REGULARITY IN SEMIGROUPS OF TRANSFORMATIONS WITH INVARIANT SETS

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Abstract: Let $T(X)$ be the semigroup of all transformations on a set X . For a fixed nonempty subset Y of X , let

$$S(X, Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}.$$

Then $S(X, Y)$ is a semigroup of total transformations on X which leave a subset Y of X invariant. In this paper, we characterize left regular, right regular and intra-regular elements of $S(X, Y)$ and consider the relationships between these elements. Moreover, we count the number of left regular elements of $S(X, Y)$ when X is a finite set.

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1. Introduction

Let X be a nonempty set and let $T(X)$ denote the set of all transformations from X into itself. Then $T(X)$ is a semigroup under the composition of maps and it is called the full transformation semigroup on X .

It is known that $T(X)$ is a regular semigroup, that is, for every $\alpha \in T(X)$, $\alpha = \alpha\beta\alpha$ for some $\beta \in T(X)$.

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In this paper, we study the subsemigroup of $T(X)$, namely $S(X, Y)$, defined by

$$S(X, Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}$$

where $\emptyset \neq Y \subseteq X$. Then $S(X, Y)$ is a semigroup of total transformations on X which leave a subset Y of X invariant. This semigroup was first introduced and studied by K.D. Magill [2] in 1966. In fact, if $Y = X$, then $S(X, Y) = T(X)$. So we may regard $S(X, Y)$ as a generalization of $T(X)$.

In 2005, S. Nenthein, P. Youngkhong and Y. Kemprasit [4] showed that $S(X, Y)$ is a regular semigroup if and only if $X = Y$ or Y contains exactly one element, and $\text{Reg } S(X, Y) = \{\alpha \in S(X, Y) : X\alpha \cap Y = Y\alpha\}$ is the set of all regular elements of $S(X, Y)$. Moreover, they count the number of regular elements of $S(X, Y)$ when X is a finite set.

In 2011, P. Honyam and J. Sanwong [3] described Green's relations and ideals on $S(X, Y)$. Here, in Section 3, we characterize left regular, right regular and intra-regular elements on $S(X, Y)$ and consider the relationships between these elements. In Section 4, we count the number of left regular elements of $S(X, Y)$ when X is a finite set.

2. Preliminaries

In this section, we list some known results, definitions and notations that will be used throughout this paper.

Let X be a set and Y a nonempty subset of X . We defined

$$S(X, Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}.$$

For $\alpha, \beta \in S(X, Y)$, we have $Y\alpha \subseteq Y$ and $Y\beta \subseteq Y$. Then $Y\alpha\beta \subseteq Y\beta \subseteq Y$, so $\alpha\beta \in S(X, Y)$. Therefore, $S(X, Y)$ is a subsemigroup of $T(X)$. From [4], we have $S(X, Y)$ is regular if and only if $X = Y$ or $|Y| = 1$. That is, in general $S(X, Y)$ is not a regular semigroup. Let

$$\text{Reg } S(X, Y) = \{\alpha \in S(X, Y) : X\alpha \cap Y = Y\alpha\}.$$

Then $\text{Reg } S(X, Y)$ is the set of all regular elements of $S(X, Y)$.

As in A. H. Clifford and G. B. Preston [1], we shall use the notation

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}$$

to mean that $\alpha \in T(X)$ and take as understood that the subscript i belongs to some (unmentioned) index set I , the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$, and that $X\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i$ for all i .

With the above notation, for any $\alpha \in S(X, Y)$ we can write

$$\alpha = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix},$$

where $A_i \cap Y \neq \emptyset$, $B_j, C_k \subseteq X \setminus Y$, $\{a_i\} \subseteq Y$, $\{b_j\} \subseteq Y \setminus \{a_i\}$ and $\{c_k\} \subseteq X \setminus Y$. Here, I is a nonempty set, but J or K can be empty. For examples: If $\alpha \in \text{Reg } S(X, Y)$, then J is an empty set. And if $\alpha \in S(X, Y) \setminus \text{Reg } S(X, Y)$, then both I and J are nonempty but K can be an empty set.

We note that for any $\alpha \in S(X, Y)$, the symbol π_α denotes the decomposition of X induced by the map α , namely $\pi_\alpha = \{x\alpha^{-1} : x \in X\alpha\}$. For a nonempty subset Z of X , we denote $\pi_\alpha(Z)$ by $\pi_\alpha(Z) = \{x\alpha^{-1} : x \in X\alpha \cap Z\}$. Thus $\pi_\alpha(Y) = \{y\alpha^{-1} : y \in X\alpha \cap Y\}$.

