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SYMMETRIC BI-DERIVATIONS ON $TM-$ALGEBRAS

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Abstract: Recently an algebra based on propositional calculi was introdued by Tamilarasi and Mekalai in the year 2010 known as $TM-$algebras, see [6]. In our paper [1] we introduced the notion of derivation on TM-algebras. In this paper, we introduce the notion of symmetric bi-derivation on TM-algebras and study some of its properties.

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1. Introduction

It is well known that BCK and BCI-algebras are two classes of algebras of logic. They were introduced by Imai and Iseki [3] and have been extensively investigated by many researchers. Recently another algebra based on propositional calculi was introdued by Tamilarasi and Mekalai [6] in the year 2010 known as $TM-$algebras.

Motivated by the notion of derivations on rings and near-rings Jun and Xin [4] studied the notion of derivation on BCI-algebras. In our paper [1],

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we introduced the notion of derivation on $TM-$algebras. In [5], the authors have discussed the notion of symmetric bi-derivation on BCI-algebras. This motivated us to introduce the notion of symmetric bi-derivation on TM-algebras in this paper. We study the properties of symmetric bi-derivations on TM-algebras and prove that the set of all symmetric bi-derivations on a TM-algebra forms a semigroup under a suitably defined binary composition.

2. Preliminaries

In this section, we recall some basic definitions and results that are needed for our work.

Definition 2.1. A $TM-$algebra $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

1. $x * 0 = x$
2. $(x * y) * (x * z) = z * y \ \forall \ x, y, z \in X$.

Definition 2.2. A $TM-$algebra X is said to be associative if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$.

Definition 2.3. For any $TM-$algebra $(X, *, 0)$. We define the set $G(X) = \{x \in X \mid 0 * x = x\}$.

Remark 2.4. In a $TM-$algebra X, by definition, $x \wedge y = y * (y * x)$. However, in a TM-algebra, $x = y * (y * x)$. Hence, in a TM-algebra, we have $x \wedge y = x \ \ \forall \ x, y \in X$.

Definition 2.5. [2] Let X be a $TM-$algebra. If we define an operation $+$, called addition, as $x + y = x * (0 * y)$ for all $x, y \in X$, then $(X, +)$ is an abelian group with identity 0 and the additive inverse $-x = 0 * x \ \forall \ x \in X$.

Remark 2.6. If we have a $TM-$algebra $(X, *, 0)$ it follows from the above definition that $(X, +)$ is an abelian group with $-y = 0 * y \ \forall \ y \in X$. Then we have $x - y = x * y \ \forall \ x, y \in X$. On the other hand if we choose an abelian group $(X, +)$ with an identity 0 and define $x * y = x - y$, we get a $TM-$algebra $(X, *, 0)$ where $x + y = x * (0 * y) \ \ \forall \ x, y \in X$.

Definition 2.7. Let $(X, *, 0)$ be a $TM-$algebra. A self map $d : X \to X$ is said to be a $(l, r)-$derivation on X, if $d(x * y) = (d(x) * y) \wedge (x * d(y))$. d is said to be a $(r, l)-$derivation on X, if $d(x * y) = (x * d(y)) \wedge (d(x) * y)$. It is said to be a derivation on X if d is both a $(l, r)-$derivation and a $(r, l)-$derivation on X.

3. Symmetric BI-Derivations

We start this section with the definition of Cartesian product of $TM-$algebras.

Definition 3.1. Let X, Y be $TM-$algebras. An operation $*$ on the Cartesian product $X \times Y$ of X, Y is defined as follws.

1. $(x_1, y_1) * (x_2, y_2) = (x_1 * x_2, y_1 * y_2)$.

2. $(0, 0) = 0$,

Lemma 3.2. Cartesian product of two $TM-$algebras is again a $TM-$algebra.

