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INDUCED PATH DECOMPOSITION AND HOLE IN SIERPINSKY GRAPH

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Abstract: Path decomposition and Path width which are closely analogous to tree decomposition and tree width play a key role in the theory of graph minors. They have many applications in VLSI Design, Graph Drawing, Compiler Design and Linguistics [1]. Many problems in graph algorithm can be solved effectively on graphs of bounded path width by using dynamic programming on a path decomposition of the graph [2]. Decomposition may also be used to measure the space complexity of dynamic programming algorithms on graphs of bounded tree width [3]. Once path decomposition has been found, a topological ordering of width w (if one exists) can be found using dynamic programming in linear time [4]. In this paper we discuss about the induced path decomposition, the hole and the positions of every vertex of the Sierpinsky Graph w.r.t the hole.

Key Words: induced path, acyclic induced path, induced path decomposition, hole, sierpinsky graph

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1. Introduction

In graph theory an induced path in an undirected graph is a path which is an induced sub graph of G with a sequence of vertices in G such that each two adjacent vertices in the sequence is connected by an edge in G whereas *each two non-adjacent vertices in the sequence is not connected by any edge in* G. An induced path is a sort of *spinal* of the original graph. A path P in a connected graph G is called maximally extended path if its extension at either end by appending an edge is not possible or in other words there is no new vertex other than the one that already belongs to P which is adjacent to one of the two end vertices of P [5].If G is a connected graph with vertex set $V(G)$ and edge set $E(G)$ then a collection $\psi = \{H_1, H_2, H_3, \ldots, H_r\}$ of subsets of V(G) with each H_i as a path or cycle and every edge of $E(G)$ lying exactly in one H_i is called a decomposition of G.

2. Sierpinsky Graph

An equilateral triangle S_1 is divided into four equal triangles namely one down triangle ∇_2 in the middle and three up triangles Δ_2. Except the middle down triangle ∇_2 each other three up triangles Δ_2 are again divided into four equal triangles ∇_3 and three Δ_3. This process of keeping the down triangle undivided and dividing the up triangles again into four is continued up to the required iteration[Fig1(a)].

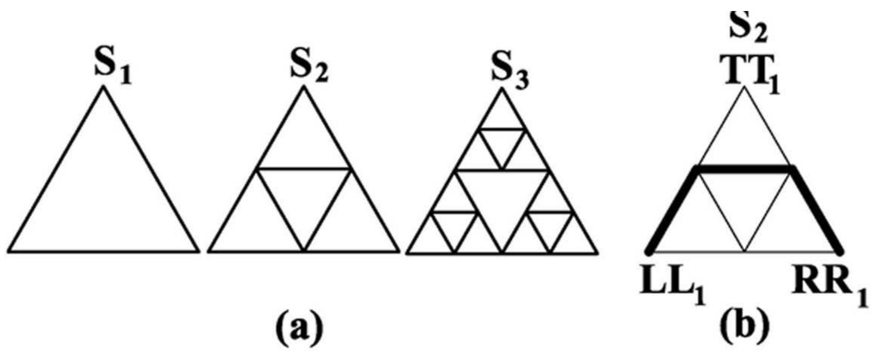

Figure 1: (a)Sierpinsky Graph (b)Longest Induced path in S_2

For $k > 2$, the structure of S_k can be viewed as $\nabla_2 + 3\nabla_3 + 3^2\nabla_4 + \ldots + 3^{k-2}\nabla_k + 3^{k-1}\Delta_k$.

S_k consists of three copies of S_{k-1} which we call as of left , right and top components denoted by S^L_{k-1} , S^R_{k-1} and S^T_{k-1} respectively. The extreme vertices of S^P_{k-1} are denoted by TT_{k-1}, TL_{k-1}, TR_{k-1} Similarly LT_{k-1}, LL_{k-1}, LR_{k-1} and RT_{k-1}, RL_{k-1}, RR_{k-1} are defined.We note that $TL_{k-1} = LT_{k-1}; LR_{k-1} = RL_{k-1}; TR_{k-1} = RT_{k-1}$.