The following theorem is given by S. Nenthein, P. Youngkhong and Y. Kemprasit [4].

Theorem 2.1. *The following statements hold for the semigroup $S(X, Y)$.*

(1) *For $\alpha \in S(X, Y)$, α is a regular element of $S(X, Y)$ if and only if $X\alpha \cap Y = Y\alpha$.*

(2) *$S(X, Y)$ is regular if and only if either $Y = X$ or $|Y| = 1$.*

Green's relations on $S(X, Y)$ are given by P. Honyam and J. Sanwong [3], which are needed in characterizing left regular, right regular and intra-regular elements on $S(X, Y)$.

Lemma 2.2. *Let $\alpha, \beta \in S(X, Y)$. Then:*

(1) *$\alpha\mathcal{L}\beta$ if and only if $X\alpha = X\beta$ and $Y\alpha = Y\beta$.*

(2) *$\alpha\mathcal{R}\beta$ if and only if $\pi_\alpha = \pi_\beta$ and $\pi_\alpha(Y) = \pi_\beta(Y)$.*

(3) *$\alpha\mathcal{J}\beta$ if and only if $|X\alpha| = |X\beta|$, $|Y\alpha| = |Y\beta|$ and $|X\alpha \setminus Y| = |X\beta \setminus Y|$.*

3. Left Regular, Right Regular and Intra-Regular Elements

In this section, we give necessary and sufficient conditions for elements in $S(X, Y)$ to be left regular, right regular and intra-regular. We first recall that:

an element $\alpha \in S(X, Y)$ is left regular [right regular] if $\alpha = \beta\alpha^2$ [$\alpha = \alpha^2\beta$] for some $\beta \in S(X, Y)$ and it is intra-regular if $\alpha = \lambda\alpha^2\mu$ for some $\lambda, \mu \in S(X, Y)$. Since the identity map $1_X \in S(X, Y)$, it follows that $S(X, Y)^1 = S(X, Y)$. For each $\alpha \in S(X, Y)$, we denote $\mathcal{L}$ -class, $\mathcal{R}$ -class and $\mathcal{J}$ -class containing α by L_α, R_α and J_α respectively.

Theorem 3.1. *Let $\alpha \in S(X, Y)$. Then the following statements are equivalent:*

- (1) α is left regular.
- (2) $X\alpha = X\alpha^2$ and $Y\alpha = Y\alpha^2$.
- (3) $\alpha^2 \in L_\alpha$.

Proof. It is clear that (2) $\Leftrightarrow$ (3) by Lemma 2.2(1). If α is left regular, then $\alpha = \beta\alpha^2$ for some $\beta \in S(X, Y)$ and we have $\alpha^2 = \alpha \cdot \alpha$, so $\alpha\mathcal{L}\alpha^2$ and that $\alpha^2 \in L_\alpha$. Conversely, if $\alpha^2 \in L_\alpha$, then $\alpha = \beta\alpha^2$ for some $\beta \in S(X, Y)$. So α is left regular. $\square$

Similar to Theorem 3.1, by using Lemma 2.2(2), we obtain the following theorem.

Theorem 3.2. *Let $\alpha \in S(X, Y)$. Then the following statements are equivalent:*

- (1) α is right regular.
- (2) $\pi_\alpha = \pi_{\alpha^2}$ and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$.
- (3) $\alpha^2 \in R_\alpha$.

In general, there is no relationships between left regular elements and right regular elements as shown in the following example.

Example 1. Let X be the set of all natural numbers.

- (1) Let $Y = \{1, 2, 3, 4\}$ and define α by

$$n\alpha = \begin{cases} n & , n \in \{1, 2\}; \\ 4 & , n = 3; \\ 3 & , n \in \{4, 5\}; \\ n - 1 & , \text{otherwise.} \end{cases}$$

Then $\alpha \in S(X, Y)$ and

$$n\alpha^2 = \begin{cases} n & , n \in Y; \\ 4 & , n = 5; \\ 3 & , n = 6; \\ n - 2 & , \text{otherwise.} \end{cases}$$

Thus $X\alpha = X\alpha^2$ is the set of all natural numbers and $Y\alpha = \{1, 2, 3, 4\} = Y\alpha^2$. So α is left regular. But α is not right regular, since

$$\pi_\alpha(Y) = \{\{1\}, \{2\}, \{3\}, \{4, 5\}\} \neq \{\{1\}, \{2\}, \{3, 6\}, \{4, 5\}\} = \pi_{\alpha^2}(Y).$$

(2) Let Y the set of all positive even integers, and let α be defined by $n\alpha = n + 2$ for all $n \in X$. Then $\alpha \in S(X, Y)$ and $n\alpha^2 = n + 4$ for all $n \in X$. Thus $\pi_\alpha(Y) = \{\{2n\}\} = \pi_{\alpha^2}(Y)$ and $\pi_\alpha = \{\{n\}\} = \pi_{\alpha^2}$. So α is right regular. But α is not left regular, since $Y\alpha = \{2n + 2\} \neq \{2n + 4\} = Y\alpha^2$.

