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Γ-SEMIGROUPS WITH LOCAL UNITS AND MORITA EQUIVALENCE OF SEMIGROUPS WITH LOCAL UNITS

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Abstract: In this paper we define local units for Γ-semigroups and show that two semigroups L and R with local units are Morita equivalent if and only if there exists a Γ-semigroup A with local units whose operator semigroups are isomorphic to L and R. We apply this result to study some interplays between operator semigroups of Γ-semigroups with local units by using Morita theories for semigroups.

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Key Words: Γ-semigroup, Morita equivalence

1. Introduction

If A and Γ are two non-empty sets then A is said to be a Γ-semigroup if there exist mappings $A\times\Gamma\times A\to A$, denoted by $(a,\gamma,b)\mapsto a\gamma b$, and $\Gamma\times A\times\Gamma\to\Gamma$, denoted by $(\alpha,a,\beta)\mapsto\alpha a\beta$, satisfying $(a\alpha b)\beta c=a(\alpha b\beta)c=a\alpha(b\beta c)$ for all $a,b,c\in A$ and $\alpha,\beta\in\Gamma$. This notion was introduced by M.K. Sen in 1981 as a generalization of semigroup. In this paper we extend the results of [7] in the sense that unities have been replaced by local units. For related notions of Γ-semigroups reader is referred to [2]. We refer to [8] for the notions of S-act, biact, act morphism, tensor product, Morita equivalence, Morita context and the notations of US-Act and FS-Act. For the notions of semigroups we refer

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to [1, 3, 4, 5]. Throughout this paper *mappings act from right.*

2. Main Results

Definition 1. Let A be a Γ-semigroup. Then A is said to have right local units if for each $x \in A$ there exists an idempotent element $[\gamma, f]$ in the right operator semigroup R of A such that $x\gamma f = x$. Dually we define left local units of A and the local units of Γ. A Γ-semigroup A is defined to have local units if both A and Γ have right and left local units. Clearly a Γ-semigroup A with unities[2] is a Γ-semigroup with local units.

Theorem 2. *Let A be a Γ-semigroup with local units and its left and right operator semigroups be respectively L and R. Then:*

(1) L and R are semigroups with local units;

(2) ${}_LA_R$ and ${}_R\Gamma_L$ are unitary biacts;

(3) ${}_LA \in FL$-Act and ${}_R\Gamma \in FR$-Act.

Proof. (1) The proof is evident from Definition 1.

(2) Let us define the maps $L \times A \to A$ and $A \times R \to A$ by $[a, \alpha]b := a\alpha b$ and $a[\beta, b] := a\beta b$ respectively. Then, for all $a, b, c \in A$ and $\alpha, \beta \in \Gamma$, we have $a([\alpha, b][\beta, c]) = a[\alpha, b\beta c] = a\alpha(b\beta c) = (a\alpha b)\beta c = (a[\alpha, b])[\beta, c]$. So A is a right R-act. Again, since A has local units, for every $a \in A$, there exist an idempotent element $[\gamma, f] \in R$, depending on a, such that $a[\gamma, f] = a$. Hence $A \in$ Act-UR. Similarly we can show that $A \in UL$-Act.

Now consider $a \in A$, $[c, \alpha] \in L$ and $[\beta, b] \in R$. Then, from the generalized associative property of the Γ-semigroup A, we have $[c, \alpha](a[\beta, b]) = [c, \alpha](a\beta b) = c\alpha(a\beta b) = (c\alpha a)\beta b = ([c, \alpha]a)[\beta, b]$. Hence ${}_LA_R$ is a unitary biact. Similar series of arguments lead us to that ${}_R\Gamma_L$ is a unitary biact.

(3) According to the Lemma 8.2 of [8], the UL-act ${}_LA$ is in FL-Act if and only if $L \otimes A \cong A$ as biacts. Again, by the Proposition 3.2 of [8], the map $\kappa : L \otimes A \to A$ defined by $[b, \beta] \otimes a \mapsto [b, \beta]a$ is a surjective L-morphism. So it is sufficient to prove the injectivity of κ to complete our theorem.

