CONTRA $p_s$-CONTINUOUS FUNCTIONS

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Abstract: A new form of contra-continuity, called contra $p_s$-continuity, is introduced. It is shown that this class of functions is strictly between contra-complete continuity and contra precontinuity. Characterizations and properties of these functions are established. Relationships between these functions and other related classes of functions are also developed.

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1. Introduction

Contra-continuous functions were introduced by Dontchev (see [3]) in 1996. Since then many variations of contra-continuity have been investigated. In 2002 Jafari and Noiri (see [5]) introduced the concept contra-precontinuity and a weak form of contra-precontinuity, called almost precontinuity, was developed by Ekici (see [4]) in 2004. Recently the notions of a $p_s$-open set and a $p_s$-continuous function have been introduced by Khalaf and Assad (see [6]). In this note the concept of a $p_s$-open set is used to develop a new weak form of contra-precontinuity, which we call contra $p_s$-continuity. It is established that this class of functions is strictly between contra-complete continuity and contra-precontinuity. It is also shown that contra $p_s$-continuity implies a weak form of semi-continuity.
2. Preliminaries

The symbols $X$, $Y$, and $Z$ represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set $A$ are signified by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. A set $A$ is preopen (respectively, semi-open, regular open) if $A \subseteq \text{Int}(\text{Cl}(A))$, (respectively, $A \subseteq \text{Cl}(\text{Int}(A))$, $A = \text{Int}(\text{Cl}(A))$). A set $A$ is preclosed (respectively, semi-closed, regular closed) provided its complement is preopen (respectively, semi-open, regular open). A set $A$ is $p_s$-open (see [6]) if $A$ is preopen and a union of semi-closed sets. A set is $p_s$-closed if its complement is $p_s$-open. The semi-interior (respectively, $p_s$-interior (see [6])) of a set $A$, denoted by $\text{sInt}(A)$ (respectively, $p_s\text{Int}(A)$) is the union of all semi-open (respectively, $p_s$-open) sets contained in $A$ and the semi-closure (respectively, $p_s$-closure (see [7])) of $A$, denoted by $\text{sCl}(A)$, (respectively, $p_s\text{Cl}(A)$) is the intersection of all semi-closed (respectively, $p_s$-closed) sets containing $A$.

**Definition 1.** A function $f : X \to Y$ is said to be contra-completely continuous if $f^{-1}(F)$ is regular open for every closed subset $F$ of $Y$.

**Definition 2.** A function $f : X \to Y$ is said to be contra-precontinuous (see [5]) if $f^{-1}(V)$ is preclosed for every open subset $V$ of $Y$.

**Definition 3.** A function $f : X \to Y$ is said to be $p_s$-continuous (see [6]) (respectively almost $p_s$-continuous (see [7])) if, for every $x \in X$ and every open subset $V$ of $Y$ containing $f(x)$, there exists a $p_s$-open subset $U$ of $X$ containing $x$ such that $f(U) \subseteq V$ (respectively, $f(U) \subseteq \text{Int}(\text{Cl}(V))$).

Note that a function is $p_s$-continuous if and only if the inverse image if every open set is $p_s$-open (see [7]).

3. Contra $p_s$-Continuous Functions

We define a function $f : X \to Y$ to be contra $p_s$-continuous provided that, for every open subset $V$ of $Y$, $f^{-1}(V)$ is $p_s$-closed.

**Definition 4.** Let $A$ be a subset of a space $X$. The kernel of $A$ (see [9]), denoted by $\text{ker}(A)$, is the intersection of all open subsets of $X$ containing $A$.

**Lemma 1.** (see [5]) The following statements hold for subsets $A$ and $B$ of a space $X$:

(a) $x \in \text{ker}(A)$ if and only if $A \cap F \neq \emptyset$ for every closed subset $F$ of $X$ containing $x$. 

...
(b) $A \subseteq \ker(A)$ and $A = \ker(A)$ if $A$ is open in $X$.

(c) If $A \subseteq B$, then $\ker(A) \subseteq \ker(B)$.

**Theorem 5.** For a function $f : X \rightarrow Y$, the following statements are equivalent:

(a) $f$ is contra $p_s$-continuous.