Theorem 1. *For all k≥1 there exist longest induced paths in S_k of length 3^{k-1}.*

Proof. Choose a longest path $P^{RR_2}_{LL_2}$ in S_2 from LL_1 to RR_1 as shown in fig1(b). Assume that a longest path $P^{RR_{k-1}}_{LL_{k-1}}$ in S_{k-1} exists from LL_{k-2} to RR_{k-2}.

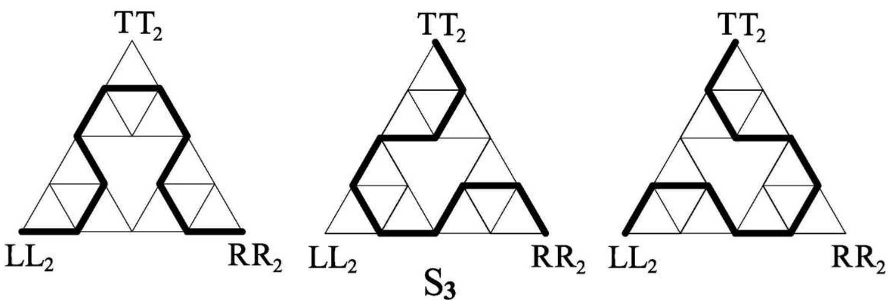

Figure 2: Longest Induced Paths in S_3

Consider S_k.

We construct a longest path $P^{RR_k}_{LL_k}$ in S_k from LL_{k-1} to RR_{k-1} as follows (see fig 2 for S_3)

$P^{RR_k}_{LL_k} = P^{LT_{k-1}}_{LL_{k-1}} * P^{TR_{k-1}}_{LT_{k-1}} * P^{RR_{k-1}}_{TR_{k-1}}$.

Similarly the second is from RR_k to TT_k as

$P^{TT_k}_{RR_k} = P^{LR_{k-1}}_{RR_{k-1}} * P^{TL_{k-1}}_{LR_{k-1}} * P^{TT_{k-1}}_{TL_{k-1}}$

and the third is from TT_k to LL_k as

$P^{LL_k}_{TT_k} = P^{LT_{k-1}}_{TT_{k-1}} * P^{TR_{k-1}}_{LT_{k-1}} * P^{RR_{k-1}}_{TR_{k-1}}$. □

3. Decomposition of a Graph

A decomposition of a graph G is a collection $\Psi = \{H_1, H_2, H_3, \ldots, H_r\}$ of sub graphs of G such that every edge of $E(G)$ belongs to exactly one H_i. In this section we introduce the notion of induced acyclic path decomposition number and determine the value of this parameter for Sierpinsky Graph.

Definition 2. If $\Psi = \{H_1, H_2, H_3, \ldots, H_r\}$ is a decomposition of G with each H_i as a path or cycle then Ψ is called a path decomposition of G. The minimum cardinality of path decomposition of G is called the path decomposition number of G and is denoted by $\pi(G)$. The parameter π was introduced by Arumugam [6].

Definition 3. If $\Psi = \{H_1, H_2, H_3, \ldots, H_r\}$ is a decomposition of G with each H_i as a path and not a cycle then Ψ is called an acyclic path decomposition of G. The minimum cardinality of acyclic path decomposition of G is called the acyclic path decomposition number of G and is denoted by $\pi_{ia}(G)$.The parameter π_{ia} was introduced by Harary and further studied by Harary and Schwenk [7].

Definition 4. If $\Psi = \{H_1, H_2, H_3, \ldots, H_r\}$ is an acyclic path decomposition of G with two paths having at most one vertex in common then it is known as simple acyclic path decomposition. The minimum cardinality of simple acyclic path decomposition of G is called the simple acyclic path decomposition number of G and is denoted by $\pi_{as}(G)$. The parameter π_{as} was introduced by Arumugam and Sahul Hamid [8] who used $\pi_s(G)$ for simple acyclic path decomposition number and called simple acyclic path decomposition as a simple path cover.

Definition 5. If every vertex of G is an internal vertex of at most one member of the decomposition then the acyclic path decomposition is called acyclic graphoidal covering and its minimum cardinality number is $\eta_a(G)$[9].