Let $[a, \alpha], [b, \beta] \in L$ and $c, d \in A$ such that $[a, \alpha]c = [b, \beta]d$, i.e., $a\alpha c = b\beta d$. Then there exist idempotents $[e, \delta] \in L$ and $[\gamma, f] \in R$ such that $e\delta b = b$ and

$c\gamma f = c$ respectively. So we have

$$\begin{array}{llll} [a,\alpha]\otimes c & = [a,\alpha]\otimes c\gamma f & = [a,\alpha][c,\gamma]\otimes f & = [a\alpha c,\gamma]\otimes f \\ & = [b\beta d,\gamma]\otimes f & = [e\delta b\beta d,\gamma]\otimes f & = [e,\delta][b\beta d,\gamma]\otimes f \\ & = [e,\delta]\otimes b\beta d\gamma f & = [e,\delta]\otimes a\alpha c\gamma f & = [e,\delta]\otimes a\alpha c \\ & = [e,\delta]\otimes b\beta d & = [e,\delta][b,\beta]\otimes d & = [b,\beta]\otimes d. \end{array}$$

So κ is injective and hence the theorem. □

Theorem 3. *With the same notation as in the above theorem the following hold:*

(1) L and R are Morita equivalent;

(2) ${}_LA$ and ${}_R\Gamma$ are respectively generators for FL-Act and FR-Act;

(3) $L \cong L \otimes End_R(\Gamma)$ and $R \cong R \otimes End_L(A)$ as semigroups;

(4) ${}_LA_R \cong L \otimes Hom_R(\Gamma, R)$ and ${}_R\Gamma_L \cong R \otimes Hom_L(A, L)$.

Proof. By Theorem 2, (1) L and R are semigroups with local units; (2) ${}_LA_R$ and ${}_R\Gamma_L$ are unitary biacts; (3) ${}_LA \in FL$-Act and ${}_R\Gamma \in FR$-Act. Now let us define the maps $\tau : A \otimes \Gamma \to L$ and $\mu : \Gamma \otimes A \to R$ as $(a \otimes \alpha)\tau = [a, \alpha]$ and $(\alpha \otimes a)\mu = [\alpha, a]$ respectively. The fact that these maps are well-defined, can be proved by routine verification. Also it is clear that these mappings are surjective. Now we see that $(a\otimes\alpha)\tau b = [a, \alpha]b = a\alpha b = a[\alpha, b] = a(\alpha\otimes b)\mu$ and $\alpha(a \otimes \beta)\tau = \alpha[a, \beta] = \alpha a\beta = [\alpha, a]\beta = (\alpha \otimes a)\mu\beta$.

Thus $\langle L, R,_L A_R,_R \Gamma_L, \tau, \mu\rangle$ is a unitary Morita context with τ and μ surjective. Hence the theorem follows from Theorem 8.3 of [8]. □

Theorem 4. *Let L and R be two Morita equivalent semigroups with local units. Then there exists a Γ-semigroup A with local units whose left and right operator semigroups are respectively isomorphic to L and R.*

Proof. As L and R are Morita equivalent, the categories FL-Act and FR-Act are equivalent categories via functors, say $F : FL$-Act $\to FR$-Act and $G : FR$-Act $\to FL$-Act. Now let $A = G(R)$ and $\Gamma = F(L)$. Then, according to the Theorem 7.3 of [8], we can define $<\ \ >: A\otimes_R \Gamma \to L$ and $\lceil\ \rceil : \Gamma\otimes_L A \to R$, such that the sixtuple $\langle L, R,_L A_R,_R \Gamma_L, <\ \ >, \lceil\ \rceil\rangle$ defines a Morita context. Furthermore $A \otimes_R \Gamma$ and $\Gamma \otimes_L A$ become semigroups if we put respectively $(a \otimes \alpha)(b \otimes \beta) = a \otimes \lceil\alpha, b\rceil\beta$ and $(\alpha \otimes a)(\beta \otimes b) = \alpha\otimes < a, \beta > b$. Also $<\ \ >$ and $\lceil\ \rceil$ are surjective semigroup morphisms.

Now we define $A \times \Gamma \times A \to A$ and $\Gamma \times A \times \Gamma \to \Gamma$ respectively as $(a, \gamma, b) \mapsto a\gamma b := a\lceil \gamma, b\rceil$ and $(\alpha, x, \beta) \mapsto \alpha x \beta := \alpha < x, \beta >$. Then we deduce the following equalities:

$$(a\alpha b)\beta c = (a\lceil \alpha, b\rceil)\lceil \beta, c\rceil = a(\lceil \alpha, b\rceil\lceil \beta, c\rceil) = a\lceil \alpha, < b, \beta > c\rceil.$$

$$a\alpha(b\beta c) = a\lceil \alpha, b\lceil \beta, c\rceil\rceil = a\lceil \alpha, < b, \beta > c\rceil.$$

$$a(\alpha b\beta)c = a\lceil \alpha < b, \beta >, c\rceil = a\lceil \alpha, < b, \beta > c\rceil.$$

Hence A is a Γ-semigroup. Let its left and right operator semigroups be L' and R'. In order to prove that these are isomorphic to L and R we first define $\phi : L' \to L$ by $([a, \alpha])\phi =< a, \alpha >$. By routine verification we see that ϕ is well-defined. Now $< a, \alpha >=< b, \beta >$ implies $\gamma a\alpha = \gamma b\beta$ and $a\alpha x = b\beta x$ for all $x \in A,\ \gamma \in \Gamma$. So $[a, \alpha] = [b, \beta]$. Hence ϕ is injective. Again, the map ϕ is surjective semigroup morphism since $<\ \ >$ is also so.