(b) For every closed subset $F$ of $Y$, $f^{-1}(F)$ is $p_s$-open.

(c) For every $x \in X$ and every closed subset $F$ of $Y$ containing $f(x)$, there exists a $p_s$-open subset $U$ of $X$ containing $x$ such that $f(U) \subseteq F$.

(d) $f(\text{psCl}(A)) \subseteq \ker(f(A))$ for every subset $A$ of $X$.

(e) $\text{psCl}(f^{-1}(B)) \subseteq f^{-1}(\ker(B))$ for every subset $B$ of $Y$.

**Proof.** The implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are clear.

(c) $\Rightarrow$ (d) Let $A \subseteq X$ and let $y \in f(\text{psCl}(A))$. Suppose $y \notin \ker(f(A))$. Then there exists an open subset $V$ of $Y$ such that $f(A) \subseteq V$ and $y \notin V$. Let $x \in \text{psCl}(A)$ such that $y = f(x)$. Then $f(x) \in Y - V$, which is closed in $Y$. By (c) there exists a $p_s$-open subset $U$ of $X$ for which $x \in U$ and $f(U) \subseteq Y - V$. Since $f(A) \subseteq V$, $A \cap U = \emptyset$. Since $U$ is $p_s$-open, it follows that $x \notin \text{psCl}(A)$. This contradiction proves that $y \in \ker(f(A))$.

(d) $\Rightarrow$ (e) Let $B \subseteq Y$. It follows from (d) that

$$f(\text{psCl}(f^{-1}(B))) \subseteq \ker(f(f^{-1}(B))) \subseteq \ker(B)$$

and thus $\text{psCl}(f^{-1}(B)) \subseteq f^{-1}(\ker(B))$.

(e) $\Rightarrow$ (a) Let $V$ be an open subset of $Y$. Using (e) we obtain $\text{psCl}(f^{-1}(V)) \subseteq f^{-1}(\ker(V)) = f^{-1}(V)$ and, since $\text{psCl}(f^{-1}(A))$ is $p_s$-closed, it follows that $f^{-1}(V)$ is $p_s$-closed. \hfill $\Box$

**Definition 6.** A function $f : X \rightarrow Y$ is said to be usc-continuous if, for every closed subset $F$ of $Y$, $f^{-1}(F)$ is a union of semi-closed sets.

The implications below follow from the definition of a $p_s$-open set and the fact that regular open sets are both preopen and semi-closed.

contra-complete cont. $\Rightarrow$ contra $p_s$-cont. $\Rightarrow$ contra-precont.

$$\Downarrow$$

usc-continuous

The following examples show that none of the above implications are reversible.

**Example 7.** Let $X$ be the real numbers and let $\sigma$ be the usual topology on $X$. Let $A = (0, 1) \cup (1, 2)$ and let $\tau = \{X, \emptyset, X - A\}$. The identity mapping $f : (X, \sigma) \rightarrow (X, \tau)$ is contra $p_s$-continuous but not contra-completely continuous.
Example 8. Let $X$ be the real numbers with the usual topology. The identity mapping on $X$ is obviously semi-continuous and hence usc-continuous. However, it is not contra $p_s$-continuous.

Example 9. Assume $X = \{a, b, c\}$ has the topology $\sigma = \{X, \emptyset, \{a\}\}$ and let $\tau = \{X, \emptyset, \{b, c\}\}$. The identity mapping $f : (X, \sigma) \rightarrow (X, \tau)$ is contra-precontinuous but not contra $p_s$-continuous.

Since the identity on the real numbers with the usual topology is $p_s$-continuous but not contra $p_s$-continuous (Example 8), $p_s$-continuity does not imply contra $p_s$-continuity. Also, since the function in Example 7 is contra $p_s$-continuous, but not $p_s$-continuous, $p_s$-continuity and contra $p_s$-continuity are independent.

Definition 10. A space $X$ is said to be a C-space (see [2]) if every open subset of $X$ is a union of closed sets.