For convenience, from now on if $\alpha \in S(X, Y)$, the notations Y' and X' are for $Y\alpha$ and $X\alpha \setminus Y\alpha$ respectively. To prove that $\alpha \in S(X, Y)$ is left regular if and only if α is right regular when $X\alpha$ is finite, we begin with the following three lemmas.

Lemma 3.3. *Let $\alpha \in S(X, Y)$. If $X\alpha = X\alpha^2$ is finite and $Y\alpha = Y\alpha^2$, then*

- (1) $(X \setminus Y)\alpha \subseteq (X \setminus Y) \cup Y'$;
- (2) $X' \subseteq X \setminus Y'\alpha^{-1}$;
- (3) $|y'\alpha^{-1} \cap Y'| = 1$ for all $y' \in Y'$;
- (4) $|x'\alpha^{-1} \cap X'| = 1$ for all $x' \in X'$.

Proof. Assume that $X\alpha = X\alpha^2$ is finite and $Y\alpha = Y\alpha^2$. So we can write

$$\alpha = \begin{pmatrix} A_1 & \dots & A_n & B_1 & \dots & B_m & C_1 & \dots & C_t \\ a_1 & \dots & a_n & b_1 & \dots & b_m & c_1 & \dots & c_t \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $B_j, C_k \subseteq X \setminus Y$; $a_i, b_j \in Y, c_k \in X \setminus Y$ for all $i = 1, \dots, n$, $j = 1, \dots, m$ and $k = 1, \dots, t$. Then $Y' = \{a_1, \dots, a_n\}$ and $X' = \{b_1, \dots, b_m, c_1, \dots, c_t\}$.

(1) We prove that $J = \emptyset$ where $J = \{1, \dots, m\}$. Suppose that there is $j \in J$ such that $B_j\alpha = \{b_j\}$ where $b_j \in Y$. Then $B_j\alpha^2 = (B_j\alpha)\alpha = \{b_j\}\alpha$. Assume

that $b_j \in A_{i_0}$ for some i_0 . Thus $B_j\alpha^2 = \{b_j\}\alpha = \{a_{i_0}\}$. Since $Y\alpha^2 = Y\alpha = \{a_1, \dots, a_n\}$, there is A_k such that $A_k\alpha^2 = \{a_{i_0}\}$. That is $A_k\alpha^2 = B_j\alpha^2$. But $A_k\alpha \neq B_j\alpha$, so $|X\alpha^2| \leq n + (m-1) + t < |X\alpha|$, which implies that $X\alpha^2 \subsetneq X\alpha$, a contradiction. Thus $J = \emptyset$. So we can write

$$\alpha = \begin{pmatrix} A_1 & \dots & A_n & C_1 & \dots & C_t \\ a_1 & \dots & a_n & c_1 & \dots & c_t \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $C_k \subseteq X \setminus Y$; $a_i \in Y, c_k \in X \setminus Y$ for all $i = 1, \dots, n$ and $k = 1, \dots, t$. Then $Y' = \{a_1, \dots, a_n\}$ and $X' = \{c_1, \dots, c_t\}$. Therefore, $(X \setminus Y)\alpha \subseteq (X \setminus Y) \cup Y'$.

(2) Now, we have $X' = \{c_1, \dots, c_t\} \subseteq X \setminus Y$ and $X \setminus Y'\alpha^{-1} = C_1 \cup \dots \cup C_t$. Suppose that there is $c_p \in X' \cap Y'\alpha^{-1}$. So $c_p \in A_{i_0}$ for some i_0 . Then $C_p\alpha^2 = \{a_{i_0}\}$. Since $Y\alpha^2 = Y\alpha$, there is A_q such that $A_q\alpha^2 = \{a_{i_0}\}$, so we get $C_p\alpha^2 = \{a_{i_0}\} = A_q\alpha^2$, thus $|X\alpha^2| \leq n + (t-1) < |X\alpha|$ which implies that $X\alpha^2 \subsetneq X\alpha$, a contradiction. Therefore, $X' \cap Y'\alpha^{-1} = \emptyset$, hence $X' \subseteq X \setminus Y'\alpha^{-1}$.

