we have the path $x - (-y) - y$ as before. Now if v is a unit, then since $x + u - v = z_2$ and $y + u + v = z_1$, we have the path $x - (u - v) - (u + v) - y$, since u is a zero-divisor.

So the above three cases show that $\Gamma_2(R)$ is 2-connected. □

Corollary 2.12. *(i) $\Gamma_2(\mathbb{Z}_n)$, when connected, is 2-connected.*

(ii) $\Gamma_2(R_1 \times R_2 \times \cdots \times R_n)$, where R_i's are finite commutative rings $n > 1, n \in \mathbb{N}$, is 2-connected.

Remark 2.13. (i) Let F be a finite field with $|F| = p^n$ for some prime p and $n \in \mathbb{N}$. Now F has no zero-divisors. So two vertices can be adjacent if and only if their corresponding elements are additive inverses of each other. As a result, $\Gamma_2(F)$ is a disjoint union of $(p^n - 1)/2$ copies of K_2.

(ii) Γ_2-graph taken over direct products of finite fields is connected from Theorem 2.8. In particular, the subgraph of $\Gamma_2(F_1 \times F_2)$ induced by the zero divisors is complete, because zero-divisors there are either of the form $(a, 0)$ or $(0, b)$ and hence any two of them are adjacent. Again, if any one of the fields F_i is $\mathbb{Z}_2$, then the subgraph induced by the units is complete. Also, subgraph of $\Gamma_2(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3)$ induced by the units is a complete graph.

(iii)Let $n > 1$ be a positive integer . Then $K_{2^n-1} \cong \Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2($n copies$))\cong \Gamma_2(R)$ where R is any boolean ring with 2^n elements.

Proposition 2.14. *$\Gamma_2(R) \cong \Gamma_2(S)$ does not imply $R \cong S$. Here we give two examples in favour of this proposition :*
(i) $\Gamma_2(\mathbb{Z}_2[x, y]/ < x^2, xy, y^2 >) \cong \Gamma_2(\mathbb{Z}_8) \cong K_3 \cup K_4$
(ii) $\Gamma_2(\mathbb{Z}_p[x]/ < x^n >) \cong \Gamma_2(\mathbb{Z}_{p^n}) \cong K_{p^{n-1}-1} \cup (p - 1)/2$ copies of $K_{p^{n-1},p^{n-1}}$.

3. Degrees and Other Properties of $\Gamma_2(\mathbb{Z}_n)$

Now we look at $\Gamma_2(\mathbb{Z}_n)$ in more details. We start with the degree patterns of $\Gamma_2(\mathbb{Z}_n)$. For this graph maximum possible degree of any vertex is $(n-2)$, since there are $(n-1)$ vertices in the graph, and a vertex is not adjacent to itself. Also note that $\Gamma_2(\mathbb{Z}_2)$ is a single vertex graph and $\Gamma_2(\mathbb{Z}_3)$ has an isolated vertex. In our next result, we show that $\Gamma_2(\mathbb{Z}_n)$ has no isolated vertex for any other value of n and also give the necessary and sufficient condition for attaining the maximum possible degree.

Theorem 3.1. *For $n > 4$, $\Gamma_2(\mathbb{Z}_n)$ has no vertex of degree 0. A vertex can have maximum degree i.e. $(n-2)$, if and only if the prime-power factorization of n has more than one prime and there is an idempotent x such that $2x = 0$.*

Proof. $\Leftarrow$ (Necessity of the condition) :

Case I: Let $n = p^m$ for some prime p. Here subgraph of $\Gamma_2(\mathbb{Z}_n)$ induced by zero-divisors is complete and the subgraph induced by units is union of complete bipartite graphs . So a vertex cannot have degree 0. Again Since the subgraph of $\Gamma_2(\mathbb{Z}_n)$ induced by zero-divisors is disjoint from the subgraph induced by units here, a vertex cannot be adjacent to all other vertices and hence we do not have a vertex of degree $(n-2)$.