Theorem 11. If $f : X \rightarrow Y$ is contra $p_s$-continuous and $Y$ is a C-space, then $f$ is $p_s$-continuous.

Proof. Let $x \in X$ and let $V$ be an open subset of $Y$ containing $f(x)$. Since $Y$ is a C-space, there exists a closed subset $F$ of $Y$ such that $x \in F \subseteq V$. Then, since $f$ is contra $p_s$-continuous, Theorem 5 implies that there exists a $p_s$-open subset $U$ of $X$ containing $x$ such that $f(U) \subseteq F$. Hence $f(U) \subseteq V$, which proves that $f$ is $p_s$-continuous.

Corollary 12. If $f : X \rightarrow Y$ is contra $p_s$-continuous and $Y$ is either regular or $T_1$, then $f$ is $p_s$-continuous.

Definition 13. A function $f : X \rightarrow Y$ is said to be $p_s$-preopen (respectively, M-preopen (see [8]), if for every $p_s$-open (respectively, preopen) subset $U$ of $X$, $f(U)$ is preopen.

Theorem 14. If $f : X \rightarrow Y$ is $p_s$-preopen and contra $p_s$-continuous, then $f$ is almost $p_s$-continuous.

Proof. Let $x \in X$ and let $V$ be an open subset of $Y$ containing $f(x)$. Then by Theorem 5 there exists a $p_s$-open subset $U$ of $X$ for which $x \in U$ and $f(U) \subseteq \text{Cl}(V)$. Since $f$ is $p_s$-preopen, $f(U)$ is preopen and therefore $f(U) \subseteq \text{Int}(\text{Cl}(V))$, which proves that $f$ is almost $p_s$-continuous.

Corollary 15. If $f : X \rightarrow Y$ is M-preopen and contra $p_s$-continuous, then $f$ is almost $p_s$-continuous.

Corollary 16. If $f : X \rightarrow Y$ has the property that images of semi-closed sets are preopen and $f$ is contra $p_s$-continuous, then $f$ is almost $p_s$-continuous.
Definition 17. A function \( f : X \to Y \) is said to be almost weakly \( p_s \)-continuous if for every open subset \( V \) of \( Y \), \( f^{-1}(V) \subseteq p_s \text{Int}(\text{Cl}(f^{-1}(\text{Cl}(V)))) \).

Theorem 18. If \( f : X \to Y \) is contra \( p_s \)-continuous, then \( f \) is almost weakly \( p_s \)-continuous.

Proof. Assume \( V \) is an open subset of \( Y \). Then, since \( f^{-1}(\text{Cl}(V)) \) is \( p_s \)-open in \( X \), \( f^{-1}(V) \subseteq f^{-1}(\text{Cl}(V)) \subseteq p_s \text{Int}(\text{Cl}(f^{-1}(\text{Cl}(V)))) \), which proves that \( f \) is almost weakly \( p_s \)-continuous. \( \square \)

Definition 19. The \( p_s \)-frontier of a subset \( A \) of a space \( X \) is the set given by \( p_s \text{Fr}(A) = p_s \text{Cl}(A) \cap p_s \text{Cl}(X - A) \).

Theorem 20. Let \( f : X \to Y \) be a function and let \( x \in X \). Then \( f \) is not contra \( p_s \)-continuous at \( x \) if and only if \( x \) is a member of the \( p_s \)-frontier of the inverse image of a closed subset of \( Y \) containing \( f(x) \).

Proof. (\( \Rightarrow \)) Assume \( f \) is not contra \( p_s \)-continuous at \( x \). Then by Theorem 5 there exists a closed subset \( F \) of \( Y \) such that \( f(x) \in F \) and, for every \( p_s \)-open subset \( U \) of \( X \) containing \( x \), \( f(U) \not\subseteq F \). Hence \( U \cap f^{-1}(Y - F) \not= \emptyset \) for every \( p_s \)-open subset \( U \) of \( X \) containing \( x \), which implies that \( x \in p_s \text{Cl}(f^{-1}(Y - F)) \) and hence \( x \in p_s \text{Cl}(f^{-1}(F)) \cap p_s \text{Cl}(f^{-1}(Y - F)) = p_s \text{Fr}(f^{-1}(F)) \).

