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ADDITIVE QUADRATIC FUNCTIONAL EQUATION ARE STABLE IN BANACH SPACE: A FIXED POINT APPROACH

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Abstract: In this paper, the authors established the generalized Ulam - Hyers stability of a mixed type Additive Quadratic(AQ)-functional equation

$$f(x+2y+3z)+f(x-2y+3z)+f(x+2y-3z)+f(x-2y-3z) = 4f(x)+8[f(y)+f(-y)]+18[f(z)+f(-z)]$$

in Banach spaces using fixed point approach.

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1. Introduction

S.M. Ulam [25] is the pioneer for the famous stability problem in functional equations. In 1940, while he was delivering a talk before the Mathematics Club of University of Wisconsin, he proposed a number of unsolved problems. Among those was the following question concerning the stability of homomorphisms:

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"Let G be group and H be a metric group with metric $d(.,.)$. Given $\epsilon > 0$ does there exists a $\delta > 0$ such that if a function $f : G \to H$ satisfies

$$d\left(f(xy), f(x)f(y)\right) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $a : G \to H$ with

$$d\left(f(x), a(x)\right) < \varepsilon$$

for all $x \in G$."

In 1941, D. H. Hyers [9] gave an affirmative answer to the question of S.M. Ulam for Banach spaces. In 1950, T. Aoki [2] was the second author to treat this problem for additive mappings. In 1978, Th.M. Rassias [21] succeeded in extending Hyers' Theorem by weakening the condition for the Cauchy difference controlled by $(||x||^p + ||y||^p)$, $p \in [0, 1)$, to be unbounded.

In 1982, J.M. Rassias [19] replaced the factor $||x||^p + ||y||^p$ by $||x||^p||y||^q$ for $p, q \in R$. A generalization of all the above stability results was obtained by P. Gavruta [8] in 1994 by replacing the unbounded Cauchy difference by a general control function $\varphi(x, y)$.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi etal., [24] by considering the summation of both the sum and the product of two $p-$ norms. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7, 13, 16, 20, 24]) and reference cited there in.

The generailzed Ulam - Hyers - Rassias stability for various quadratic functional equations; like

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(z+x) \tag{1.1}$$

$$f(x+y+z) + f(x-y) + f(y-z) + f(z-x) = 3f(x) + 3f(y) + 3f(z) \tag{1.2}$$

$$f\left(\sum_{i=1}^{n} x_i\right) + \sum_{i=1}^{n} f(x_i - x_j) = nf\left(\sum_{i=1}^{n} x_i\right), \quad n \geq 2 \tag{1.3}$$

was investigated by S.M. Jung [12], J.H. Bae, K.W. Jun [5]. For further readings one can refer [6, 10, 15, 22]. Infact, Y.S. Jung, I.S. Chang [14] discussed the Hyers - Ulam - Rassias stability for the new cubic type functional equation

$$\begin{aligned} &f(x+y+2z) + f(x+y-2z) + f(2x) + f(2y) \\ &\quad = 2[f(x+y) + 2f(x+z) + 2f(y+z) + 2f(x-z) + 2f(y-z)] \end{aligned} \tag{1.4}$$

with the fixed point alternative.

Recently, M. Arunkumar et al., [3] first time introduced and investigated the solution and generalized Ulam-Hyers stability of a 2 - variable AC - mixed type functional equation

$$f(2x+y,2z+w)-f(2x-y,2z-w) = 4[f(x+y,z+w)-f(x-y,z-w)]-6f(y,w) \tag{1.5}$$

having solutions

$$f(x,y) = ax + by \tag{1.6}$$

and

$$f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 \tag{1.7}$$

in Banach space via direct and fixed point approach. Very recently, M.Arunkumar and P. Agilan [4] established the general solution and generalized Ulam - Hyers stability of a mixed type Additive Quadratic(AQ)-functional equation

$$\begin{aligned} f(x+2y+3z)+f(x-2y+3z)+f(x+2y-3z)+f(x-2y-3z) \\ = 4f(x)+8[f(y)+f(-y)]+18[f(z)+f(-z)] \end{aligned} \tag{1.8}$$

in Banach spaces a direct method.

In this paper, the authors has proved the generalized Ulam - Hyers stability of a mixed type Additive Quadratic (AQ)-functional equation (1.8) in Banach spaces with the help of fixed point method.

Now we will recall the fundamental results in fixed point theory.