Case II: Let $n = p_1^{q_1} p_2^{q_2} ... p_k^{q_k}$ where $k \geq 2$. Here $\Gamma_2(\mathbb{Z}_n)$ is connected and hence no vertex can have degree 0. If possible, let x be a vertex of degree $n-2$. Now $x \neq 1$, since 1 is not adjacent to -2. In fact, x cannot be a unit, because then x is not adjacent to $-2x$ (Note that $-2x \neq 0, x$). Hence x must be a zero-divisor. So $x \leftrightarrow 1$. Since 0 is not taken as a vertex, $x \neq 0$, so $x.1 \neq 0$. So we must have $x + 1 = z_1$ for some zero divisor z_1. We show that $z_1 \neq 0$. Let $z_1 = 0$. Then $x = -1$. So -1 is adjacent to all other vertices. Note that the number of units in $\mathbb{Z}_n$ is $\phi(n) = \phi(p_1^{q_1} p_2^{q_2} ... p_k^{q_k}) \geq 4$. So there exists a unit other than $1, -1, 2$. Let that unit be y . Since $-1 \leftrightarrow y$, we have that $y - 1$ is a zero-divisor. So $1-y$ is a zero-divisor. Again $1-y \neq 0, -1$ and so $-1 \leftrightarrow (1-y)$. Hence $-1 + 1 - y = -y$ is a zero-divisor i.e. y is a zero-divisor. This is a contradiction. So $z_1 \neq 0$. Now $x - z_1 = -1$ which is not a zero-divisor. So since $x \leftrightarrow (-z_1)$, we have $xz_1 = 0$. Note that $x \neq -z_1$ as that would have implied that $2x = -1$, thus contradicting the fact that x is a zero-divisor.

Similarly $x \leftrightarrow -1 \Rightarrow (x-1) = z_2$ for some zero-divisor z_2. Procceding as before, $xz_2 = 0$.

Now $xz_1 = 0 \Rightarrow x(x+1) = 0 \Rightarrow x^2 + x = 0$.....(i)

and $xz_2 = 0 \Rightarrow x(x-1) = 0 \Rightarrow x^2 = x$ . ....(ii)

From (i) and (ii), we have $2x = 0$ and $x^2 = x$. So the necessary condition to have a $(n-2)$ degree vertex is to have more than one prime factor in the prime-power factorization of n and to have an idempotent element x such that $2x = 0$.

$\Rightarrow$ (Sufficiency of the condition) : Let x be an element such that $x^2 = x$ and $2x = 0$. Now $2x = 0 \Rightarrow 2|n$ and $(n/2)|x$. So $x = (n/2)$. Again $x^2 = x \Rightarrow n|x(x-1)$. Now $x(x-1) = (n^2)/4 - n/2$. So $n|x(x-1) \Rightarrow (n^2-2n)/4 = kn$ for some integer k. This gives that $(n-2)/4$ is an integer, say t. So $n = 2+4t = 2(2t+1) = 2r$ where r is odd. Now we have $2 \leftrightarrow x$. Let z be a zero-divisor other than 2 and x. So x and z have a common factor p_i, because $x = n/2 = r$ and z being not 2, has a common factor with $n/2 = r$.(Note that power of 2 in the prime power factorization of n is 1 , since $n = 2r$ and r is odd). So $x+z$ is a zero-divisor and hence $x \leftrightarrow z$. Again, let u be a unit. So u is odd. Now x being odd, $x+u$ is even, and hence a zero divisor. So $x \leftrightarrow u$. So x is adjacent to all other vertices and hence $deg(x) = n-2$. □

Next we consider the degree pattern of a unit in $\Gamma_2(\mathbb{Z}_n)$.

Theorem 3.2. *Let u be a unit in $\mathbb{Z}_n$. Then in $\Gamma_2(\mathbb{Z}_n)$, $deg(u) = (n-\phi(n)-1)$ or $(n-\phi(n))$ according as n is even or odd.*

Proof. Consider a unit u. Clearly it is adjacent to precisely the vertices of the form $(z-u)$, where z is any zero divisor. Now $|\{z-u : z \in Z(\mathbb{Z}_n)| = |\{z : z \in Z(\mathbb{Z}_n)\}| = n-\phi(n)$. Now $u \in \{z-u : z \in Z(\mathbb{Z}_n)\}$ according as n is even or odd. Hence $deg(u) = n-\phi(n)$ if n is odd. □

Corollary 3.3. *$\Gamma_2(\mathbb{Z}_n)$ is not Eulerian for any positive integer n.*

Proof. Let n be even. So from Theorem 3.2, we have that $deg(1) = n-\phi(n)-1 =$ an odd number. Again, if n is odd, $deg(1) = n-\phi(n) =$ an odd number in this case. So $deg(1)$ is always odd. So $\Gamma_2(\mathbb{Z}_n)$ cannot be eulerian for any n. □

Next , we have some results about the zero-divisors in $\Gamma_2(\mathbb{Z}_n)$.

