A FAMILY OF $p$-VALENT ANALYTIC FUNCTIONS DEFINED BY A FRACTIONAL CALCULUS OPERATOR

Mamta Pathak\(^1\), Poonam Sharma\(^2\)

Babu Banarasi Das Group of Educational Institutions
Faizabad Road, Lucknow, INDIA

\(^2\)Deparment of Mathematics and Astronomy
University of Lucknow
Lucknow, INDIA

Abstract: In this Paper a family $S(\alpha, \beta, \mu, p)$ of $p$-valent analytic functions involving fractional calculus operator $\Omega_{z}^{\mu,p}$ is studied and a sufficient coefficient condition for functions belonging to the family $S(\alpha, \beta, \mu, p)$ is proved and it is shown that this coefficient condition is necessary for its subfamily $TS(\alpha, \beta, \mu, p)$. Coefficient estimate, growth theorem and results on partial sums are obtained for the family $S(\alpha, \beta, \mu, p)$. Also an integral inequality is proved for functions belonging to the family $TS(\alpha, \beta, \mu, p)$.

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1. Prelimnaries

Let $A(p)$ denotes a family of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}z^{p+k} \quad (p \in N = \{1, 2, 3, \ldots\})$$

which are analytic and $p$-valent in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$.

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1\} \text{ and } T(p) \text{ denotes a subfamily of } A(p) \text{ whose members are of the form:}

\begin{equation}
 f(z) = z^p - \sum_{k=1}^{\infty} |a_{p+k}| z^{p+k} \quad (p \in N = \{1, 2, 3, \ldots\}).
\end{equation}

Denote } A(1) \equiv S.

The convolution or Hadamard product of } f(z) \text{ given by (1) and } g(z) \in A(p) \text{ given by}

\begin{equation}
 g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}
\end{equation}

is defined as:

\begin{equation}
 (f \ast g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}
\end{equation}

which is analytic and } p \text{-valent in the unit disk } U.

Let } S^*(p, \alpha) \text{ and } K(p, \alpha) \text{ denote respectively the family of starlike and convex functions } f(z) \in A(p) \text{ satisfying}

\begin{equation}
 \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \text{ and } \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad 0 \leq \alpha < p
\end{equation}

respectively. Again, let } \beta-UST(\alpha, p) \text{ and } \beta-UCV(\alpha, p) \text{ denote respectively the family of } \beta \text{-uniformly starlike and } \beta \text{-uniformly convex functions } f(z) \in A(p) \text{ satisfying}

\begin{equation}
 \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta \left| \frac{zf'(z)}{f(z)} - p \right| + \alpha
\end{equation}

and

\begin{equation}
 \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| + \alpha
\end{equation}

respectively for some } \beta \geq 0, \quad 0 \leq \alpha < p \text{ and } z \in U. \text{ Clearly } 0-UST(\alpha, p) \equiv S^*(p, \alpha), \ 0-UCV(\alpha, p) \equiv K(p, \alpha). \text{ Denote } S^*(1, \alpha) \equiv S^*(\alpha), \ K(1, \alpha) \equiv K(\alpha) \text{ and } \beta-UST(\alpha, 1) \equiv \beta-UST(\alpha), \ \beta-UCV(\alpha, 1) \equiv \beta-UCV(\alpha).

Saitoh [8] introduced a Carlson-Shaffer type operator } L_p(a, c) \text{ for } f(z) \in A(p) \text{ which is defined as:

\begin{equation}
 L_p(a, c)f(z) := \phi_p(a, c, z) \ast f(z)
\end{equation}

where } \phi_p(a, c, z) \text{ is defined as:
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\[ \phi_p(a, c, z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k} \]  

(9)

for $a \in R_+$, $c \in R/Z_0 = \{0, -1, -2, -3, \ldots\}$ and $(a)_k$ is the Pochhammer symbol defined as:

\[ (a)_k = \frac{\Gamma(a + k)}{\Gamma(a)} = a(a + 1)(a + 2)\ldots(a + k - 1), \quad k \in N = 1, 2, \ldots \]

The Riemann-Liouville fractional derivative operator [9] of order $\mu$ ($0 \leq \mu \leq 1$) for analytic function $f(z)$ is defined as:

\[ D_z^\mu f(z) = \frac{1}{\Gamma(1 - \mu)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)\mu} dt, \quad 0 \leq \mu < 1 \]  

(10)

\[ D_z^1 f(z) = f'(z) \]

which is analytic in simply connected region of $z$-plane containing the origin, multiplicity of $(z-t)^\mu$ is removed by taking $\log(z-t)$ to be real when $(z-t) > 0$ and is well defined in the unit disk $U$.