Proof. Let X and Y be two $TM-$algebras. Consider the cartesian product $X \times Y$.

$$(x, y) * (0, 0) = (x * 0, y * 0) = (x, y),$$

$$\begin{aligned}((x_1, y_1) * (x_2, y_2)) * ((x_1, y_1) * (x_3, y_3)) &= (x_1 * x_2, y_1 * y_2) * ((x_1 * x_3, y_1 * y_3)) \\ &= ((x_1 * x_2) * (x_1 * x_3), (y_1 * y_2) * (y_1 * y_3)) \\ &= ((x_3 * x_2), (y_3 * y_2)) \\ &= (x_3, y_3) * (x_2, y_2)\end{aligned}$$

Therefore $(X \times Y, *, 0)$ is a $TM-$algebra.

Definition 3.3. Let X be a $TM-$algebra. A mapping $D : X \times X \to X$ is a symmetric map if $D(x, y) = D(y, x)$ holds for all pairs of elements $x, y \in X$.

Example 3.4. Let $(X, *, 0)$ be a $TM-$algebra with the Cayley table.

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

The map $D : X \times X \to X$ defined by $D(x, y) = x * (0 * y)$ is a symmetric map.

Definition 3.5. Let X be a $TM-$algebra and $D : X \times X \to X$ be a symmetric mapping. A mapping $d : X \to X$ defined by $d(x) = D(x, x)$ is called trace of D.

Example 3.6. Let $(X, *, 0)$ be a $TM-$algebra with the Cayley table.

$*$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

The map $D : X \times X \to X$ defined by $D(x, y) = x*(0*y) = x+y$ is a symmetric map.

Since $x = 0,\ D(0, 0) = 0 + 0 = 0$. $x = 1,\ D(1, 1) = 1 + 1 = 3$. $x = 2$, $D(2, 2) = 2 + 2 = 3$. $x = 3,\ D(3, 3) = 3 + 3 = 0$.

Thus the mapping $d : X \to X$ given by $d(x) = D(x, x) = \begin{cases} 0 & if\ x = 0, 3 \\ 3 & if\ x = 1, 2 \end{cases}$ is the trace of the symmetric mapping D.

Definition 3.7. Let X be a $TM-$algebra and $D : X \times X \to X$ be a symmetric mapping. If D satisfies the identity, $D(x * y, z) = (D(x, z) * y) \wedge (x * D(y, z))$ for all $x, y, z \in X$, then D is called left-right symmetric bi-derivation. ((l, r) symmetric bi-derivation)

If D satisfied the identity, $D(x * y, z) = (x * D(y, z)) \wedge (D(x, z) * y)$ for all $x, y, z \in X$, then D is called right-left symmetric bi-derivation. ($(r, l)-$symmetric bi-derivation)

If D is both an (l, r)symmetric bi-derivation and an (r, l) symmetric bi-derivation then D is called a symmetric bi-derivation.

Example 3.8. Consider in example 3.6. Define a mapping $D : X \times X \to X$ by $D(x, y) = x * (0 * y)$ for all $x, y \in X$. Then D is a $(l, r)-$symmetric Bi-derivation.

Example 3.9. Consider in example 3.4. Define $D(x, y) = x * (0 * y)$ for all $x, y \in X$ is a symmetric map. Then D is a symmetric Bi-derivation.

Example 3.10. Consider the $TM-$algebra with the Cayley-Table as in exapmle 3.4. Define the symmetric map $D : X \times X \to X$ such that

$D(x, x) = 3$, if $x = 0, 1, 2, 3$.
$D(0, 3) = D(3, 0) = D(1, 2) = D(2, 1) = 0$.
$D(0, 2) = D(2, 0) = D(1, 3) = D(3, 1) = 1$.
$D(0, 1) = D(1, 0) = D(2, 3) = D(3, 2) = 2$.
Then D is a symmetric Bi-derivation.

Proposition 3.11. Let X be a $TM-$algebra. Define a symmetric map $D : X \times X \to X$ by $D(x, y) = x+y$ for all $x, y \in X$. Then D is a $(l, r)-$symmetric

Bi-derivation.