Definition 6. An acyclic induced path decomposition is a collection of sub graphs H_i, $\Psi = \{H_1, H_2, H_3, \ldots, H_r\}$ with each H_i as induced path. The minimum cardinality of the acyclic induced path decomposition is called as the acyclic induced path decomposition number of G and is denoted by $\pi_{ia}(G)$[10].

In a tree, every path decomposition is induced path decomposition and in fact it has been proved that trees are the only graphs in which every path decomposition is induced.

Theorem 7. *For all $k > 1$, $\Psi = \{IP_{LR}, IP_{RT}, IP_{TL}\}$ is an induced path decomposition (IPD) for S_k with each IP_{LR},IP_{RT},IP_{TL} as induced path and $|\Psi|$ =3.*

Proof. By theorem 1 of Section 2, for all $k > 1$, we can consider an induced path from left corner vertex L to right corner vertex R that can be denoted as

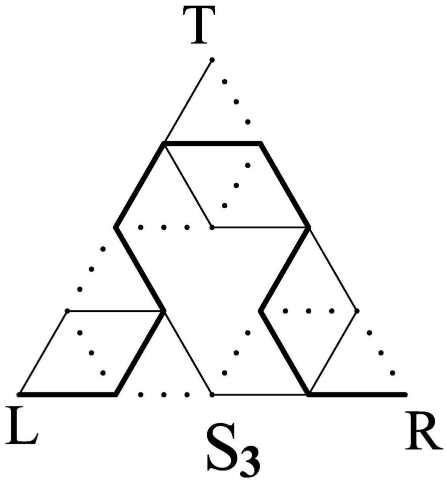

Figure 3: Induced Path Decomposition in S_3

IP_{LR}, second an induced path from right corner vertex R to top corner vertex T that can be denoted as IP_{RT} and thirdly an induced path from top corner vertex T to left corner vertex L that can be denoted as IP_{TL}[fig2].

Obviously we observe that these three induced paths IP_{LR},IP_{RT} and IP_{TL} fully cover S_k and every edge of S_k is found exactly in one of the three induced paths $IP_{LR}, IP_{RT}, IP_{TL}$[Fig3]. Hence by the above definition of induced path decomposition, we have $\Psi = \{IP_{LR}, IP_{RT}, IP_{TL}\}$ as an induced path decomposition of S_k.

We also observe that other than the three terminal vertices L,R and T, *every vertex of* S_k *is an internal vertex* of exactly two induced paths and hence by the above definition of acyclic graphoidal covering of a graph that *every vertex of a graph should be an internal vertex of at most one member of the decomposition* we have $\Psi = \{IP_{LR}, IP_{RT}, IP_{TL}\}$ as an acyclic path covering which is not graphoidal of S_k.

□

Theorem 8. *For all* $k > 1$,$\Psi = \{IP_{LR}, IP_{RT}, IP_{TL}\}$ *is only an acyclic induced path decomposition and not a simple acyclic decomposition of* S_k.

Proof. In S_k, for all $k > 1$, we have 3 distinct induced paths $IP_{LR}, IP_{RT}, IP_{TL}$ covering S_k completely. We already know that there are $\frac{3}{2}(3^{k-1} + 1)$ vertices in S_k. We have three sets of paths $\{IP_{LR}, IP_{RT}\}$, $\{IP_{RT}, IP_{TL}\}$,$\{IP_{LR}, IP_{TL}\}$