(3) Assume that $|y'\alpha^{-1} \cap Y'| \neq 1$ for some $y' \in Y'$. Suppose that there are a_i and a_j both belong to A_l for some $l \in \{1, \dots, n\}$ where $a_i \neq a_j \in Y'$. Then $A_i\alpha^2 = \{a_l\} = A_j\alpha^2$. Thus $A_i\alpha^2 = A_j\alpha^2$, so $|Y\alpha^2| \leq n-1$ where $|Y\alpha| = n$, a contradiction. So a_i and a_j belong to different A_l for all $i \neq j$. Since $|Y'| = n$ is finite, we obtain $Y' \cap A_l \neq \emptyset$ for all l and that $|y'\alpha^{-1} \cap Y'| = 1$ for all $y' \in Y'$.

(4) Assume that $|x'\alpha^{-1} \cap X'| \neq 1$ for some $x' \in X'$. Suppose that there are c_u and $c_v \in C_k$ for some $k \in \{1, \dots, t\}$ where $c_u \neq c_v \in X'$. Then $C_u\alpha^2 = \{c_k\} = C_v\alpha^2$ and thus $C_u\alpha^2 = C_v\alpha^2$, a contradiction, since $X\alpha^2 = X\alpha$. So c_u and c_v belong to different C_k for all $u \neq v$. Since $|X'| = t$ is finite, it follows that $X' \cap C_k \neq \emptyset$ for all k . Therefore, $|x'\alpha^{-1} \cap X'| = 1$ for all $x' \in X'$. $\square$

Lemma 3.4. *Let $\alpha \in S(X, Y)$. If $\pi_\alpha = \pi_{\alpha^2}$ is finite and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$ then, $\pi_\alpha(Y\alpha) = \pi_{\alpha^2}(Y\alpha)$.*

Proof. Assume that $\pi_\alpha = \pi_{\alpha^2}$ is finite and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$. We have $\pi_\alpha(Y\alpha) = \{y\alpha^{-1} : y \in X\alpha \cap Y\alpha\}$ and $\pi_{\alpha^2}(Y\alpha) = \{y(\alpha^2)^{-1} : y \in X\alpha^2 \cap Y\alpha\}$. Since π_α is finite, we can write

$$\alpha = \begin{pmatrix} A_1 & \dots & A_n & B_1 & \dots & B_m & C_1 & \dots & C_t \\ a_1 & \dots & a_n & b_1 & \dots & b_m & c_1 & \dots & c_t \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $B_j, C_k \subseteq X \setminus Y$; $a_i, b_j \in Y, c_k \in X \setminus Y$ for all $i = 1, \dots, n$, $j = 1, \dots, m$ and $k = 1, \dots, t$. Since $\pi_\alpha = \pi_{\alpha^2}$ and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$, we have

$$\alpha^2 = \begin{pmatrix} A_1 & \dots & A_n & B_1 & \dots & B_m & C_1 & \dots & C_t \\ a'_1 & \dots & a'_n & b'_1 & \dots & b'_m & c'_1 & \dots & c'_t \end{pmatrix}$$

where $a'_i, b'_j \in Y$ and $c'_k \in X \setminus Y$. So $Y\alpha^2 = \{a'_1, \dots, a'_n\}$, while $Y\alpha = \{a_1, \dots, a_n\}$. Since $Y\alpha \subseteq Y$, we obtain $Y\alpha^2 \subseteq Y\alpha$. So $\{a'_1, \dots, a'_n\} \subseteq \{a_1, \dots, a_n\}$. Since $Y\alpha$ is finite, we have $\{a'_1, \dots, a'_n\} = \{a_1, \dots, a_n\}$. Thus $X\alpha \cap Y\alpha = X\alpha^2 \cap Y\alpha$. Therefore, $\pi_\alpha(Y\alpha) = \{A_1, \dots, A_n\} = \pi_{\alpha^2}(Y\alpha)$. $\square$

Lemma 3.5. *Let $\alpha \in S(X, Y)$. If $\pi_\alpha = \pi_{\alpha^2}$ is finite and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$, then*

- (1) $(X \setminus Y)\alpha \subseteq (X \setminus Y) \cup Y'$;
- (2) $X' \subseteq X \setminus Y' \alpha^{-1}$;
- (3) $|y' \alpha^{-1} \cap Y'| = 1$ for all $y' \in Y'$;
- (4) $|x' \alpha^{-1} \cap X'| = 1$ for all $x' \in X'$.

