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WEAKLY $n-$PRIME IDEAL OF POSETS

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Abstract: In this paper, we study the weakly $n-$prime ideals of poset and shown that for a weakly $3-$prime ideal I of P, if I has $*-$property(for any $a, b \in P\backslash J$, we have either $a = b$ or $L(a, b) = \{0\}$), then there are at most two prime ideals of P that are minimal over I. there are at most two prime ideals of P that are minimal over I.

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1. Introduction

Throughout this paper, $(P, \leq)$ denotes a poset with smallest element 0. For basic terminology and notation for posets, we refer [1]. For $M \subseteq P$, let $L(M) := \{x \in P : x \leq m \text{ for all } m \in M\}$ denotes the lower cone of M in P, and dually let $U(M) := \{x \in P : m \leq x \text{ for all } m \in M\}$ be the upper cone of M in P.

For $A, B \subseteq P$ we shall write $L(A, B)$ instead of $L(A \cup B)$ and dually for upper cones. If $M = \{x_1, ..., x_n\}$ is finite, then we use the notation $L(x_1, ..., x_n)$ instead of $L(\{x_1, ..., x_n\})$ (and dually). By an ideal we mean a non-empty subset $I \subseteq P$ such that if $b \in I$ and $a \leq b$, then $a \in I$. Following [2], an ideal I is called $n-$prime if for pairwise distinct elements $x_1, x_2, ..., x_n \in P$, if $L(x_1, x_2, ..., x_n) \subseteq I$, then at least $(n-1)$ of $n-$ subsets $L(x_2, x_3, ..., x_n), L(x_1, x_3, ..., x_n), ..., L(x_1, x_2, ..., x_{n-1})$ is a subset of I.

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An ideal I is called weakly $n-$prime if for pairwise distinct elements $x_1, x_2, ...,$ $x_n \in P$, if $L(x_1, x_2, ..., x_n) \subseteq I$, then at least one of $n-$ subsets $L(x_2, x_3, ..., x_n)$, $L(x_1, x_3, ..., x_n), ..., L(x_1, x_2, ..., x_{n-1})$ is a subset of I. As a special case we get I is prime (i.e.,2-prime) if and only if $x_1 \in I$ or $x_2 \in I$ whenever $L(x_1, x_2) \subseteq I$, and I is prime if and only if I weakly prime. It is clear that every n prime is weakly n-prime, but converse need not be true, in general as the following example shows.

Example 1.1. Let $P = \{1, 2, 3, 4, 5, 6\}$ and R be the relation on P given by $(a, b) \in R$ if a is a factor b. Then (P, R) is a poset and $\{1\}$ is a weakly 3-prime ideal of P, but it is not a 3-prime ideal as $L(2, 3, 6) \subseteq \{1\}$.

An ideal minimal in the set of all weakly $n-$prime ideals containing some given ideal I is called a minimal weakly $n-$prime ideal of I. A non-empty set M of P is called a $m - system$ if for any $x_1, x_2 \in M$, there exists $t \in L(x_1, x_2)$ such that $t \in M$. It is easy to verify that for any ideal I of P, I is $2-$prime if and only if $P \backslash I$ is an $m-$system, and also $U(x)$ is a $m-$system for any $x \in P$. Let I be an ideal of P. Then $P(I)$ denotes the intersection of all prime ideals of P containing I, and intersection of prime ideals of P. Following [2], let I be an ideal of P. Then the extension of I by $x \in P$ is meant the set $< x, I >= \{a \in P \ : \ L(a, x) \subseteq I\}$.