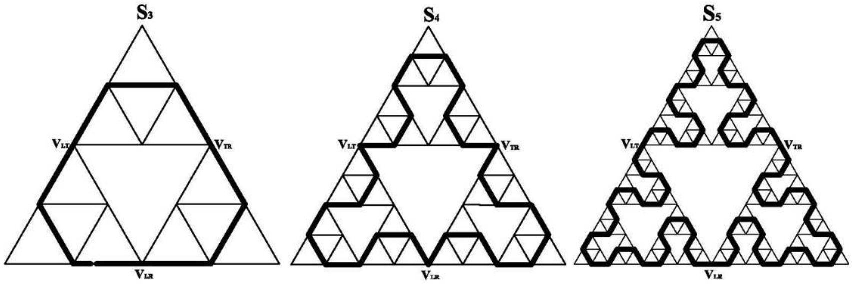

Figure 4: Longest Induced Cycle in S_3, S_4, S_5

taken two at a time. We can see that in S_2 , any two paths have two i.e.$\frac{1}{2}(3^{2-1}+1)$ vertices in common. In S_3, any two paths have five i.e.$\frac{1}{2}(3^{3-1}+1)$ vertice in common[Fig3]. In S_4, any two paths have fourteen i.e.$\frac{1}{2}(3^{4-1}+1)$ vertices as common.In general for all $k > 1$ we see that any two paths in S_k have $\frac{1}{2}(3^{k-1}+1)$ vertices in common. Hence $\Psi = \{IP_{LR}, IP_{RT}, IP_{TL}\}$ is not a simple acyclic decomposition of S_k and $\Psi = \{IP_{LR}, IP_{RT}, IP_{TL}\}$ is an induced path cover of S_k with $|\Psi| = \pi_{ia}(S_k) = 3$.

□

4. Induced Cycle (Hole)

Definition 9. An induced path $P = \{v_0, v_1, v_2, v_3, \ldots, v_{r-1}, v_r\}$ with terminal vertices as the same i.e. $v_0 = v_r$ is an induced cycle denoted as IC. The Induced cycles are also called chordless cycles or holes (when length greater than four).

An antihole is a hole in the complement of G. Arumugam discussed about the significance of the position of every vertex of G with respect to any cycle C in G. Here we find the longest induced cycle of the Sierpinsky Graph and discuss about the positions of every vertex of S_k with respect to the induced cycle IC_k.

Theorem 10. *For all $k > 2$, in S_k there exists unique longest induced cycle of length 3^{k-1}.*

Proof. For $k = 1$, no induced cycle can exist as the length of the induced path is '1' which cannot be closed.

For $k = 2$, none of these $\{P_{2,1}, P_{2,2}, P_{2,3}\}$ longest induced paths' terminal vertices can be made to coinside to form a cycle as it will result in three cosecutive members of the sequence to be in the same down triangle.

For $k = 3$, consider $\{P_{2,1}, P_{2,2}, P_{2,3}\}$ the three distinct induced paths in S_2. We know that S_3 is just three copies of S_2 denoted as components Δ_L,Δ_R,Δ_T with V_{LT}, V_{TR}, V_{RL} as the common vertices of the components Δ_L and Δ_T, Δ_T and Δ_R,Δ_R and Δ_L respectively. We start an induced path from the vertex V_{RL} and trace the induced path $P_{2,2}$ to reach V_{LT}, from V_{LT} we trace $P_{2,1}$ to reach V_{TR} and from V_{TR} we trace $P_{2,3}$ to reach V_{RL}. Thus we get a longest induced cycle started from V_{RL} and ended up with V_{RL} with no addition of any new vertex or new edge. We denote this induced cycle as IC_3 in S_3.This cycle being exactly the length of the three induced paths $\{P_{2,1}, P_{2,2}, P_{2,3}\}$ in S_2, its total length in S_3 is $3(3) = 3^{3-1}$[fig4].

Proceeding this way, by induction on k, if we have three longest induced paths $\{P_{k-1,1}, P_{k-1,2}, P_{k-1,3}\}$ each of length 3^{k-2} in S_{k-1}, then S_k being just three copies of the components of S_{k-1} with V_{LT}, V_{TR}, V_{RL} as the common vertices of the components Δ_L and Δ_T, Δ_T and Δ_R,Δ_R and Δ_L respectively we start a path from the vertex V_{RL} and trace the path $P_{k-1,2}$ to reach V_{LT} then from V_{LT} we trace $P_{k-1,1}$ to reach V_{TR} and from V_{TR} we trace $P_{k-1,3}$ to reach V_{RL}. Thus we get a longest induced cycle started from V_{RL} and ended up with V_{RL} with no addition of new vertex or new edge. We denote this longest induced cycle as IC_k in S_k. This cycle being exactly the length of the three induced paths $P_{k-1,1}, P_{k-1,2}, P_{k-1,3}$ in S_{k-1}, its total length is $3(3^{k-2}) = 3^{k-1}$. This longest cycle is unique as no other cycle of length 3^{k-1} can be considered at all in S_k. □