Proof. Since π_α is finite, so we can write

$$\alpha = \begin{pmatrix} A_1 & \dots & A_n & B_1 & \dots & B_m & C_1 & \dots & C_t \\ a_1 & \dots & a_n & b_1 & \dots & b_m & c_1 & \dots & c_t \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $B_j, C_k \subseteq X \setminus Y$; $a_i, b_j \in Y, c_k \in X \setminus Y$ for all $i = 1, \dots, n$, $j = 1, \dots, m$ and $k = 1, \dots, t$. Then $Y' = \{a_1, \dots, a_n\}$ and $X' = \{b_1, \dots, b_m, c_1, \dots, c_t\}$. Since $\pi_\alpha = \pi_{\alpha^2}$ is finite and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$, we have by Lemma 3.4 that $\pi_{\alpha^2}(Y\alpha) = \pi_\alpha(Y\alpha) = \{A_1, \dots, A_n\}$.

(1) Suppose that there is $j \in J$ such that $B_j\alpha = \{b_j\}$ where $b_j \in Y$. Then $B_j\alpha^2 = \{b_j\}\alpha$. Assume that $b_j \in A_{i_0}$ for some i_0 , so $B_j\alpha^2 = \{a_{i_0}\}$. Since $a_{i_0} \in X\alpha^2$ and $a_{i_0} \in Y\alpha$, we obtain $a_{i_0} \in X\alpha^2 \cap Y\alpha$, so $a_{i_0}(\alpha^2)^{-1} \in \pi_{\alpha^2}(Y\alpha) = \{A_1, \dots, A_n\}$. Thus there is $A_k \in \pi_{\alpha^2}(Y\alpha)$ such that $A_k\alpha^2 = \{a_{i_0}\}$. Hence $B_j\alpha^2 = \{a_{i_0}\} = A_k\alpha^2$ and therefore $B_j \cup A_k \subseteq a_{i_0}(\alpha^2)^{-1}$. That means $\pi_{\alpha^2}(Y) \neq \pi_\alpha(Y)$ which is a contradiction. Then $J = \emptyset$. So we can write

$$\alpha = \begin{pmatrix} A_1 & \dots & A_n & C_1 & \dots & C_t \\ a_1 & \dots & a_n & c_1 & \dots & c_t \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $C_k \subseteq X \setminus Y$; $a_i \in Y, c_k \in X \setminus Y$ for all $i = 1, \dots, n$ and $k = 1, \dots, t$. Hence $Y' = \{a_1, \dots, a_n\}$ and $X' = \{c_1, \dots, c_t\}$. Therefore, $(X \setminus Y)\alpha \subseteq (X \setminus Y) \cup Y'$.

(2) We have $X' = \{c_1, \dots, c_t\} \subseteq X \setminus Y$ and $X \setminus Y' \alpha^{-1} = C_1 \cup \dots \cup C_t$. Suppose that there is $c_p \in X' \cap Y' \alpha^{-1}$. Then $c_p \in A_{i_0}$ for some i_0 . So $C_p\alpha^2 = \{a_{i_0}\}$. Since $a_{i_0} \in X\alpha^2$ and $a_{i_0} \in Y\alpha$, we obtain $a_{i_0} \in X\alpha^2 \cap Y\alpha$, so $a_{i_0}(\alpha^2)^{-1} \in \pi_{\alpha^2}(Y\alpha) =$

$\{A_1, \dots, A_n\}$. Thus there is $A_k \in \pi_{\alpha^2}(Y\alpha)$ such that $A_k\alpha^2 = \{a_{i_0}\}$. Hence $C_p\alpha^2 = \{a_{i_0}\} = A_k\alpha^2$ and therefore $C_p \cup A_k \subseteq a_{i_0}(\alpha^2)^{-1}$, which implies that $\pi_{\alpha^2}(Y) \neq \pi_{\alpha}(Y)$, a contradiction. So $X' \cap Y'\alpha^{-1} = \emptyset$. Therefore, $X' \subseteq X \setminus Y'\alpha^{-1}$.