(\( \Leftarrow \)) Let \( x \in X \) and assume that \( x \in p_s \text{Fr}(f^{-1}(F)) \) for some closed subset \( F \) of \( Y \) containing \( f(x) \). Suppose \( f \) is contra \( p_s \)-continuous at \( x \). Then there exists a \( p_s \)-open subset \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq F \). Then we see that \( x \in U \subseteq f^{-1}(F) \) and hence that \( x \not\in p_s \text{Cl}(Y - f^{-1}(F)) \) and thus \( x \not\in p_s \text{Fr}(f^{-1}(F)) \). Therefore \( f \) is not contra \( p_s \)-continuous at \( x \). \( \square \)

4. Properties

Recall that the graph of a function \( f : X \to Y \) is the subset \( G(f) = \{(x, y) : y = f(x)\} \) of the product space \( X \times Y \).

Definition 21. A function \( f : X \to Y \) is said to have a \( p_s \)-closed graph if for every \((x, y) \in X \times Y - G(f)\) there exists a \( p_s \)-open set \( U \) of \( X \) such that \( x \in U \subseteq X \) and an open set \( V \) such that \( y \in V \subseteq Y \) for which \((x, y) \in U \times V \subseteq X \times Y - G(f)\).

Theorem 22. If \( f : X \to Y \) is contra \( p_s \)-continuous and \( Y \) is \( T_2 \), then \( G(f) \) is \( p_s \)-closed.
Proof. Assume \((x, y) \in X \times Y - G(f)\). Then, since \(y \neq f(x)\), there exist disjoint open subsets \(V\) and \(W\) of \(X\) and \(Y\), respectively, such that \(f(x) \in V\) and \(y \in W\). Since \(f\) is contra \(p_s\)-continuous, there exists a \(p_s\)-open subset \(U\) of \(X\) such that \(x \in U\) and \(f(U) \subseteq \text{Cl}(V)\). Because \(\text{Cl}(V) \cap W = \emptyset\), \(f(U) \cap W = \emptyset\) and thus \((x, y) \in U \times W \subseteq X \times Y - G(f)\). Therefore \(G(f)\) is \(p_s\)-closed.

**Definition 23.** A function \(f : X \to Y\) is said to have a contra \(p_s\)-closed graph if, for every \((x, y) \in X \times Y - G(f)\), there exists a \(p_s\)-open set \(U\) of \(X\) such that \(x \in U \subseteq X\) and a closed set \(F\) such that \(y \in F \subseteq Y\) for which \((x, y) \in U \times F \subseteq X \times Y - G(f)\).

**Theorem 24.** If \(f : X \to Y\) is contra \(p_s\)-continuous and \(Y\) is Urysohn, then \(G(f)\) is contra \(p_s\)-closed.

Proof. Assume \((x, y) \in X \times Y - G(f)\). Then, since \(y \neq f(x)\), there exist open subsets \(V\) and \(W\) of \(X\) and \(Y\), respectively, such that \(f(x) \in V\) and \(y \in W\) and \(\text{Cl}(V) \cap \text{Cl}(W) = \emptyset\). Since \(f\) is contra \(p_s\)-continuous, there exists a \(p_s\)-open subset \(U\) of \(X\) containing \(x\) such that \(f(U) \subseteq \text{Cl}(V)\). Then we have \((x, y) \in U \times \text{Cl}(W) \subseteq X \times Y - G(f)\), which proves that \(G(f)\) is contra \(p_s\)-closed.

**Theorem 25.** Assume \(Y\) is an Urysohn space and that \(f_i : X_i \to Y\) for \(i = 1, 2\) is contra \(p_s\)-continuous for each \(i\). The set \(A = \{(x_1, x_2) : f_1(x_1) = f_2(x_2)\}\) is \(p_s\)-closed in the product space \(X_1 \times X_2\).