Theorem 11. *In S_k where $k > 2$, the number $N(v_{lc})$ of vertices left (inside) of the induced cycle IC_k is given by the formula $N(v_{lc}) = 3^{k-2} - 3^{k-3} + 3^{k-4} + \ldots \pm 3$ where v_{lc} denotes the vertex lying left of the induced cycle IC_k.*

Proof. In S_3 the induced cycle IC_3 is of length 3^2 and there are 3 vertices lying left of the induced cycle. i.e.$N(v_{lc}) = 3^{k-2} = 3^{3-2} = 3$.In S_4, the induced cycle IC_4 is of length 3^3 and there are 6 vertices lying left of the induced cycle *i.e*$N(v_{lc}) = 3^{k-2} - 3^{k-3} = 3^2 - 3 = 6$. In S_5,the induced cycle IC_5 is of length 3^4 and there are 21 vertices lying left of the induced cycle i.e.$N(v_{lc}) = 3^{k-2} - 3^{k-3} + 3^{k-4} = 3^3 - 3^2 + 3 = 21$. In S_6, the induced cycle IC_6 is of length 3^5 and there are 60 vertices lying left of the induced cycle . i.e $N(v_{lc}) = 3^{k-2} - 3^{k-3} + 3^{k-4} - 3^{k-5} = 3^4 - 3^3 + 3^2 - 3 = 60$.In S_7 the induced cycle

IC_7 is of length 3^6 and there are 183 vertices of S_7 lying left of the induced cycle. i.e.$N(v_{lc}) = 3^{k-2} - 3^{k-3} + 3^{k-4} - 3^{k-5} + 3^{k-6} = 3^5 - 3^4 + 3^3 - 3^2 + 3 = 183$.Hence in S_k, we have $N(v_{lc}) = 3^{k-2} - 3^{k-3} + 3^{k-4} - 3^{k-5} + \ldots \pm 3$ according as k is odd or even. □

Theorem 12. *In S_k where $k > 2$, the number $N(v_{rc})$ of vertices right (outside)of the induced cycle IC_k is given by the formula $N(v_{rc}) = \frac{1}{2}[3+3^{k-1} - 2(3^{k-2} + 3^{k-3} + 3^{k-4} + \ldots \pm 3)]$ where v_{rc} denotes the vertex lying right of the induced cycle IC_k.*

Proof. The total number of vertices in S_k is $\frac{3}{2}[3^{k-1} + 1]$.By theorem 11 we have the number of vertices lying on the induced cycle as 3^{k-1}. By the previous theorem, the number of vertices left of the cycle is given by the formula $N(v_{lc}) = 3^{k-2} - 3^{k-3} + 3^{k-4} + \ldots \pm 3$.Hence the number of vertices lying right of the hole is given by the formula $N(v_{rc}) = \frac{1}{2}[3+3^{k-1} - 2(3^{k-2} - 3^{k-3} + 3^{k-4} - 3^{k-5} + \ldots \pm 3)]$.

In S_3, $N(v_{rc}) = \frac{1}{2}[3 + 3^2 - 2(3^1] = 3$.
In S_4, $N(v_{rc}) = \frac{1}{2}[3 + 3^3 - 2(3^2 - 3^1)] = 9$
In S_5, $N(v_{rc}) = \frac{1}{2}[3 + 3^4 - 2(3^3 - 3^2 + 3)] = 21$
In S_6, $N(v_{rc}) = \frac{1}{2}[3 + 3^5 - 2(3^4 - 3^3 + 3^2 - 3)] = 63$
Hence the number of vertices right of the hole is given by the formula
$N(v_{rc}) = \frac{1}{2}[3 + 3^{k-1} - 2(3^{k-2} - 3^{k-3} + 3^{k-4} - 3^{k-5} + \ldots \pm 3)]$. □

5. Conclusion

A path decomposition of a graph is a collection of sub graphs whose union is G. There are various types of path decompositions with specific conditions imposed on the members of the decomposition. Here we determined the acyclic induced path decomposition number $\pi_{ia}(S_k)$ for all $k > 1$.We also discussed about the longest induced cycle IC_k and the positions of every vertex of S_k w.r.t to IC_k .Such investigations for tetrahedral and butterfly graph remain as challenges.

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