(3) Assume that $|y'\alpha^{-1} \cap Y'| \neq 1$ for some $y' \in Y'$. Suppose that there are a_i and a_j both belong to A_l for some $l \in \{1, \dots, n\}$ where $a_i \neq a_j \in Y'$. Then $A_i\alpha^2 = \{a_l\} = A_j\alpha^2$. Thus $A_i \cup A_j \subseteq a_l(\alpha^2)^{-1}$. Since $a_l \in X\alpha^2 \cap Y$, we obtain $a_l(\alpha^2)^{-1} \in \pi_{\alpha^2}(Y)$, so $|\pi_{\alpha^2}(Y)| \leq n - 1$ where $|\pi_{\alpha}(Y)| = n$. Thus $\pi_{\alpha}(Y) \neq \pi_{\alpha^2}(Y)$, which is a contradiction. That means a_i and a_j belong to different A_l for all $i \neq j$. Hence $Y' \cap A_l \neq \emptyset$ for all l since $|Y'| = n$ is finite. Therefore, $|y'\alpha^{-1} \cap Y'| = 1$ for all $y' \in Y'$.

(4) Assume that $|x'\alpha^{-1} \cap X'| \neq 1$ for some $x' \in X'$. Suppose that there are c_u and $c_v \in C_k$ for some $k \in \{1, \dots, t\}$ where $c_u \neq c_v \in X'$. Then $C_u\alpha^2 = \{c_k\} = C_v\alpha^2$. So $C_u \cup C_v \subseteq c_k(\alpha^2)^{-1}$. This implies that $|\pi_{\alpha^2}| \leq n + (t - 1) < |\pi_{\alpha}|$ which is a contradiction. Thus c_u and c_v belong to different C_k for all $u \neq v$. And $X' \cap C_k \neq \emptyset$ for all k since $|X'| = t$ is finite. Therefore, $|x'\alpha^{-1} \cap X'| = 1$ for all $x' \in X'$. $\square$

Corollary 3.6. *Let $\alpha \in S(X, Y)$ be such that $X\alpha$ is finite. If α is left regular, then α is regular.*

Proof. Assume that α is left regular. Then $X\alpha = X\alpha^2$ and $Y\alpha = Y\alpha^2$. Since $Y\alpha \subseteq Y$ and $Y\alpha \subseteq X\alpha$, we obtain $Y\alpha \subseteq X\alpha \cap Y$. Now, we prove $X\alpha \cap Y \subseteq Y\alpha$ by letting $y \in X\alpha \cap Y$. If $y \notin Y\alpha$, then $y \in (X \setminus Y)\alpha \subseteq (X \setminus Y) \cup Y'$ by Lemma 3.3(1). Thus $y \in X \setminus Y$ or $y \in Y'$. Since $y \in Y$, it follows that $y \in Y' = Y\alpha$ which is a contradiction. Therefore, $y \in Y\alpha$ and that $X\alpha \cap Y \subseteq Y\alpha$. So $X\alpha \cap Y = Y\alpha$ which implies that α is regular by Theorem 2.1. $\square$

Corollary 3.7. *Let $\alpha \in S(X, Y)$ be such that π_{α} is finite. If α is right regular, then α is regular.*

Proof. Assume that α is right regular. Then $\pi_{\alpha} = \pi_{\alpha^2}$ is finite and $\pi_{\alpha}(Y) = \pi_{\alpha^2}(Y)$. By using the same proof as given for Corollary 3.6 and Lemma 3.5(1), we obtain $X\alpha \cap Y = Y\alpha$ and hence α is regular as required. $\square$

Theorem 3.8. *Let $\alpha \in S(X, Y)$ be such that $X\alpha$ is finite. Then α is left regular if and only if α is right regular.*

Proof. Assume that α is left regular. Then $X\alpha = X\alpha^2$ and $Y\alpha = Y\alpha^2$. Since $X\alpha$ is finite, we may write $X\alpha = \{a_1, \dots, a_n, c_1, \dots, c_t\}$ where $Y' = \{a_1, \dots, a_n\}$. By Lemma 3.3(1), we can write

$$\alpha = \begin{pmatrix} A_1 & \dots & A_n & C_1 & \dots & C_t \\ a_1 & \dots & a_n & c_1 & \dots & c_t \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $C_k \subseteq X \setminus Y$; $a_i \in Y, c_k \in X \setminus Y$ for all $i = 1, \dots, n$ and $k = 1, \dots, t$. Since $|y'\alpha^{-1} \cap Y'| = |A_i \cap \{a_1, \dots, a_n\}| = 1$ for all $i = 1, \dots, n$, there is a permutation σ on the set $\{1, \dots, n\}$ such that $a_i \in A_{i\sigma}$ for all i . So we obtain $A_i\alpha^2 = \{a_{i\sigma}\}$. Similarly, since $|x'\alpha^{-1} \cap X'| = |C_k \cap \{c_1, \dots, c_t\}| = 1$ for all $k = 1, \dots, t$, there is a permutation δ on the set $\{1, \dots, t\}$ such that $c_k \in C_{k\delta}$ for all k . Thus $C_k\alpha^2 = \{c_{k\delta}\}$. So

$$\alpha^2 = \begin{pmatrix} A_1 & \dots & A_n & C_1 & \dots & C_t \\ a_{1\sigma} & \dots & a_{n\sigma} & c_{1\delta} & \dots & c_{t\delta} \end{pmatrix}.$$

That is $\pi_\alpha(Y) = \{A_1, \dots, A_n\} = \pi_{\alpha^2}(Y)$ and $\pi_\alpha = \{A_1, \dots, A_n, C_1, \dots, C_t\} = \pi_{\alpha^2}$. Hence α is right regular.