2. Weakly Prime Ideals

Theorem 2.1. *Let P be a poset. If I is weakly $n-$prime ideal of P, then it is also $(n + 1)$ prime of P.*

Proof. Let $x_1, x_2, ..., x_n, x_{n+1}$ be pairwise distinct elements of P and let $L(x_1, x_2, ..., x_{n+1}) \subseteq I$. Suppose $L(x_1, x_2, ..., x_n) \nsubseteq I$ and $L(x_2, x_3, ..., x_{n+1}) \nsubseteq I$. Then there exists $a \in L(x_1, x_2, ..., x_n) \nsubseteq I$ and $b \in L(x_2, x_3, ..., x_n) \nsubseteq I$ with $a \neq b$ and $b \notin \{x_1, x_2, ...x_n\}$. Now

$$L(x_1, x_2, x_3, ..., x_{n-1}, b) \subseteq L(x_1, x_2, ..., x_n, x_{n+1}) \subseteq I.$$

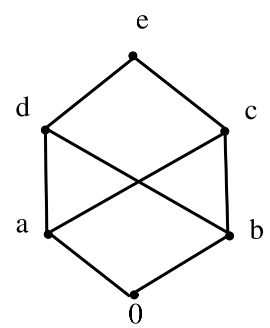
Since I is weakly n-prime and $L(x_1, x_2, ..., x_{n-1}) \nsubseteq I$, we have at least one $n-1$ of n-subsets $L(x_2, x_3, ..., b)$, $L(x_2, x_4, ..., b), ..., L(x_2, x_3, ...x_{n-1}, b)$ is a subset of I which implies $b \in I$ as $b \in L(x_1, x_2, ..., x_{n+1})$, a contradiction. So at most one n of $n + 1$-subsets $L(x_2, x_3, ..., x_{n+1})$, $L(x_1, x_3, ..., x_{n+1})$, $L(x_1, x_2, ..., x_{n-1})$ is a subset of I, which implies I is a $(n + 1)$ prime ideal of P. □

Corollary 2.2. *([1] Theorem 3) Let I be an ideal of P. If I is n-prime, then it is also $(n + 1)$-prime.*

Theorem 2.3. *Let I be an ideal, $n \geq 3$. If $< x, I >$ is weakly $(n-1)$-prime ideal for each $x \in P \backslash I$, then I is weakly n-prime.*

Proof. Let $x_1, x_2, ..., x_n, x_n$ be pairwise distinct n-elements of $P \backslash I$ and let $L(x_1, x_2, ..., x_n) \subseteq I$. Then $L(x_1, x_2, ..., x_{n-1}) \subseteq < x_n, I >$. Since $< x_n, I >$ is weakly $(n-1)$ prime ideal of P, we have at least one of $(n-1)$ subset $L(x_2, x_3, ..., x_{n-3})$, $L(x_1, x_3, ..., x_{n-1}, ..., L(x_1, x_2, ..., x_{n-2})$ is a subset of $< x_n, I >$ which gives I is weakly n-prime. □

Example 2.4. Consider the set $P = \{0, a, b, c, d, e\}$ and define the partial order relation $\leq$ as follows:



Here $I = \{0\}$ is a weakly $3-$prime ideal of P, but $< d, I >= \{0\}$ is not a 2-prime ideal of P.

The above example shows that the converse of Theorem 2.3 does not hold in general, but we have the following theorem.

Theorem 2.5. *Let I be a $3-$prime ideal of P and J is a prime ideal of P that properly contains I. If $J \subseteq < x, I >$ for any $x \in J \backslash I$, then $< x, I >$ is a prime ideal of P.*

Proof. Let $L(y, z) \subseteq < x, I >$ for some $y, z \in P$. Then $L(x, y, z) \subseteq I$. Suppose $y \notin < x, I >$ and $z \notin < x, I >$. Then $L(x, y) \nsubseteq I$ and $L(x, z) \nsubseteq I$. Since $J \subseteq < x, I >$, we have $y \notin J$ and $z \notin J$, which imply $L(y, z) \nsubseteq J$ as J is prime ideal. Since I is weakly $3-$prime and $L(x, y, z) \subseteq I$, we have $L(y, z) \subseteq I \subseteq J$ which implies $z \in J \subseteq < x, I >$, a contradiction. □

Note that if we consider $I = \{0\}$ and $J = \{o, a, b, c, d\}$ in Example 2.4, then $J \nsubseteq < c, I >$ for $c \in J \backslash I$ and $< c, I >$ is not a prime ideal of P. Therefore, the condition $J \nsubseteq < c, I >$ is not superficial in Theorem 2.5.