The image of power function $z^k$ under the operator defined in (10) is given as:

\[ D_z^\mu (z^k) = \frac{\Gamma(k+1)}{\Gamma(k-\mu+1)} z^{k-\mu}, \quad 0 \leq \mu \leq 1. \]  

(11)

Thus, the normalized operator $\Omega_z^{\mu,p} : A(p) \to A(p)$ is defined as:

\[ \Omega_z^{\mu,p} f(z) = z^p \frac{\Gamma(p+1-\mu)}{\Gamma(p+1)} D_z^\mu f(z), \quad 0 \leq \mu \leq 1, \quad p \in N \]  

(12)

and its series expansion using (12) for $f(z) \in A(p)$ of the form (1) is given as:

\[ \Omega_z^{\mu,p} f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} \phi_p^\mu(k) z^{p+k} \]  

(13)

where

\[ \phi_p^\mu(k) = \frac{(p+1)_k}{(p+1-\mu)_k}. \]  

(14)

Involving operator $\Omega_z^{\mu,p}$ given in (13), a generalized family $S(\alpha, \beta, \mu, p)$ of functions $f(z) \in A(p)$ for some $\beta \geq 0, \quad 0 \leq \alpha < p, \quad 0 \leq \mu \leq 1$ is defined by

\[ \text{Re} \left\{ \frac{z(\Omega_z^{\mu,p} f(z))'}{\Omega_z^{\mu,p} f(z)} \right\} > \beta \left| \frac{z(\Omega_z^{\mu,p} f(z))'}{\Omega_z^{\mu,p} f(z)} - p \right| + \alpha. \]  

(15)
It is examined that if \( f(z) \in S(\alpha, \beta, \mu, p) \) with \( 0 \leq \beta \leq \frac{\alpha}{p}, \ 0 \leq \alpha < p, \) then

\[
\Omega_{z}^{\mu,p} f(z) \in S^{*}(p, \frac{\alpha - \beta p}{1 - \beta}).
\] (16)

Note that \( S(\alpha, \beta, 0, p) \equiv \beta - UST(\alpha, p) \) and \( S(\alpha, \beta, 1, p) \equiv \beta - UCV(\alpha, p) \). Also, \( TS(\alpha, \beta, \mu, p) \equiv S_{\mu}(\alpha, \beta, \mu, p) \cap T(p) \).

Further, it is observed that \( S(\alpha, 0, \mu, p) \equiv S_{\mu}(\alpha, 1, -1, \mu) \), a family studied in chapter 2.

2. Coefficient Conditions

In this section, a sufficient coefficient condition for functions belonging to \( S(\alpha, \beta, \mu, p) \) family is given and then it is proved that this condition is necessary for its subfamily \( TS(\alpha, \beta, \mu, p) \).

**Theorem 1.** Let \( f(z) \in A(p) \) of the form (1), satisfies

\[
\sum_{k=1}^{\infty} \frac{k(1 + \beta) + (p - \alpha)}{(p - \alpha)} \phi^{\mu}_{p}(k)|a_{p+k}| \leq 1,
\] (17)

for \( \beta \geq 0, \ 0 \leq \alpha < p, \ 0 \leq \mu \leq 1, \ \phi^{\mu}_{p}(k) = \frac{(p+1)_{k}}{(p+1-\mu)_{k}} \), then \( f(z) \in S(\alpha, \beta, \mu, p) \).

**Proof.** Let the inequality (17) holds true, then it is to prove that

\[
\text{Re} \left\{ \frac{z (\Omega_{z}^{\mu,p} f(z))'}{\Omega_{z}^{\mu,p} f(z)} - p \right\} > \beta \left| \frac{z (\Omega_{z}^{\mu,p} f(z))'}{\Omega_{z}^{\mu,p} f(z)} - p \right| + \alpha - p \]

or,

\[
\beta \left| \frac{z (\Omega_{z}^{\mu,p} f(z))'}{\Omega_{z}^{\mu,p} f(z)} - p \right| - \text{Re} \left\{ \frac{z (\Omega_{z}^{\mu,p} f(z))'}{\Omega_{z}^{\mu,p} f(z)} - p \right\} < p - \alpha.
\]