Theorem 3.4. *(i) Let z be a non-zero zero-divisor in $\mathbb{Z}_n$, such that $z^2 = 0$. Then $deg(z) = n-\phi(n)-2$.*

(ii)Let z be a non-zero zero-divisor in $\mathbb{Z}_n$, where n is squarefree. Then $deg(z) = n-\phi(n)-2-\phi(p_1p_2\cdots p_l)$, where $p_1, p_2, \cdots, p_l$ are distinct prime factors of z.

Proof. These can be proved using Theorem 3.3 and 3.4 of [8]. □

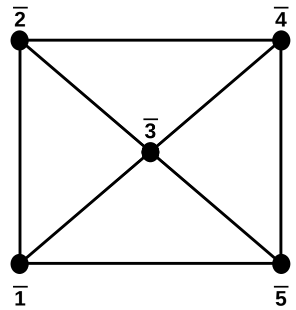

Figure 1: $\Gamma_2(\mathbb{Z}_6)$

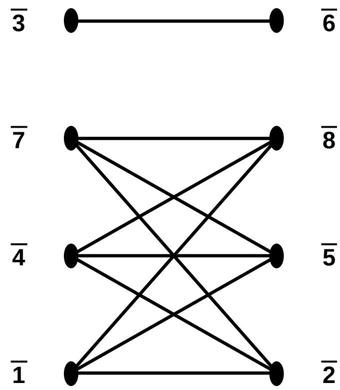

Figure 2: $\Gamma_2(\mathbb{Z}_9)$

Now we consider the girth and planarity in $\Gamma_2(\mathbb{Z}_n)$.

Theorem 3.5.

$$girth(\Gamma_2(\mathbb{Z}_n)) = \begin{cases} 4 & \text{if } n = 9 \\ \infty & \text{if } n = 2,3,4 \text{ or } n \text{ is a prime} \\ 3 & \text{otherwise.} \end{cases}$$

Proof. $\Gamma_2(\mathbb{Z}_2)$ is a single vertex graph. It is clear that $girth(\Gamma_2(\mathbb{Z}_n))$ is ∞ for $n = 2,3,4$ or when n is a prime(from Remark 2.13). Now we first consider the cases when $n = p^m(> 4)$, where p is a prime and $m > 1$. If $p = 2$, then from Theorem 2.7, the subgraph of $\Gamma_2(\mathbb{Z}_n)$ induced by the zero-divisors is a $K_{2^{m-1}-1}$ graph. Now $2^m > 4 \implies 2^{m-1} - 1 \geq 3$. So a 3-cycle exists. Next, let $p \geq 3$. In this case $p^m \geq 9$. Now from Figure 2, $\Gamma_2(\mathbb{Z}_9) = K_2 \cup K_{3,3}$, so although it has no 3-cycle, it has a 4-cycle and hence girth is 4. For $p^m > 9$, the subgraph of $\Gamma_2(\mathbb{Z}_{p^m})$ induced by zero-divisors is a $K_{p^{m-1}-1}$ graph from Theorem 2.6. Now $p^m > 9$ and $p \geq 3 \implies p^{m-1} - 1 > 3$. So a 3-cycle exists and the girth is 3. Now we come to the cases where n has more than one prime factor. Let one of the prime factors of n, say p_i is greater than 3. Then if p_1 is the least prime factor of n, then $n \geq p_1 p_i \geq 5p_1$ and hence we have a 3-cycle $p_1 - 2p_1 - 3p_1 - p_1$. Note that even if $p_1 > 3$, still this gives the same 3-cycle. Again if 2 and 3 are the only prime factors then for $n > 6$ we have the 3-cycle $2 - 4 - 6 - 2$ and for $n = 6$ we have the 3-cycle $2 - 3 - 4 - 2$. Hence the result. □

Remark 3.6. From the proof of Theorem 3.5, it is clear that the subgraph of $\Gamma_2(\mathbb{Z}_n)$ induced by the zero-divisors has girth 3 for all n except when $n = 4, 9$ or when n is a prime (in those cases the girth is ∞). Again, from part (5) of Theorem 3.14 in [1], girth of the subgraph induced by the units is ≤ 4 if the subgraph contains a cycle.