Proof. Suppose \((x_1, x_2) \in (X_1 \times X_2) - A\). Then \(f_1(x_1) \neq f_2(x_2)\) and, since \(Y\) is Urysohn, there exist open sets \(V_1\) and \(V_2\) containing \(f_1(x_1)\) and \(f_2(x_2)\), respectively, such that \(\text{Cl}(V_1) \cap \text{Cl}(V_2) = \emptyset\). Since products of \(p_s\)-open sets are \(p_s\)-open, \(f_1^{-1}(\text{Cl}(V_1)) \times f_2^{-1}(\text{Cl}(V_2))\) is \(p_s\)-open in \(X_1 \times X_2\). Then we see that \((x_1, x_2) \in f_1^{-1}(\text{Cl}(V_1)) \times f_2^{-1}(\text{Cl}(V_2)) \subseteq (X_1 \times X_2) - A\). Since unions of \(p_s\)-open sets are \(p_s\)-open (see [6]), it follows that \(A\) is closed.

Recall that a set \(A\) is called regular semi-open if \(A = \text{sInt}(\text{sCl}(A))\).

**Lemma 2.** (see [6]) If \(X\) is a space, \(A\) is a regular semi-open subset of \(X\) and \(B\) is a \(p_s\)-open subset of \(X\), then \(A \cap B\) is \(p_s\)-open in \(A\).

**Theorem 26.** If \(f : X \to Y\) is contra \(p_s\)-continuous and \(A\) is a regular semi-open subset of \(X\), then \(f|_A : A \to Y\) is contra \(p_s\)-continuous.

Proof. Let \(F\) be a closed subset of \(Y\). Since \(f^{-1}(F)\) is \(p_s\)-open in \(X\), using Lemma 2 we have \(f^{-1}|_A(F) = f^{-1}(F) \cap A\) is \(p_s\)-open in \(A\). Thus \(f|_A : A \to Y\) is contra \(p_s\)-continuous.
The proof of the following theorem is straightforward.

**Theorem 27.** If \( f : X \rightarrow Y \) is contra \( p_s \)-continuous and \( g : Y \rightarrow Z \) is continuous, then \( g \circ f : X \rightarrow Z \) is contra \( p_s \)-continuous.

**Theorem 28.** Let \( f_\alpha : X \rightarrow Y_\alpha \) be a function for every \( \alpha \in A \) and let \( f : X \rightarrow \prod_{\alpha \in A} Y_\alpha \) be the product function given by \( f(x) = (f_\alpha(x))_{\alpha \in A} \) for every \( x \in X \). If \( f \) is contra \( p_s \)-continuous, then \( f_\alpha \) is contra \( p_s \)-continuous for every \( \alpha \in A \).

Proof. Let \( \beta \in A \) and let \( p_\beta : \prod_{\alpha \in A} Y_\alpha \rightarrow Y_\beta \) be the \( \beta \)th projection function. Since \( f_\beta = p_\beta \circ f \), it follows from Theorem 27 that \( f_\beta \) is contra \( p_s \)-continuous.

**Theorem 29.** If \( f : X \rightarrow Y \) is a function, \( g : Y \rightarrow Z \) is injective and closed, and \( g \circ f : X \rightarrow Z \) is contra \( p_s \)-continuous, then \( f \) is contra \( p_s \)-continuous.

Proof. Let \( F \) be a closed subset of \( Y \). Since \( g \) is closed, \( g(F) \) is closed in \( Z \). Then, since \( g \circ f \) is contra \( p_s \)-continuous and \( g \) is injective, we see that \( f^{-1}(F) = f^{-1}(g^{-1}(g(F))) \) is \( p_s \)-open in \( X \), which proves that \( f \) is contra \( p_s \)-continuous.

**Definition 30.** A function \( f : X \rightarrow Y \) is said to be \( M-p_s \)-open if for every \( p_s \)-open subset \( U \) of \( X \), \( f(U) \) is \( p_s \)-open in \( Y \).

The proof of the following result is analogous to the above proof and is omitted.

**Theorem 31.** If \( f : X \rightarrow Y \) is surjective and \( M-p_s \)-open, \( g : Y \rightarrow Z \) is a function, and \( g \circ f : X \rightarrow Z \) is contra \( p_s \)-continuous, then \( g \) is contra \( p_s \)-continuous.

**References**