Since, \( \text{Re}(w) \) or \( \text{Re}(-w) \leq |w| \) for some complex number \( w \), it follows that

\[
\beta \left| \frac{z (\Omega_{z}^{\mu,p} f(z))'}{\Omega_{z}^{\mu,p} f(z)} - p \right| - \text{Re} \left\{ \frac{z (\Omega_{z}^{\mu,p} f(z))'}{\Omega_{z}^{\mu,p} f(z)} - p \right\} \leq (1 + \beta) \left| \frac{z (\Omega_{z}^{\mu,p} f(z))'}{\Omega_{z}^{\mu,p} f(z)} - p \right|
\]
\[ (1 + \beta) \frac{\sum_{k=1}^{\infty} k \phi_p^\mu(k)|a_{p+k}|}{1 - \sum_{k=1}^{\infty} \phi_p^\mu(k)|a_{p+k}|} \]

whose right hand side expression is bounded above by \((p - \alpha)\) if, inequality (17) holds, which proves Theorem 1.

**Theorem 2.** Let \( f(z) \in T(p) \) be of the form (2), then \( f(z) \in TS(\alpha, \beta, \mu, p) \) if and only if

\[ \sum_{k=1}^{\infty} \frac{(k(1 + \beta) + (p - \alpha)) \phi_p^\mu(k)|a_{p+k}|}{(p - \alpha)} \leq 1, \quad (18) \]

for \( \beta \geq 0, \ 0 \leq \alpha < p, \ 0 \leq \mu \leq 1 \) and \( \phi_p^\mu(k) = \frac{(p+1)_k}{(p+1-\mu)_k} \).

**Proof.** In view of Theorem 1, it needs only to prove necessary part. Let \( f(z) \in TS(\alpha, \beta, \mu, p) \), then \( f(z) \) of the form (2) satisfies

\[ \Re \left\{ \frac{z \left( \Omega_z^{\mu,p} f(z) \right)'}{\Omega_z^{\mu,p} f(z)} \right\} > \beta \left| \frac{z \left( \Omega_z^{\mu,p} f(z) \right)'}{\Omega_z^{\mu,p} f(z)} - p \right| + \alpha \]

or,

\[ \beta \left| \frac{p - \sum_{k=1}^{\infty} (p + k) \phi_p^\mu(k)|a_{p+k}|z^k}{1 - \sum_{k=1}^{\infty} \phi_p^\mu(k)|a_{p+k}|z^k} \right| - p < \Re \left\{ \frac{p - \sum_{k=1}^{\infty} (p + k) \phi_p^\mu(k)|a_{p+k}|z^k}{1 - \sum_{k=1}^{\infty} \phi_p^\mu(k)|a_{p+k}|z^k} - \alpha \right\}. \]

Since, \( \Re(w) \) or \( \Re(-w) \leq |w| \),

\[ \Re \beta \left\{ \frac{p - \sum_{k=1}^{\infty} (p + k) \phi_p^\mu(k)|a_{p+k}|z^k}{1 - \sum_{k=1}^{\infty} \phi_p^\mu(k)|a_{p+k}|z^k} \right\} < \Re \left\{ \frac{p - \sum_{k=1}^{\infty} (p + k) \phi_p^\mu(k)|a_{p+k}|z^k}{1 - \sum_{k=1}^{\infty} \phi_p^\mu(k)|a_{p+k}|z^k} - \alpha \right\}. \]

Letting \( z \to 1^- \) along the real axis, it gives the desired inequality

\[ \sum_{k=1}^{\infty} \frac{(k(1 + \beta) + (p - \alpha)) \phi_p^\mu(k)|a_{p+k}|}{(p - \alpha)} \leq 1. \]

This proves Theorem 3.
3. Coefficient Estimate for the Family $S(\alpha, \beta, \mu, p)$

**Theorem 3.** Let $f(z) \in A(p)$ of the form (1) be in the family $S(\alpha, \beta, \mu, p)$, $0 \leq \beta \leq \frac{\alpha}{p}$, $0 \leq \alpha < p$, then

$$|a_{p+k}| \leq \frac{1}{k! \phi_p^\mu(k)} \prod_{j=1}^{k} \left( j - 1 + \frac{2(p-\alpha)}{(1-\beta)} \right), \quad k \geq 1$$

(19)

where $\phi_p^\mu(k) = \frac{(p+1)_k}{(p+1-\mu)_k}$.