Conversely, assume that α is right regular. Then $\pi_\alpha = \pi_{\alpha^2}$ and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$. Since $X\alpha$ is finite, we obtain $\pi_\alpha = \pi_{\alpha^2}$ is finite. Then by Lemma 3.5(1), we can write

$$\alpha = \begin{pmatrix} A_1 & \dots & A_n & C_1 & \dots & C_t \\ a_1 & \dots & a_n & c_1 & \dots & c_t \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $C_k \subseteq X \setminus Y$; $a_i \in Y, c_k \in X \setminus Y$ for all $i = 1, \dots, n$ and $k = 1, \dots, t$. Since $|y'\alpha^{-1} \cap Y'| = 1$ for all $y' \in Y'$ and $|x'\alpha^{-1} \cap X'| = 1$ for all $x' \in X'$. So by the same proof as given above

$$\alpha^2 = \begin{pmatrix} A_1 & \dots & A_n & C_1 & \dots & C_t \\ a_{1\sigma} & \dots & a_{n\sigma} & c_{1\delta} & \dots & c_{t\delta} \end{pmatrix}$$

where σ is a permutation on the set $\{1, \dots, n\}$ and δ is a permutation on the set $\{1, \dots, t\}$. Thus $Y\alpha = \{a_1, \dots, a_n\} = Y\alpha^2$ and $X\alpha = \{a_1, \dots, a_n, c_1, \dots, c_t\} = X\alpha^2$. Hence α is left regular. $\square$

Theorem 3.9. *Let $\alpha \in S(X, Y)$. Then the following statements are equivalent:*

- (1) α is intra-regular.
- (2) $|X\alpha| = |X\alpha^2|, |Y\alpha| = |Y\alpha^2|$ and $|X\alpha \setminus Y| = |X\alpha^2 \setminus Y|$.
- (3) $\alpha^2 \in J_\alpha$.

Proof. It is clear that (2) $\Leftrightarrow$ (3) by Lemma 2.2(3). If α is intra-regular, then $\alpha = \beta\alpha^2\gamma$ for some $\beta, \gamma \in S(X, Y)$ and we have $\alpha^2 = \alpha \cdot \alpha \cdot 1_X$, so $\alpha \mathcal{J} \alpha^2$ and that $\alpha^2 \in J_\alpha$. Conversely, if $\alpha^2 \in J_\alpha$, then $\alpha = \beta\alpha^2\gamma$ for some $\beta, \gamma \in S(X, Y)$. So α is intra-regular. $\square$

Since $L_\alpha \subseteq J_\alpha$ and $R_\alpha \subseteq J_\alpha$, the following corollary is a direct consequences of Theorem 3.1, Theorem 3.2 and Theorem 3.9.

Corollary 3.10. *Let $\alpha \in S(X, Y)$. Then α is intra-regular if one of the following conditions holds.*

- (1) α is left regular.
- (2) α is right regular.

Theorem 3.11. *Let $\alpha \in S(X, Y)$ be such that $X\alpha$ is a finite set. Then the following statements are equivalent:*

- (1) α is left regular.
- (2) α is right regular.
- (3) α is intra-regular.

Proof. (1) $\Leftrightarrow$ (2) follows from Theorem 3.8. (2) $\Rightarrow$ (3) is Corollary 3.10(2). Now, assume that α is intra-regular. Then $|X\alpha| = |X\alpha^2|$ and $|Y\alpha| = |Y\alpha^2|$. Since $X\alpha^2 \subseteq X\alpha$, $|X\alpha^2| = |X\alpha|$ and $X\alpha$ is finite, we obtain $X\alpha = X\alpha^2$. Similarly, we have $Y\alpha = Y\alpha^2$. Therefore, α is left regular by Theorem 3.1. $\square$

The following examples show that the finiteness of $X\alpha$ in Theorem 3.11 is necessary.