Theorem 2.6. *Let M be a non-void $m-$ system in P and J is an ideal of P with $J \cap M = \Phi$. Then J is contained in a prime ideal $I (\neq P)$ with $I \cap M = \Phi$.*

Proof. Let $\mathbf{A} = \{J : J$ is an ideal of P with $J \cap M = \Phi\}$. Then $\mathbf{A} \neq \Phi$ and there exists a maximal element $I \in \mathbf{A}$ with $M \cap I = \Phi$. Let $x \notin I$ and $y \notin I$ be distinct elements of P. Then $I \subset I \cup L(x)$ and $I \subset I \cup L(y)$ which gives $M \cap L(x) \neq \Phi$ and $M \cap L(y) \neq \Phi$. So, there exist $a \in M \cap L(x)$ and $b \in M \cap L(y)$ such that $t \in M$ for some $t \in L(a, b)$. Since $a \leq x$ and $b \leq y$, we have $t \in L(x, y)$. Thus $L(x, y) \cap M \neq \Phi$ and hence $L(x, y) \nsubseteq I$. □

An ideal J of P is said to have $*-$ property if for any $a, b \in P \backslash J$, we have either $a = b$ or $L(a, b) = \{0\}$.

Theorem 2.7. *Let $J \subseteq I$ be ideals of P, where I is prime. If J has $*-$ property, then the following conditions are equivalent:*
(i) I is a minimal prime ideal of J.
(ii) For each $x \in I$, there exists $y \in P \backslash I$ and $t \in U(x)$ such that $L(t, y) \subseteq J$.

Proof. Let I be a minimal prime ideals of J and suppose that there exists $x \in I$ such that $L(y_i, t_j) \nsubseteq J$ for all $y_i \in P \backslash I$ and $t_j \in U(x)$. Then there exists $a_{ij} \in L(y_i, t_j)$ such that $a_{ij} \notin J$ for $y_i \in P \backslash I$ and $t_j \in U(x)$. Now we collect $M = \{a_{ij} : \ a_{ij} \in L(y_i, t_j) \backslash J$ for $y_i \in P \backslash I$ and $t_j \in U(x)\}$. Then $M = P \backslash I$ is a m-system of P with $M \cap J = \Phi$, by Theorem 2.6, there exists a prime ideal I_1 containing J with $I_1 \cap M = \Phi$. If $x \in I_1$, then $L(x, y_i) \subseteq I_1$ for every $y_i \in P \backslash I$. But, there exists $p \in L(x, y_i) \backslash J$ with $p \in M$ which gives $I_1 \cap M \neq \Phi$, a contradiction. So $x \notin I_1$. Now let $i_1 \in I_1$ and suppose $i_1 \notin I$. Then $i_1 \in P \backslash I$ and $L(i_1, x) \subseteq I_1$, but $L(i_1, x) \nsubseteq J$, a contradiction. Thus $I_1 \subset I$, which is also a contradiction to minimality of I.

Conversely, let I_1 be a prime ideal of P with $J \subseteq I_1 \subseteq I$. Let $x \in I$. Then there exists $y \in P \backslash I$ and $t \in U(x)$ such that $L(t, y) \subseteq J \subseteq I_1$. Since $y \notin I_1$, we have $t \in I_1$ which implies $x \in I_1$. Thus $I \subseteq I_1$ and hence I is a minimal prime ideal of J. □

The following example shows that the condition J has $*-$property is not superficial in Theorem 2.7.