Theorem 1.1. *(Banach's contraction principle) Let (X, d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

(A_1) $$d(Tx, Ty) \leq Ld(x, y)$$

for some (Lipschitz constant) $L < 1$. Then,

(i) The mapping T has one and only fixed point $x^* = T(x^*)$;

(ii)The fixed point for each given element x^* is globally attractive, that is

(A_2) $$lim_{n\to\infty} T^n x = x^*,$$

for any starting point $x \in X$;

(iii) One has the following estimation inequalities:

(A_3) $$d(T^n x, x^*) \leq \frac{1}{1-L} \ d(T^n x, T^{n+1}x), \forall \ \ n \geq 0, \forall \ \ x \in X;$$

(A_4) $$d(x, x^*) \leq \frac{1}{1-L} \ d(x, x^*), \forall \ \ x \in X.$$

Theorem 1.2. *[17](The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*

(B_1) $$d(T^n x, T^{n+1}x) = \infty \quad \forall \ n \geq 0,$$

or

(B_2) *there exists a natural number n_0 such that:*
(i) $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$;
(ii) *The sequence $(T^n x)$ is convergent to a fixed point y^* of T*
(iii) *y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;*
(iv) $d(y^*, y) \leq \frac{1}{1-L}$ $d(y, Ty)$ *for all $y \in Y$.*

In Section 2, the generalized Ulam - Hyers stability of the functional equation (1.8) for odd case is proved. In Section 3, the generalized Ulam - Hyers stability of the functional equation (1.8) for even case is present. The generalized Ulam - Hyers stability of the functional equation (1.8) for mixed case is established in Section 4.

Hereafter through out this paper let us assume V be a vector space and B Banach space respectively. Define a mapping $Df : X \to Y$ by

$$\begin{aligned} Df(x, y, z) &= f(x + 2y + 3z) + f(x - 2y + 3z) + f(x + 2y - 3z) + f(x - 2y - 3z) \\ &\quad - 4f(x) - 8[f(y) + f(-y)] - 18[f(z) + f(-z)] \end{aligned}$$

for all $x, y, z \in X$.

2. Stability Results: Odd Case

In this section, the generalized Ulam - Hyers stability of the Mixed type AQ-functional equation (1.8) for odd case is provided.

Theorem 2.1. *Let $f_a : V \to B$ be a mapping for which there exist a function $\alpha : V^3 \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \frac{\alpha(\mu_i^k x, \mu_i^k y,\ \mu_i^k z)}{\mu_i^k} = 0 \tag{2.1}$$

where $\mu_i = 3$ if $i = 0$ and $\mu_i = \frac{1}{3}$ if $i = 1$ such that the functional inequality with

$$\|Df_a(x, y, z)\| \leq \alpha(x, y, z) \tag{2.2}$$

for all $x, y, z \in V$. If there exists $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{3}, \frac{x}{3}, 0\right),$$

has the property

$$\beta(x) \leq L\ \mu_i\ \beta\left(\frac{x}{\mu_i}\right) \tag{2.3}$$

for all $x \in V$. Then there exists unique additive function $A : V \to B$ satisfying the functional equation (1.8) and

$$\|f_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\beta(x) \tag{2.4}$$

holds for all $x \in V$.

Proof. Consider the set $X = \{p/p : V \to B,\ p(0) = 0\}$ and introduce the generalized metric on X,

$$d(p, q) = \inf\{K \in (0, \infty) :\| p(x) - q(x) \|\leq K\beta(x), x \in V\}.$$

It is easy to see that (X, d) is complete. Define $T : X \to X$ by $Tp(x) = \frac{1}{\mu_i}p(\mu_i x)$, for all $x \in V$. Now $p, q \in X$,

$$\begin{aligned} d(p, q) \leq K \Rightarrow & \| p(x) - q(x) \|\leq K\beta(x), x \in V. \\ \Rightarrow & \left\|\frac{1}{\mu_i}p(\mu_i x) - \frac{1}{\mu_i}q(\mu_i x)\right\| \leq \frac{1}{\mu_i}K\beta(\mu_i x), x \in V, \\ \Rightarrow & \left\|\frac{1}{\mu_i}p(\mu_i x) - \frac{1}{\mu_i}q(\mu_i x)\right\| \leq LK\beta(x), x \in V, \\ \Rightarrow & \| Tp(x) - Tq(x) \|\leq LK\beta(x), x \in V, \\ \Rightarrow & d(Tp, Tq) \leq LK. \end{aligned}$$