Theorem 3.7. *$\Gamma_2(\mathbb{Z}_n)$ is planar if and only if $n = 4, 6, 8$ or n is a prime.*

Proof. When n is a prime, then $\Gamma_2(\mathbb{Z}_n)$ is a disjoint union of copies of K_2 (from Theorem 2.13). So the graph is planar when n is a prime. From Figure 1, $\Gamma_2(\mathbb{Z}_6)$ is planar and so is $\Gamma_2(\mathbb{Z}_4)(\cong K_1 \cup K_2)$. Again from Theorem 2.7, $\Gamma_2(\mathbb{Z}_8)$ is $\cong K_3 \cup K_4$ and hence is planar. Now let $n = 2^m$ where $2^m > 8$. From Theorem 2.7, the subgraph of $\Gamma_2(\mathbb{Z}_n)$ induced by the units is a $K_{2^{m-1}}$. Clearly $2^{m-1} \geq 8$ and hence the graph contains a K_5, so it cannot be a planar. For $p \geq 3$, $\Gamma_2(\mathbb{Z}_{p^m})$, where $m > 1$, contains a $K_{p^{m-1},p^{m-1}}$ from Theorem 2.6. Since $p^{m-1} \geq 3$, the graph contains a $K_{3,3}$ and hence cannot be planar. Now we come to the cases where n has more than one prime factor. First let n be even. If $n = 10$, then the subgraph induced by $\{2, 4, 6, 8, 5\}$ forms a K_5 and for $n \geq 12$, the subgraph induced by $\{2, 4, 6, 8, 10\}$ forms a K_5. So the graph is not planar in these cases. Now let n be odd. Then if the prime factors of n, arranged in increasing order are $p_1, p_2, \ldots, p_k$, then $p_2 \geq 5$. So $n \geq 5p_1$. Here the subgraph induced by vertices $\{p_1, 2p_1, 3p_1, 4p_1, n/p_1\}$ forms a K_5 and hence the graph is not planar. Hence the result. □

4. $\Gamma_2(R)$ over Non-Commutative Rings and Infinite Rings

Let us consider the $\Gamma_2(R)$ graphs for non-commutative rings. Redmond[7] defined an undirected zero-divisor graph for a non-commutative ring [7], and that graph is always connected (unless $Z(R) \neq \{0\}$)[5]. Definition of $\Gamma_2(R)$ graphs clearly makes room for non-commutative rings. A particular case of interest is the ring of matrices with entries from any commutative ring.

Theorem 4.1. *$\Gamma_2(M_n(R))$, where R is a commutative ring and n is a natural number > 1, is connected.*

Proof. Note that a matrix A is either a left or a right zero-divisor in $M_n(R)$ for a commutative ring R if and only if $det(A) \in Z(R)$ (cf. Theorem 9.1 of [4]). So in $M_n(R)$, the singular matrices , i.e., the matrices with determinant 0, are zero-divisors. Now from Theorem 2.2 of [5], it is clear that between any two zero-divisors, there is a path. Hence, we try to find an adjacent zero-divisor for an arbitrary unit. If it can be found, then that implies that the graph is connected.

Let us consider a unit in $M_n(R)$, say $A = (a_{ij})$. Since it is a unit, $detA \neq 0$. Now consider the matrix $B = (b_{ij})$ defined by $b_{1i} = -a_{1i}$ for all $i = 1, 2, .., n$ and all other elements are 0. So $detB = 0$, i.e., B is a zero-divisor. Note that since $detA \neq 0$, $a_{1j} \neq 0$ for at least one j $\in \{1, 2.., n\}$. So for that j, $b_{1j} \neq 0$. Hence B is not the zero matrix. Now all the elements of the first row of $A+B$ is zero, so $det(A+B) = 0$, i.e., $A+B$ is a zero-divisor and hence B is adjacent to A. So every unit has an adjacent zero-divisor. So $\Gamma_2(M_n(R))$ is connected. □

Finally, we have a look at the graph $\Gamma_2(R)$ where R is an infinite ring. The first ring that naturally comes in our mind is the ring $\mathbb{Z}$. Now $\Gamma_2(\mathbb{Z})$ is disjoint union of infinite number of copies of K_2 since two vertices u, v are adjacent if and only if $u + v = 0$. However, when we move to the direct product of $\mathbb{Z}$ with any ring R, then the graph is connected.

Theorem 4.2. *Let R be a ring $\neq \{0\}$. Then $\Gamma_2(\mathbb{Z} \times R)$ is connected with diameter ≤ 2 and girth 3. Moreover it contains a cycle of length k for any natural number $k > 2$.*

Proof. First part of the theorem statement follows from Theorem 2.8. Now it is easy to see that for any positive integer $k(> 2)$, there is a cycle of length k viz. $(1, 0) - (2, 0) \ldots - (k, 0) - (1, 0)$ as a subgraph of this graph. In particular, its girth is 3. □

Remark 4.3. If R in the above theorem contains at least one element u which is not a zero-divisor (which happens for rings wih unity), then diameter is necessarily 2, because $(1, 0)$ is not adjacent to $(1, u)$ in that case. e.g.- $\Gamma_2(\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z})$ is connected with diameter 2 and girth 3.

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