Example 2.8. Let $P = \{0, 1, 2, 3, 4, 5, 6\}$ and R be the "'less than or equal" relation on P. Then (P, R) is a poset and $I = \{0, 1\}$ is a minimal prime ideal of $J = \{0\}$. But for $1 \in I$, there exists no $y \in P \backslash I$ and $t \in U(x)$ such that $L(t, y) \subseteq J$.

Lemma 2.9. *[1] For any ideal I of P, we have $P(I) = I$.*

Proof. Let $x \in P(I)$ for any ideal I of P. If $I \cap U(x) = \Phi$, then there exists a prime ideal I_1 containing I with $I_1 \cap U(x) = \Phi$ which implies $x \notin I_1$, a contradiction. □

From Theorem 2.6 and Lemma 2.9, we conclude that for any I of P, if $x \in I$, we have $U(x) \subseteq I$ and $P(P) \subseteq I$, where $P(P)$ is the intersection of all prime ideals of P.

Theorem 2.10. *Suppose I is weakly $3-$prime ideal of P. If I has $*-$property, then there are at most two prime ideals of P that are minimal over I.*

Proof. Suppose that there are at least three prime ideals of P that are minimal over I. Let I_1 and I_2 be two distinct minimal $2-$prime ideals of P over I. Hence there is $x_1 \in I_1 \backslash I_2$ and $x_2 \in I_2 \backslash I_1$. We now claim that $L(x_1, x_2) \subseteq I$.

Since $x_1 \in I_1$, by Theorem 2.7, there is $c_2 \in P \backslash I_1$ such that $L(t, c_2) \subseteq I$ for some $t \in U(x_1)$. Then $L(c_2, x_1) \subseteq I$ as $L(c_2, x_1) \subseteq L(c_2, t)$. Similarly, we can get $L(c_1, x_2) \subseteq I$ some $c_1 \in P \backslash I_2$. Observe that $c_1 \in I_1 \backslash I_2$; $c_2 \in I_2 \backslash I_1$ and $x_1, x_2 \notin I$. By Theorem 5 of [2], we have $< x_1,\ I >$ is a $2-$prime ideal of P. Since $L(c_1, x_2) \subseteq < x_1,\ I >$, we have $L(x_2, x_1) \subseteq I$. Now, suppose that there is a $2-$prime ideal I_3 of P that is minimal over I such that I_3 is neither I_2 nor I_3. Let $y_1 \in I_1 \backslash (I_2 \cup I_3); y_2 \in I_2 \backslash (I_1 \cup I_3)$ and $y_3 \in I_3 \backslash (I_1 \cup I_2)$. Then by previous argument, we have $L(y_1, y_2) \subseteq I$, but neither $y_1 \in I_3$ nor $y_2 \in I_3$, a contradiction. □

From Theorem 2.10, we have the following question. If I is a $n-$prime ideals of P and has $*-$property, then whether there are at most two $n-1$ prime ideals of P that are minimal over I, for any $n > 3$.

Theorem 2.11. *Let I be a $3-$prime ideal of P. If I has $*-$property, then one of the following statements must hold:*

(i) I is a prime ideal of P.

(ii) $I = I_1 \cap I_2;\ L(i_1, i_2) \subseteq I$ for all $i_1 \in I_1$ and $i_2 \in I_2$, where I_1 and I_2 are the distinct prime ideals of P that are minimal over I.

Proof. By Lemma 2.9 and Theorem 2.10, we have $P(I) = I$ is a $2-$prime ideal of P or $I = P(I) = I_1 \cap I_2$, where I_1 and I_2 are distinct minimal $2-$prime ideals of I. Suppose $I = I_1 \cap I_2$ and let $x \in L(i_1, i_2)$ for $i_1 \in I_1$ and $i_2 \in I_2$. Then $x \leq i_1$ and $x \leq i_2$. If $i_1 \in I_1 \backslash I_2$ and $i_2 \in I_2 \backslash I_1$, then $L(i_1, i_2) \subseteq I$ by the proof of Theorem 2.7, and so $x \in I$. □

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