This implies $d(Tp, Tq) \leq Ld(p, q)$, for all $p, q \in X$. i.e., T is a strictly contractive mapping on X with Lipschitz constant L. Replacing (x, y, z) by $(x, x, 0)$ in (2.2) and using oddness of f, we get

$$\left\|f_a(x) - \frac{1}{3}f_a(3x)\right\| \leq \frac{1}{6}\alpha(x, x, 0) \tag{2.5}$$

for all $x \in V$. Using (2.3) for the case $i = 0$ it reduces to

$$\left\|f_a(x) - \frac{1}{3}f_a(3x)\right\| \leq \frac{1}{3}\beta(x)$$

for all $x \in V$.

$$\text{i.e.,} \qquad d(f_a, Tf_a) \leq \frac{1}{3} \Rightarrow d(f_a, Tf_a) \leq \frac{1}{3} = L = L^1 < \infty.$$

Again replacing $x = \frac{x}{3}$ in (2.5), we get

$$\left\| f_a(x) - 3f\left(\frac{x}{3}\right) \right\| \leq \frac{1}{2}\alpha\left(\frac{x}{3}, \frac{x}{3}, 0\right).$$

for all $x \in V$. Using (2.3) for the case $i = 1$ it reduces to

$$\left\| f_a(x) - 3f\left(\frac{x}{3}\right) \right\| \leq \beta(x)$$

for all $x \in V$.

$$\text{i.e.,} \qquad d(f_a, Tf_a) \leq 1 \Rightarrow d(f_a, Tf_a) \leq 1 = L^0 < \infty.$$

In above cases, we arrive

$$d(f_a, Tf_a) \leq L^{1-i}$$

Therefore $(B_2(i))$ holds.

By $(B_2(ii))$, it follows that there exists a fixed point A of T in X such that

$$A(x) = \lim_{k \to \infty} \frac{f_a(\mu_i^k x)}{\mu_i^k} \qquad \forall \ x \in V. \tag{2.6}$$

In order to prove $A : V \to B$ is Additive. Replacing (x, y, z) by $\left(\mu_i^k x, \mu_i^k y, \mu_i^k z\right)$ in (2.2) and dividing by μ_i^k, it follows from (2.1) and (2.6), A satisfies (1.8) for all $x, y, z \in V$. i.e., A satisfies the functional equation (1.8).

By $(B_2(iii))$, A is the unique fixed point of T in the set $Y = \{f_a \in X : d(Tf_a, A) < \infty\}$, using the fixed point alternative result A is the unique function such that

$$\|f_a(x) - A(x)\| \leq K\beta(x)$$

for all $x \in V$ and $K > 0$. Finally by $(B_2(iv))$, we obtain

$$d(f_a, A) \leq \frac{1}{1-L} d(f_a, Tf_a)$$

this implies

$$d(f_a, A) \leq \frac{L^{1-i}}{1-L}.$$

Hence we conclude that

$$\| f_a(x) - A(x) \| \leq \frac{L^{1-i}}{1-L}\beta(x).$$

for all $x \in V$. This completes the proof of the theorem. □

From Theorem 2.1, we obtain the following corollary concerning the stability for the functional equation (1.8).

Corollary 2.2. *Let $f : V \to B$ be a mapping and there exits real numbers λ and s such that*

$$\|Df_a(x,y,z)\| \leq \begin{cases} \lambda, \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s < 1 \text{ or } s > 1; \\ \lambda\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & s < \frac{1}{3} \text{ or } s > \frac{1}{3}; \end{cases}$$

for all $x, y, z \in V$, then there exists a additive function $A : V \to B$ such that

$$\|f_a(x) - A(x)\| \leq \begin{cases} \dfrac{\lambda}{4}, \\ \dfrac{\lambda||x||^s}{|3-3^s|}, \\ \dfrac{\lambda||x||^{3s}}{|3-3^{3s}|} \end{cases} \tag{2.7}$$

for all $x \in V$.

Proof. Setting

$$\alpha(x,y,z) = \begin{cases} \lambda \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s\right\}, \\ \lambda\left\{||x||^s||y||^s||z||^s + (||x||^{3s} + ||y||^{3s} + ||z||^{3s})\right\} \end{cases}$$

for all $x, y, z \in V$. Now

$$\frac{\alpha(\mu_i^k x, \mu_i^k y, \mu_i^k z)}{\mu_i^k} = \begin{cases} \dfrac{\lambda}{\mu_i^k} \\ \dfrac{\lambda}{\mu_i^k}\left\{||\mu_i^k x||^s + ||\mu_i^k y||^s + ||\mu_i^k z||^s\right\}, \\ \dfrac{\lambda}{\mu_i^k}\left\{||\mu_i^k x||^s\ ||\mu_i^k y||^s\ ||\mu_i^k z||^s\left\{||\mu_i^k x||^{3s} + ||\mu_i^k y||^{3s} + ||\mu_i^k z||^{3s}\right\}\right\} \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } k \to \infty, \\ \to 0 \text{ as } k \to \infty, \\ \to 0 \text{ as } k \to \infty. \end{cases}$$