Hence L and L' are isomorphic as semigroups. Analogously, R' is isomorphic to R.

Again, from the construction, we see that ${}_{L'}A_{R'}$ and ${}_{R'}\Gamma_{L'}$ are unitary biacts. Then for each $a \in A$ and $\alpha \in \Gamma$ there exist idempotents $[e_a, \delta_a],\ [e_\alpha, \delta_\alpha] \in L'$ and $[\gamma_a, f_a],\ [\gamma_\alpha, f_\alpha] \in R'$ such that $e_a\delta_a a = a, a\gamma_a f_a = a; \gamma_\alpha f_\alpha \alpha = \alpha, \alpha e_\alpha \delta_\alpha = \alpha$.

So Γ-semigroup A has local units. This completes the proof. □

Now we combine two preceding results into one.

Theorem 5. *Two semigroups L and R with local units are Morita equivalent if and only if there exists a Γ-semigroup A with both A and Γ having local units whose operator semigroups are isomorphic to L and R.*

As applications of Theorem 5 we use recent works of Lawson in [5] and of Laan and Marki in [4] to find out some interplays between the operator semigroups of a Γ-semigroup with local units. In what follows A is a Γ-semigroup with local units with L and R as its left and right operator semigroups respectively.

Corollary 6. *(1) L is periodic if and only if R is periodic.*

(2) L is regular if and only if R is regular.

(3) Greatest commutative images of L and R are isomorphic semigroups, if L and R have common two-sided local units.

Proof. By Theorem 5, L and R are Morita equivalent semigroups with local units. Also recall that Morita equivalence coincides with strong Morita equivalence for the case of semigroups with local units (see [8]). Consequently, the proof of (1), (2) and (3) from Proposition 1, Proposition 2 and Theorem 4 of [4] respectively. □

By a similar argument, we obtain the following theorem, using Theorem 3 and Theorem 6 of [4] respectively.

Corollary 7. *(1) There is an isomorphism $\Phi : Id(L) \to Id(R)$ between the lattices of ideals of L and R. Moreover, this isomorphism preserves finitely generated ideals and principal ideals.*

(2) If L and R have common joint local units, then there exists an isomorphism $\Pi : Con(L) \to Con(R)$ between the congruence lattices of L and R such that if $\rho \in Con(L)$ then the semigroups L/ρ and $R/\Pi(\rho)$ are Morita equivalent. Moreover, it preserves Rees congruence, finitely generated congruence and principal congruence.

Corollary 8. *(1) Each local submonoid of L is isomorphic to a local submonoid of R and vice-versa.*

(2) Cardinalities of the set of regular $\mathcal{D}$-classes in L and R are same.

(3) If L is a monoid, then there is an idempotent $r \in R$ such that $R = RrR$ and $rRr \cong L$. Thus R is an enlargement of L.

Proof. Since L and R are Morita equivalent semigroups with local units, by Theorem 5, we get these results from Theorem 5.1 and 5.2 of [5]. □

The following result is a consequence of Theorem 5.5 of [5] in view of our Theorem 5.

Corollary 9. *(1) L is a group if and only if R is completely simple. (2) L is an inverse semigroup if and only if R is regular and locally inverse. (3) L is a semilattice if and only if R is regular, locally a semilattice and $R/\mathcal{J}$ is a meet semilattice under subset inclusion. (4) L is an orthodox semigroup if and only if R is regular and locally orthodox. (5) L is an $\mathcal{L}$-unipotent semigroup if and only if R is regular and locally $\mathcal{L}$-unipotent. (6) L is an E-solid semigroup if and only if R is regular and locally E-solid. (7) L is a union of groups if and only if R is regular locally a union of groups and $R/\mathcal{J}$ is a meet semilattice under subset inclusion.*

Combining Theorem 5 with Lawson's Theorem 1.1 of [5] we obtain the following characterization.

Theorem 10. *Let L and R are semigroups with local units. Then the following statements are equivalent.*

(1) L and R are Morita equivalent.

(2) Cauchy completion categories $C(L)$ and $C(R)$ are equivalent.

(3) L and R have a joint enlargement.

(5) There is a Γ-semigroup with local units whose left and right operator semigroups are isomorphic to L and R respectively.

Concluding Remark: Theorem 6 of [7] becomes a special case of Theorem 5 of this paper. But a remarkable difference in the proof of these two theorems is that a direct constructive approach had been taken in [7] while we have accomplished the proof here via Morita context to avoid complexity due to the restriction in S-Act category.

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