Proof.

$$\begin{aligned} D(x*y,z) &= (x*y)+z \quad for\ all\ x,y,z \in X \\ &= (x*y)*(0*z) \\ &= (x*(0*z))*y \quad (\because\ (x*y)*z=(x*z)*y) \\ &= (x+z)*y \\ &= (x*(y+z))*((x*(y+z))*((x+z)*y)) \ (\because\ y*(y*x)=x) \\ &= ((x+z)*y)\wedge(x*(y+z)) \\ &= (D(x,z)*y)\wedge(x*D(y,z)) \end{aligned}$$

This proves that D is a $(l,r)-$symmetric Bi-derivation.

Theorem 3.12. Let X be an associative $TM-$algebra. Then the symmetric map $D : X \times X \to X$ defined by $D(x,y) = x+y$ for all $x,y \in X$ is a symmetric bi-derivation.

Proof. By the above proposition, D is a $(l,r)-$symmetric bi-derivation.

$$\begin{aligned} D(x*y,z) &= (x*y)+z \\ &= (x*y)*(0*z) \\ &= (x*(0*z))*y \\ &= ((x*0)*z)*y \quad (\because\ X\ is\ associative\ ) \\ &= (x*z)*y=(x*y)*z \quad \cdots\cdots(1) \end{aligned}$$

$$\begin{aligned} (x*D(y,z))\wedge(D(x,z)*y) &= x*D(y,z) \quad (\because\ x\wedge y = y*(y*x)=x) \\ &= x*(y+z) \\ &= x*(y*(0*z)) \\ &= x*((y*0)*z) \quad (\because\ X\ is\ associative\ ) \\ &= x*(y*z) \\ &= (x*y)*z \quad \cdots\cdots(2) \quad (\because\ X\ \ is\ associative\ ) \end{aligned}$$

From (1) and (2), $D(x*y,z) = (x*D(y,z))\wedge(D(x,z)*y)$ for all $x,y,z \in X$.

This proves that D is $(r,l)-$symmetric bi-derivation and hence a symmetric bi-derivation.

Proposition 3.13. Let X be a $TM-$algebra and $D : X \times X \to X$ be a symmetric map. Then

1. If D is a $(l,r)-$symmetric bi-derivation then $D(x,y) = D(x,y) \wedge x$ for all $x, y \in X$.

2. If D is a $(r,l)-$symmetric bi-derivation then $D(x,y) = x \wedge D(x,y)$ for all $x, y \in X$ if and only if $D(0,y) = 0$ for all $y \in X$.

Proof.

1. Let D be a $(l,r)-$symmetric bi-derivation.

$$\begin{aligned}
D(x,y) &= D(x*0,y) \quad for\ all\ x,y\ inX \\
&= (D(x,y)*0) \wedge (x * D(0,y)) \\
&= D(x,y) \wedge (x * D(0,y)) \\
&= (x * D(0,y)) * ((x * D(0,y)) * D(x,y)) \\
&= (x * D(0,y)) * ((x * D(x,y)) * D(0,y)) \\
&\quad (\because\ (x*y)*z = (x*z)*y) \\
&= x * (x * D(x,y)) \qquad (\because\ (x*z)*(y*z) = x*y) \\
&= D(x,y) \wedge x
\end{aligned}$$

2. Let D be a $(r,l)-$symmetric bi-derivation and $D(0,y) = 0$ for all $y \in X$.

$$\begin{aligned}
D(x,y) &= D(x*0,y) \\
&= (x * D(0,y)) \wedge (D(x,y)*0) \\
&= (x*0) \wedge D(x,y) \\
&= x \wedge D(x,y).
\end{aligned}$$

Conversely, if $D(x,y) = x \wedge D(x,y)$ for all $x, y \in X$. Then

$$D(0,y) = 0 \wedge D(0,y) = D(0,y) * (D(0,y)*0) = D(0,y) * D(0,y) = 0.$$

Proposition 3.14. Let X be a $TM-$algebra and $D : X \times X \to X$ be a $(l,r)-$symmetric bi-derivation. Then

1. $D(a,y) = D(0,y) * (0 * a) = D(0,y) + a$ for all $a, y \in X$.

2. $D(a+b,y) = D(a,y) + D(b,y) - D(0,y)$ for all $a, b, y \in X$.