Or, equivalently,

$$|a_{p+k}| \leq \frac{\left(\frac{2(p-\alpha)}{1-\beta}\right)_k (p+1-\mu)_k}{(1)_k (p+1)_k}, \quad k \geq 1.$$

**Proof.** Since $f(z) \in S(\alpha, \beta, \mu, p)$ for $0 \leq \beta \leq \frac{\alpha}{p}$, $0 \leq \alpha < p$, it gives

$$\text{Re} \left\{ \frac{z (\Omega_z^\mu f(z))'}{\Omega_z^\mu f(z)} \right\} > \frac{(\alpha - \beta p)}{(1 - \beta)}.$$

Let $q(z) = 1 + q_1 z + q_2 z^2 + \ldots$, be defined as

$$q(z) = \frac{(1 - \beta) \left\{ \frac{z (\Omega_z^\mu f(z))'}{\Omega_z^\mu f(z)} \right\} - (\alpha - \beta p)}{(p - \alpha)}$$

(20)

which is analytic in $U$ with $q(0) = 1$ and $\text{Re}\{q(z)\} > 0$ for $z \in U$, then

$$z (\Omega_z^\mu f(z))' - \frac{(\alpha - \beta p)}{(1 - \beta)} \Omega_z^\mu f(z) = \frac{(p - \alpha)}{(1 - \beta)} q(z) \Omega_z^\mu f(z).$$

On writing their respective series expansions, it gives

$$z \left( p z^{p-1} + \sum_{k=1}^{\infty} (p+k) \phi_p^\mu(k) a_{p+k} z^{p+k-1} \right) - \frac{(\alpha - \beta p)}{(1 - \beta)} \left( z^p + \sum_{k=1}^{\infty} \phi_p^\mu(k) a_{p+k} z^{p+k} \right)$$

$$= \frac{(p - \alpha)}{(1 - \beta)} (1 + q_1 z + q_2 z^2 + \ldots) \left( z^p + \sum_{k=1}^{\infty} \phi_p^\mu(k) a_{p+k} z^{p+k} \right).$$

On comparing the coefficients of $z^{p+k}$ on both sides in the above equation,

$$k \phi_p^\mu(k) a_{p+k} = \frac{(p - \alpha)}{(1 - \beta)} \left\{ q_k + \phi_p^\mu(1) a_{p+1} q_{k-1} + \phi_p^\mu(2) a_{p+2} q_{k-2} + \ldots \right\}.$$
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\[ ... + \phi^\mu_p(k-1)a_{p+k-1}q_1 \].

Hence, on using the coefficient estimate $|q_k| \leq 2$, $k \geq 1$ for Caratheodory functions $q(z)$ [2], for $k \geq 1$,

\[ |a_{p+k}| \leq \frac{2(p-\alpha)}{k\phi^\mu_p(k)(1-\beta)} \left\{ 1 + \phi^\mu_p(1)|a_{p+1}| + \phi^\mu_p(2)|a_{p+2}| + \ldots + \phi^\mu_p(k-1)|a_{p+k-1}| \right\}. \]  

(21)

At $k = 1$, (21) gives

\[ |a_{p+1}| \leq \frac{2(p-\alpha)}{1!\phi^\mu_p(1)(1-\beta)} \]  

(22)

which proves (19) for $k = 1$, also at $k = 2$, (21) gives

\[ |a_{p+2}| \leq \frac{2(p-\alpha)}{2!\phi^\mu_p(2)(1-\beta)} \left\{ 1 + \frac{2(p-\alpha)}{(1-\beta)} \right\}. \]  

(23)

Thus, (22) and (23) prove that (19) is true for $k = 1, 2$. Let (19) is true for $k = n$

i.e.

\[ |a_{p+n}| \leq \frac{1}{n!\phi^\mu_p(n)} \prod_{j=1}^{n} \left( j - 1 + \frac{2(p-\alpha)}{(1-\beta)} \right), \quad n \geq 1. \]  

(24)

Now writing inequality (21) for $k = n + 1$,

\[ |a_{p+n+1}| \leq \frac{2(p-\alpha)}{(n+1)\phi^\mu_p(n+1)(1-\beta)} \left\{ 1 + \phi^\mu_p(1)|a_{p+1}| + \phi^\mu_p(2)|a_{p+2}| + \ldots + \phi^\mu_p(n)|a_{p+n}| \right\}. \]