i.e., (2.1) is holds. But we have $\beta(x) = \frac{1}{2}\alpha\left(\frac{x}{3}, \frac{x}{3}, 0\right)$. Hence

$$\beta(x) = \frac{1}{2}\alpha\left(\frac{x}{3}, \frac{x}{3}, 0\right) = \begin{cases} \frac{\lambda}{2} \\ \frac{\lambda}{3^s}||x||^s, \\ \frac{\lambda}{3^{3s}}||x||^{3s}. \end{cases}$$

Also,

$$\frac{1}{\mu_i}\beta(\mu_i x) = \begin{cases} \frac{\lambda}{2\mu_i} \\ \frac{\lambda}{\mu_i \cdot 3^s}||\mu_i x||^s, \\ \frac{\lambda}{\mu_i \cdot 3^{3s}}||\mu_i x||^{3s}. \end{cases}$$
$$= \begin{cases} \mu_i^{-1}\beta(x) \\ \mu_i^{s-1}\beta(x), \\ \mu_i^{3s-1}\beta(x). \end{cases}$$

Hence the inequality (2.3) holds either, $L = 3^{s-1}$ for $s < 1$ if $i = 0$ and $L = \frac{1}{3^{s-1}}$ for $s > 1$ if $i = 1$.
Now from (2.4), we prove the following cases for condition (ii).
Case:1 $L = 3^{s-1}$ for $s < 1$ if $i = 0$

$$\|f_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{\left(3^{s-1}\right)^{1-0}}{1-(3)^{s-1}}\frac{\lambda}{3^s}||x||^s = \frac{\lambda||x||^s}{3-3^s}.$$

Case:2 $L = \frac{1}{3^{s-1}}$ for $s > 1$ if $i = 1$

$$\|f_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{\left(\frac{1}{3^{s-1}}\right)^{1-1}}{1-\frac{1}{3^{s-1}}}\frac{\lambda}{3^s}||x||^s = \frac{\lambda||x||^s}{3^s-3}.$$

Similarly, we can prove, if $L = 3^{3s-1}$ for $s < \frac{1}{3}$ if $i = 0$ and $L = \frac{1}{3^{3s-1}}$ for $s > \frac{1}{3}$ if $i = 1$. Hence the proof of the corollary. □

3. Stability Results: Even Case

In this section, the generalized Ulam - Hyers stability of the Mixed type AQ-functional equation (1.8) for even case is present. The proofs of the following Theorem and Corollary is similar tracing to that of Theorem 2.1 and Corollary 2.2. Hence we omit the proofs.

Theorem 3.1. *Let $f_q : V \to B$ be a mapping for which there exist a function $\alpha : V^3 \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \frac{\alpha(\mu_i^k x, \mu_i^k y,\ \mu_i^k z)}{\mu_i^{2k}} = 0 \tag{3.1}$$

where $\mu_i = 3$ if $i = 0$ and $\mu_i = \frac{1}{3}$ if $i = 1$ such that the functional inequality with

$$\|Df_q(x, y, z)\| \le \alpha(x, y, z) \tag{3.2}$$

for all $x, y, z \in V$. If there exists $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{3}, \frac{x}{3}, 0\right),$$

has the property

$$\beta(x) \le L\ \mu_i^2\ \beta\left(\frac{x}{\mu_i}\right) \tag{3.3}$$

for all $x \in V$. Then there exists unique quadratic function $Q : V \to B$ satisfying the functional equation (1.8) and

$$\|f_q(x) - Q(x)\| \le \frac{L^{1-i}}{1-L}\beta(x) \tag{3.4}$$

holds for all $x \in V$.

Corollary 3.2. *Let $f_q : V \to B$ be a mapping and there exits real numbers λ and s such that*

$$\|Df_q(x, y, z)\| \le \begin{cases} \lambda; \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s < 1 \quad or \quad s > 1; \\ \lambda\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & s < \frac{1}{3} \quad or \quad s > \frac{1}{3}; \end{cases} \tag{3.5}$$

for all $x, y, z \in V$, then there exists a quadratic function $Q : V \to B$ such that

$$\|f_q(x) - Q(x)\| \le \begin{cases} \dfrac{\lambda}{16}, \\ \dfrac{\lambda||x||^s}{|9-3^s|}, \\ \dfrac{\lambda||x||^{3s}}{|9-3^{3s}|} \end{cases} \tag{3.6}$$

for all $x \in V$.