3. $D(a,y) = a$ for all $a, y \in X$ if and only if $D(0,y) = 0$.

Proof.

1. Let $a = 0 * (0 * a)$.

$$\begin{aligned} D(a,y) &= D(0*(0*a),y) \\ &= (D(0,y)*(0*a)) \wedge (0*D(0*a,y)) \\ &= D(0,y)*(0*a) \qquad (\because\ x \wedge y = x) \\ &= D(0,y)+a \end{aligned}$$

2. By(1)

$$\begin{aligned} D(a+b,y) &= D(0,y)+a+b \\ &= D(0,y)+a+D(0,y)+b-D(0,y) \\ &= D(a,y)+D(b,y)-D(0,y) \end{aligned}$$

3. $D(a,y) = a$ for all $a, y \in X$.

   Put $a = 0,\ D(0,y) = 0\ \ \forall y \in X$.

   Conversely if $D(0,y) = 0$, then $D(a,y) = D(0,y) + a = 0 + a = a$.

Proposition 3.15. Let X be a $TM-$algebra and $D : X \times X \to X$ be a $(r,l)-$symmetric bi-derivation. Then

1. $D(a,y) \in G(X)$ for all $a \in G(X)$.
2. $D(a,y) = a * D(0,y) = a + D(0,y)$ for all $a, y \in X$.
3. $D(a+b,y) = D(a,y) + D(b,y) - D(0,y)$ for all $a, b, y \in X$.
4. $D(a,y) = a$ for all $a, y \in X$ if and only if $D(0,y) = 0$.

Proof.

1. $0 * a = a \quad (\because\ a \in G(X))$.

$$\begin{aligned} D(a,y) &= D(0*a,y) \qquad for\ all\ a,y \in X \\ &= (0*D(a,y)) \wedge (D(0,y)*a) \\ &= (D(0,y)*a)*((D(0,y)*a)*(0*D(a,y))) \\ &= 0*D(a,y) \qquad (\because\ y*(y*x)=x) \end{aligned}$$

   This shows that $D(a,y) \in G(X)$.

2.

$$\begin{aligned} D(a,y) &= D(a*0,y) \quad for\ all\ a,y \in X \\ &= (a*D(0,y)) \wedge (D(a,y)*0) \\ &= (a*D(0,y)) \wedge D(a,y) \\ &= D(a,y)*(D(a,y)*(a*D(0,y))) \\ &= a*D(0,y) \end{aligned}$$

$$\begin{aligned} Again\ D(a,y) &= a*D(0,y) \\ &= a*D(0*0,y) \\ &= a*((0*D(0,y)) \wedge (D(0,y)*0)) \\ &= a*(0*D(0,y)) \\ &= a+D(0,y) \end{aligned}$$

3. $D(a+b,y) = a+b+D(0,y) = a+D(0,y)+b+D(0,y)-D(0,y)$
$= D(a,y)+D(b,y)-D(0,y)$.

4. If $D(0,y)=0$, then $D(a,y)=D(a*0,y)=a*D(0,y)=a*0=a$. (By (2)) Conversely if $D(a,y)=a \ \forall\ a \in X,\ D(0,y)=0$.

4. Semigroup of Symmetric Bi-Derivations

Definition 4.1. Let $\mathscr{D}_L$ denote the set of all $(l,r)-$symmetric bi-derivation on X. Define the binary operation $\wedge$ on $\mathscr{D}_L$ as follows: For $D_1, D_2 \in \mathscr{D}_L$ define $(D_1 \wedge D_2)(x,y) = D_1(x,y) \wedge D_2(x,y)$ for all $x,y \in X$.

Proposition 4.2. Let D_1 and D_2 are $(l,r)-$symmetric bi-derivation on X, then $(D_1 \wedge D_2)$ is also a $(l,r)-$symmetric bi-derivation.