On applying (22), (23) and (24),

\[ |a_{p+n+1}| \leq \frac{2(p-\alpha)}{(n+1)\phi^\mu_p(n+1)(1-\beta)} \left\{ 1 + \frac{2(p-\alpha)}{1!(1-\beta)} + \frac{2(p-\alpha)}{2!(1-\beta)} \left( 1 + \frac{2(p-\alpha)}{(1-\beta)} \right) + \ldots + \frac{1}{n!} \prod_{j=1}^{n} \left( j - 1 + \frac{2(p-\alpha)}{(1-\beta)} \right) \right\} \]

or,

\[ |a_{p+n+1}| \leq \frac{2(p-\alpha)}{(n+1)\phi^\mu_p(n+1)(1-\beta)} \left\{ \frac{1}{(n-1)!} \left( 1 + \frac{2(p-\alpha)}{(1-\beta)} \right) + \ldots \right\}. \]
\[
\left( n - 1 + \frac{2(p - \alpha)}{1 - \beta} \right) + \frac{1}{n!} \prod_{j=1}^{n} \left( j - 1 + \frac{2(p - \alpha)}{1 - \beta} \right) \right]
\]

or,

\[
|a_{p+n+1}| \leq \frac{1}{(n+1)! \phi_p^\mu(n+1)} \left\{ \frac{2(p - \alpha)}{1 - \beta} \left( 1 + \frac{2(p - \alpha)}{1 - \beta} \right) \right. \\
\left. \left( n - 1 + \frac{2(p - \alpha)}{1 - \beta} \right) \left( n + \frac{2(p - \alpha)}{1 - \beta} \right) \right\}
\]

or,

\[
|a_{p+n+1}| \leq \frac{1}{\phi_p^\mu(n+1)(n+1)!} \prod_{j=1}^{n+1} \left( j - 1 + \frac{2(p - \alpha)}{1 - \beta} \right) , n \geq 1,
\]

which shows that, the result is true for \(k = n + 1\). Thus by mathematical induction, (19) holds true for any \(n \geq 1\). This proves the result.

Taking \(\mu = 0, p = 1\) and \(\mu = 1, p = 1\) respectively in Theorem 3, following results of Owa et al. [5] are obtained.

Corollary 4. [5] Let \(f(z) \in S\) of the form (1) be in the family \(\beta\)-UST\((\alpha)\), \(\beta \geq 0, 0 \leq \alpha < 1\), then

\[
|a_{1+k}| \leq \frac{1}{k!} \prod_{j=1}^{k} \left( j - 1 + \frac{2(1-\alpha)}{(1-\beta)} \right) , k \geq 1.
\]

Corollary 5. [5] Let \(f(z) \in S\) of the form (1) be in the family \(\beta\)-UCV\((\alpha)\), \(\beta \geq 0, 0 \leq \alpha < 1\), then

\[
|a_{1+k}| \leq \frac{1}{(k+1)!} \prod_{j=1}^{k} \left( j - 1 + \frac{2(1-\alpha)}{(1-\beta)} \right) , k \geq 1.
\]

Again, taking \(\beta = 0\) in Corollary 5, following results of Robertson [6] are obtained.

Corollary 6. [6] Let \(f(z) \in S\) of the form (1) be in the family \(S^*(\alpha)\), \(0 \leq \alpha < 1\), then

\[
|a_{1+k}| \leq \frac{1}{k!} \prod_{j=1}^{k} \left( j + 1 - 2\alpha \right) , k \geq 1.
\]
Corollary 7. [6] Let \( f(z) \in S \) of the form (1) be in the family \( K(\alpha) \), \( 0 \leq \alpha < 1 \), then
\[
|a_{1+k}| \leq \frac{1}{(k+1)!} \prod_{j=1}^{k} (j + 1 - 2\alpha), \ k \geq 1.
\]

Further, taking \( \mu = 0, \beta = 0 \) and \( \mu = 1, \beta = 0 \) respectively in Theorem 3, following results of Goyal and Bhagtani [3] are obtained.