4. Stability Results: Mixed Case

In this section, the generalized Ulam - Hyers stability of the Mixed type AQ-functional equation (1.8) for mixed case is investigated.

Theorem 4.1. *Let $f : V \to B$ be a mapping for which there exist a function $\alpha : V^3 \to [0,\infty)$ with the condition (2.1) and (3.1) where $\mu_i = 3$ if $i = 0$ and $\mu_i = \frac{1}{3}$ if $i = 1$ such that the functional inequality with*

$$\|Df(x,y,z)\| \leq \alpha(x,y,z) \tag{4.1}$$

for all $x,y,z \in V$. If there exists $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{3},\frac{x}{3},0\right),$$

has the properties (2.3) and (3.3) for all $x \in V$. Then there exists unique additive function $A : V \to B$ and a unique quadratic function $Q : V \to B$ satisfying the functional equation (1.8) and

$$\|f(x) - A(x) - Q(x)\| \leq \frac{L^{1-i}}{1-L}\left(\beta(x) + \beta(-x)\right) \tag{4.2}$$

holds for all $x \in V$.

Proof. Let $f_o(x) = \dfrac{f_a(x) - f_a(-x)}{2}$ for all $x \in x$. Then $f_o(0) = 0$ and $f_o(-x) = -f_o(x)$ for all $x \in V$. Hence

$$\|Df_o(x,y,z)\| \leq \frac{\alpha(x,y,z)}{2} + \frac{\alpha(-x,-y,-z)}{2} \tag{4.3}$$

For all $x,y,z \in V$. By Theorem 2.1, we have

$$\|f_o(x) - A(x)\| \leq \frac{1}{2}\frac{L^{1-i}}{1-L}\left(\beta(x) + \beta(-x)\right) \tag{4.4}$$

for all $x \in V$. Also, let $f_e(x) = \dfrac{f_q(x) + f_q(-x)}{2}$ for all $x \in X$. Then $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ for all $x \in x$. Hence

$$\|Df_e(x,y,z)\| \leq \frac{\alpha(x,y,z)}{2} + \frac{\alpha(-x,-y,-z)}{2} \tag{4.5}$$

For all $x,y,z \in V$. By Theorem 3.1, we have

$$\|f_e(x) - Q(x)\| \leq \frac{1}{2}\frac{L^{1-i}}{1-L}\left(\beta(x) + \beta(-x)\right) \tag{4.6}$$

for all $x \in V$. Define

$$f(x) = f_e(x) + f_o(x) \tag{4.7}$$

for all $x \in V$. From (4.4),(4.6) and (4.7), we arrive

$$\begin{aligned}\|f(x) - A(x) - Q(x)\| &= \|f_e(x) + f_o(x) - A(x) - Q(x)\| \\ &\le \|f_o(x) - A(x)\| + \|f_e(x) - Q(x)\| \\ &\le \frac{1}{2}\frac{L^{1-i}}{1-L}(\beta(x) + \beta(-x)) + \frac{1}{2}\frac{L^{1-i}}{1-L}(\beta(x) + \beta(-x)) \\ &\le \frac{L^{1-i}}{1-L}(\beta(x) + \beta(-x))\end{aligned}$$

for all $x \in V$. Hence the theorem is proved. □

Using Corollaries 2.2 and 3.2 we have the following Corollary concerning the stability of (1.8).

Corollary 4.2. *Let λ and s be nonnegative real numbers. Let a function $f : V \to B$ satisfies the inequality*

$$\|Df(x,y,z)\| \le \begin{cases} \lambda \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 1,2; \\ \lambda\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & s \neq 1,2; \end{cases} \tag{4.8}$$

for all $x, y, z \in V$. Then there exists a unique additive function $A : V \to B$ and a unique quadratic function $Q : V \to B$ such that

$$\|f(x) - A(x) - Q(x)\| \le \begin{cases} \frac{\lambda}{4} + \frac{\lambda}{16} \\ \lambda||x||^s\left(\frac{1}{|3-3^s|} + \frac{1}{|9-3^s|}\right), \\ \lambda||x||^{3s}\left(\frac{1}{|3-3^{3s}|} + \frac{1}{|9-3^{3s}|}\right), \end{cases} \tag{4.9}$$

for all $x \in V$.

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