Proof. We will prove the following implication

$$(D_1 \wedge D_2)(x*y,z) = ((D_1 \wedge D_2)(x.z)*y) \wedge (x*((D_1 \wedge D_2)(y,z))).$$

$$\begin{aligned} (D_1 \wedge D_2)(x*y,z) &= D_1(x*y,z) \wedge D_2(x*y,z) \\ &= D_2(x*y,z)*(D_2(x*y,z)*D_1(x*y,z)) \\ &= D_1(x*y,z) \end{aligned}$$

$$
\begin{aligned}
&= (D_1(x,z)*y)\wedge(x*D_1(y,z))\\
&= (x*D_1(y,z))*((x*D_1(y,z))*(D_1(x,z)*y))\\
&= D_1(x,z)*y \quad \cdots\cdots(1)
\end{aligned}
$$

$$
\begin{aligned}
((D_1\wedge D_2)(x,z)*y)\wedge(x*(D_1\wedge D_2)(y,z)) &= (x*(D_1\wedge D_2)(y,z))*\\
&\quad((x*(D_1\wedge D_2)(y,z))*((D_1\wedge D_2)(x,z)*y))\\
&=(D_1\wedge D_2)(x,z)*y\\
&=(D_1(x,z)\wedge D_2(x,z))*y\\
&=(D_2(x,z)*(D_2(x,z)*D_1(x,z)))*y\\
&=D_1(x,z)*y \quad \cdots\cdots(2)
\end{aligned}
$$

Combining (1) and (2), we get $(D_1\wedge D_2)$ is a $(l,r)-$symmetric bi-derivation.

Proposition 4.3. The binary composition $\wedge$ defined on $\mathscr{D}_L$ is associative.

Proof. Let X be a $TM-$algebra. Let D_1, D_2, D_3 are $(l,r)-$symmetric bi-derivation.

$$
\begin{aligned}
((D_1\wedge D_2)\wedge D_3)(x*y,z) &= ((D_1\wedge D_2)(x*y,z))\wedge D_3(x*y,z))\\
&=(D_1(x,z)*y)\wedge(D_3(x*y,z)) \quad \text{(By proposition4.2(1))}\\
&=D_3(x*y,z)*(D_3(x*y,z)*D_1(x,z)*y)\\
&=D_1(x,z)*y, \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
(D_1\wedge(D_2\wedge D_3))(x*y,z) &= (D_1(x*y,z))\wedge((D_2\wedge D_3)(x*y,z))\\
&=(D_1(x*y,z))\wedge(D_2(x,z)*y) \quad \text{(By proposition4.2(1))}\\
&=(D_2(x,z)*y)*((D_2(x,z)*y)*(D_1(x*y,z)))\\
&=D_1(x*y,z)\\
&=(D_1(x,z)*y)\wedge(x*D_1(y,z))\\
&=(x*D_1(y,z))*((x*D_1(y,z))*D_1(x,z)*y)\\
&=D_1(x,z)*y. \qquad (2)
\end{aligned}
$$

Combining (1) and (2) we get, $(D_1\wedge D_2)\wedge D_3 = D_1(D_2\wedge D_3)$.

This proves that, $\wedge$ is associative.

Combining the above two propositions, we get the following theorem.

Theorem 4.4. $\mathscr{D}_L$ is a semigroup under the binary composition $\wedge$ defined by $(D_1\wedge D_2)(x,y) = D_1(x,y)\wedge D_2(x,y)$ for all $x,y\in X$ and $D_1, D_2\in\mathscr{D}_L$.

Analogously we can prove that,

Theorem 4.5. $\mathscr{D}_R$ is a semigroup under the binary operation $\wedge$ defined by $(D_1 \wedge D_2)(x, y) = D_1(x, y) \wedge D_2(x, y)$ for all $x, y \in X$ and $D_1, D_2 \in \mathscr{D}_R$ where $\mathscr{D}_R$ is the set of all $(r, l)-$symmetric bi-derivation.

Combining the above two theorem we get the following theorem.

Theorem 4.6. If $\mathscr{D}$ denotes the set of all symmetric bi-derivation on X, it is a semi-group under the binary operation $\wedge$ defined by $(D_1 \wedge D_2)(x, y) = D_1(x, y) \wedge D_2(x, y)$ for all $x, y \in X$ and $D_1, D_2 \in \mathscr{D}$.

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