Corollary 8. [3] Let \( f(z) \in A(p) \) of the form (1) be in the family \( S^*(p, \alpha) \), \( 0 \leq \alpha < p \), then
\[
|a_{p+k}| \leq \frac{1}{k!} \prod_{j=1}^{k} (j - 1 + 2p - 2\alpha), \ k \geq 1.
\]

Corollary 9. [3] Let \( f(z) \in A(p) \) of the form (1) be in the family \( K(p, \alpha) \), \( 0 \leq \alpha < p \), then
\[
|a_{p+k}| \leq \frac{1}{(p+k)!} \prod_{j=1}^{k} (j - 1 + 2p - 2\alpha), \ k \geq 1.
\]

4. Growth Theorem

In this section, growth result of \( f(z) \in S(\alpha, \beta, \mu, p) \) using coefficient estimate obtained in Theorem 3 and Carlson Shaffer type operator defined in (8) is proved.

Theorem 10. Let \( f(z) \in A(p) \) of the form (1) be in the family \( S(\alpha, \beta, \mu, p) \), then for \( |z| = r < 1 \),
\[
|f(z)| \leq L_p (p + 1 - \mu, p + 1) \left( \frac{r^p}{(1-r)^{2(p-\alpha)}} \right).
\]

Proof. Since \( f \in A(p) \) of the from (1) be in the family \( S(\alpha, \beta, \mu, p) \), Theorem 3 gives
\[
|a_{p+k}| \leq \frac{r^p}{(1-r)^{2(p-\alpha)}} (p + 1 - \mu)_k (p + 1)_k (1)_k (p + 1)_k.
\]
Thus,

\[ |f(z)| \leq |z|^p + \sum_{k=1}^{\infty} |a_{p+k}| |z|^{p+k} \]

\[ \leq |z|^p + \sum_{k=1}^{\infty} \frac{\left(\frac{2(p-\alpha)}{(1-\beta)}\right)_k (p+1-\mu)_k}{(1)_k (p+1)_k} |z|^{p+k} \]

\[ = \sum_{k=0}^{\infty} \frac{\left(\frac{2(p-\alpha)}{(1-\beta)}\right)_k (p+1-\mu)_k}{(1)_k (p+1)_k} |z|^{p+k} \]

\[ = \left\{ \sum_{k=0}^{\infty} \frac{(p+1-\mu)_k}{(p+1)_k} |z|^{p+k} \right\} \ast \left\{ \sum_{k=0}^{\infty} \frac{\left(\frac{2(p-\alpha)}{(1-\beta)}\right)_k}{(1)_k} |z|^{p+k} \right\} \]

\[ = \phi_p(p+1-\mu, p+1, |z|) \ast \phi_p\left(\frac{2(p-\alpha)}{(1-\beta)}, 1, |z|\right) \]

\[ = \phi_p(p+1-\mu, p+1, |z|) \ast \frac{|z|^p}{(1-|z|)^{2\left(\frac{p-\alpha}{(1-\beta)}\right)}} \]

\[ = L_p(p + 1 - \mu, p + 1) \left( \frac{r^p}{(1 - r)^{2\left(\frac{p-\alpha}{(1-\beta)}\right)}} \right). \]

This proves Theorem 10. \[\square\]

For \( \mu = 0, \beta = 0, p = 1 \) and \( \mu = 1, \beta = 0, p = 1 \) respectively in Theorem 10, following results of Banerji and Shenan [1] are obtained.

**Corollary 11.** [1] Let \( f(z) \in S \) be in the family \( S^*(\alpha) \), then

\[ |f(z)| \leq \frac{r}{(1-r)^{2(1-\alpha)}}, \text{ } |z| = r < 1. \]

**Corollary 12.** [1] Let \( f(z) \in S \) be in the family \( K(\alpha) \), then

\[ |f(z)| \leq \phi(2(1-\alpha), 2; r), \text{ } |z| = r < 1. \]

5. Partial Sums

In this section, inequalities involving partial sums of \( f(z) \in A(p) \) are obtained. Let non-zero partial sums of \( f(z) \in A(p) \) of the form (1) be defined as follows:
\[ f_0(z) = z^p \] and \[ f_n(z) = z^p + \sum_{k=1}^{n} a_{p+k} z^{p+k}, \quad k \geq 1. \] (26)

**Theorem 13.** Let \( f(z) \in A(p) \) of the form (1) satisfies

\[ \sum_{k=1}^{\infty} c_{p+k} |a_{p+k}| \phi^k_p(k) \leq 1 \] (27)

where \( c_{p+k} := \frac{(k+\beta)(p-\alpha)}{(p-\alpha)} \frac{(p+1)_k}{(p+1-\mu)_k} \), \( \beta \geq 0, 0 \leq \alpha < p, 0 \leq \mu \leq 1 \), then \( f \in S(\alpha, \beta, \mu, p) \) and

\[
\text{Re}\left\{ \frac{f(z)}{f_n(z)} \right\} > 1 - \frac{1}{c_{p+n+1}}, \\
\text{Re}\left\{ \frac{f_n(z)}{f(z)} \right\} > \frac{c_{p+n+1}}{1 + c_{p+n+1}}, \\
\text{Re}\left\{ \frac{f'(z)}{f'_n(z)} \right\} > 1 - \frac{p + n + 1}{c_{p+n+1}}.
\] (28), (29), (30)

**Proof.** Since \( f(z) \in A(p) \) of the form (1) satisfy (27), from Theorem 1, \( f(z) \in S(\alpha, \beta, \mu, p) \). Further, from (27), as it is easily seen that

\[ c_{p+n+1} > c_{p+n} > 1, \] (31)

\[ \sum_{k=1}^{n} |a_{p+k}| + c_{p+n+1} \sum_{k=n+1}^{\infty} |a_{p+k}| \leq \sum_{k=1}^{\infty} c_{p+k} |a_{p+k}| \leq 1. \] (32)

Set

\[ g_1(z) = c_{p+n+1} \left\{ \frac{f(z)}{f_n(z)} - \left( 1 - \frac{1}{c_{p+n+1}} \right) \right\} \] (33)

which is analytic in \( U \) and \( g_1(0) = 1 \).

\[
\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| = \left| \frac{c_{p+n+1} \sum_{k=n+1}^{\infty} a_{p+k} z^k}{2 + 2 \sum_{k=1}^{n} a_{p+k} z^k + c_{p+n+1} \sum_{k=n+1}^{\infty} a_{p+k} z^k} \right|
\leq \frac{c_{p+n+1} \sum_{k=n+1}^{\infty} |a_{p+k}|}{2 - 2 \sum_{k=1}^{n} |a_{p+k}| - c_{p+n+1} \sum_{k=n+1}^{\infty} |a_{p+k}|}
\leq 1
\]

if (32) holds, which readily yields that \( \text{Re}\{g_1(z)\} > 0 \), this proves assertion (28) of Theorem 10.
Similarly, set
\[ g_2(z) = (1 + c_{p+n+1}) \left\{ \frac{f_n(z)}{f(z)} - \frac{c_{p+n+1}}{1 + c_{p+n+1}} \right\} \]
and making use of (32),
\[
\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| = \left| \frac{(1 + c_{p+n+1}) \sum_{k=n+1}^{\infty} a_{p+k} z^k}{2 + 2 \sum_{k=n+1}^{\infty} a_{p+k} z^k - (1 + c_{p+n+1}) \sum_{k=n+1}^{\infty} a_{p+k} z^k} \right|
\leq \frac{(1 + c_{p+n+1}) \sum_{k=n+1}^{\infty} |a_{p+k}|}{2 - 2 \sum_{k=n}^{n} |a_{p+k}| - (c_{p+n+1} - 1) \sum_{k=n+1}^{\infty} |a_{p+k}|}
\leq 1
\]
which proves the assertion (29).

Further, set
\[ g_3(z) = \frac{c_{p+n+1}}{p + n + 1} \left\{ \frac{f'(z)}{f_n'(z)} - \left( 1 - \frac{p + n + 1}{c_{p+n+1}} \right) \right\} \]
\[ = 1 + \frac{c_{p+n+1}}{p + n + 1} \sum_{k=n+1}^{\infty} \frac{(p+k)}{p} a_{p+k} z^k \]
and
\[
\left| \frac{g_3(z) - 1}{g_3(z) + 1} \right| = \left| \frac{\frac{c_{p+n+1}}{p + n + 1} \sum_{k=n+1}^{\infty} \frac{(p+k)}{p} a_{p+k} z^k}{2 + 2 \sum_{k=n+1}^{n} \frac{(p+k)}{p} a_{p+k} z^k + \frac{c_{p+n+1}}{p + n + 1} \sum_{k=n+1}^{\infty} \frac{(p+k)}{p} a_{p+k} z^k} \right|
\leq \frac{\frac{c_{p+n+1}}{p + n + 1} \sum_{k=n+1}^{\infty} \frac{(p+k)}{p} |a_{p+k}|}{2 - 2 \sum_{k=n+1}^{n} \frac{(p+k)}{p} |a_{p+k}| - \frac{c_{p+n+1}}{p + n + 1} \sum_{k=n+1}^{\infty} \frac{(p+k)}{p} |a_{p+k}|}
\leq 1
\]
if
\[ \sum_{k=1}^{n} \frac{(p+k)}{p} |a_{p+k}| + \frac{c_{p+n+1}}{p + n + 1} \sum_{k=n+1}^{\infty} \frac{(p+k)}{p} |a_{p+k}| \leq 1 \]
(37)
which holds if the left hand side of (37) is bounded above by \( \sum_{k=1}^{\infty} c_{p+k} |a_{p+k}| \)

i.e. if
\[ \sum_{k=1}^{n} \left( c_{p+k} - \frac{(p+k)}{p} \right) |a_{p+k}| \leq 1 \]
(38)
+ \sum_{k=n+1}^{\infty} \left( c_{p+k} - \frac{c_{p+n+1}}{p + n + 1} \frac{(p + k)}{p} \right) |a_{p+k}| \geq 0.

As \frac{c_{p+k}}{p+1} is an increasing function of \( k \), (38) is true. Thus \( \text{Re}\{g_3(z)\} > 0 \) which readily yields the assertion (30).

\textbf{Remark 14.} Note that taking \( \beta = 1 \), \( p = 1 \) and \( \mu = 0 \), \( \mu = 1 \) respectively, above results of Theorem 13 coincide with results obtained by Rosy [7].

6. Integral Mean Inequality

The following subordination result due to Littlewood [4] is used in next Theorem.

\textbf{Lemma 15.} [4] Let \( f(z) \) and \( g(z) \) be analytic in \( U \) with \( f(z) \prec g(z) \), then

\[
\int_{0}^{2\pi} \left| f(re^{i\theta}) \right|^q \, d\theta \leq \int_{0}^{2\pi} \left| g(re^{i\theta}) \right|^q \, d\theta
\]

(39)

where \( q > 0 \), \( z = re^{i\theta} \) and \( 0 < r < 1 \).

\textbf{Theorem 16.} Let \( f(z) \in T(p) \) of the form (2) be in the family \( TS(\alpha, \beta, \mu, p) \), then for \( z = re^{i\theta} \), \( 0 < r < 1 \) and \( q > 0 \)

\[
\int_{0}^{2\pi} \left| f(re^{i\theta}) \right|^q \, d\theta \leq \int_{0}^{2\pi} \left| f_1(re^{i\theta}) \right|^q \, d\theta
\]

(40)

where

\[
f_1(z) = z^p - \frac{(p - \alpha)(p + 1 - \mu)}{((1 + \beta) + (p - \alpha))(p+1)} z^{p+1}.
\]

(41)

\textbf{Proof.} For

\[
f(z) = z^p - \sum_{k=1}^{\infty} |a_{p+k}| z^{p+k}
\]

and \( f_1(z) \) given by (41), to prove (40), it is equivalent to show

\[
\int_{0}^{2\pi} \left| 1 - \sum_{k=1}^{\infty} |a_{p+k}| z^k \right|^q \, d\theta \leq \int_{0}^{2\pi} \left| 1 - \frac{(p - \alpha)(p + 1 - \mu)}{((1 + \beta) + (p - \alpha))(p+1)} z \right|^q \, d\theta.
\]
Hence, by Lemma 15, it suffices to show
\[ 1 - \sum_{k=1}^{\infty} |a_{p+k}|z^k < 1 - \frac{(p - \alpha)(p + 1 - \mu)}{(1 + \beta) + (p - \alpha)} \frac{1}{(p + 1)} w(z) \]
which is true if there exists a Schwartz function \( w(z) \) such that
\[ 1 - \sum_{k=1}^{\infty} |a_{p+k}|z^k = 1 - \frac{(p - \alpha)(p + 1 - \mu)}{(1 + \beta) + (p - \alpha)} \frac{1}{(p + 1)} w(z). \quad (42) \]
Using (18) of Theorem 2,
\[ |w(z)| = \left| \frac{(1 + \beta) + (p - \alpha)}{(p - \alpha)} \phi_p^{\mu}(1) \sum_{k=1}^{\infty} |a_{p+k}|z^k \right| \]
\[ \leq |z| < 1. \]
This completes the proof of Theorem 16.

\[ \square \]

References