Example 2. Let X be the set of all natural numbers and Y the set of all positive even integers. Let α be defined by $n\alpha = n + 4$ for all $n \in X$. Then $\alpha \in S(X, Y)$ and $n\alpha^2 = n + 8$ for all $n \in X$. Thus we have $|X\alpha| = \aleph_0 = |X\alpha^2|$, $|Y\alpha| = \aleph_0 = |Y\alpha^2|$ and $|X\alpha \setminus Y| = \aleph_0 = |X\alpha^2 \setminus Y|$. So α is intra-regular. But α is not left regular, since $Y\alpha = \{2n + 4\} \neq \{2n + 8\} = Y\alpha^2$.

Now, let β be defined by

$$n\beta = \begin{cases} n & , n \in \{1, 2, 3, 5\}; \\ 2 & , n = 4; \\ n - 2 & , \text{otherwise.} \end{cases}$$

Then $\beta \in S(X, Y)$ and

$$n\beta^2 = \begin{cases} n & , n \in \{1, 2, 3, 5\}; \\ 2 & , n \in \{4, 6\}; \\ 5 & , n \in \{7, 9\}; \\ n - 4 & , \text{otherwise.} \end{cases}$$

Thus we have $|X\beta| = \aleph_0 = |X\beta^2|$, $|Y\beta| = \aleph_0 = |Y\beta^2|$ and $|X\beta \setminus Y| = \aleph_0 = |X\beta^2 \setminus Y|$. So β is intra-regular. But β is not right regular, since $\pi_\beta(Y) = \{\{2, 4\}, \{2n + 4\}\} \neq \{\{2, 4, 6\}, \{2n + 6\}\} = \pi_{\beta^2}(Y)$.

4. The Number of Left Regular Elements

Throughout this section, X is a finite set with n elements and $Y \subseteq X$ has r elements. To count the number of left regular elements of $S(X, Y)$, we recall that the number of combinations of n distinct things taken r at a time written $\binom{n}{r}$ is given by $\binom{n}{r} = \frac{n!}{(n-r)!r!}$. That is, $\binom{n}{r}$ is the number of ways that r objects can be chosen from n distinct objects.

Lemma 4.1. *Let $\alpha \in S(X, Y)$ be a left regular element. Then $\alpha|_Y : Y \rightarrow Y$ has $\sum_{k=1}^r \binom{r}{k} k!k^{r-k}$ forms.*

Proof. Since $\alpha \in S(X, Y)$ is a left regular element, we obtain $X\alpha = X\alpha^2$ is finite and $Y\alpha = Y\alpha^2$. We have $\alpha|_Y : Y \rightarrow Y$ since $Y\alpha \subseteq Y$. Suppose that $Y\alpha = Y'$ has k elements. Let $Y' = \{y'_1, \dots, y'_k\}$. Since $\alpha \in S(X, Y)$, by Lemma 3.3 we can write

$$\alpha|_Y = \begin{pmatrix} B_1 & \dots & B_k \\ y'_1 & \dots & y'_k \end{pmatrix}$$

where $\bigcup_{i=1}^k B_i = Y$ and $|B_i \cap Y'| = 1$ for all $i = 1, \dots, k$. So $\alpha|_{Y'} : Y' \rightarrow Y'$ is a permutation. Thus $\alpha|_{Y'}$ can have $k!$ forms. Since $Y\alpha = Y'$ and $|Y \setminus Y'| = r - k$, the number of ways of placing $r - k$ distinct elements into k distinct places is k^{r-k} . Hence $\alpha|_Y$ can have $\binom{r}{k} k!k^{r-k}$ forms. Since $1 \leq k \leq r$, the number of

the maps $\alpha|_Y$ when α is left regular is $\sum_{k=1}^r \binom{r}{k} k!k^{r-k}$. □

Lemma 4.2. *Let $\alpha \in S(X, Y)$ be a left regular element. If $X' \neq \emptyset$, then $\alpha|_{X'} : X' \rightarrow X'$ has $\sum_{m=1}^{n-r} \binom{n-r}{m} m!$ forms.*

Proof. Since $\alpha \in S(X, Y)$ is a left regular element, we have $|x' \alpha^{-1} \cap X'| = 1$ for all $x' \in X'$ by Lemma 3.3(4). Then $\alpha|_{X'} : X' \rightarrow X'$ is a permutation. Suppose that X' has m elements. Let $X' = \{x'_1, \dots, x'_m\}$ and δ is the permutation on the set $\{1, \dots, m\}$. So we can write

$$\alpha|_{X'} = \begin{pmatrix} x'_1 & \dots & x'_m \\ x'_{1\delta} & \dots & x'_{m\delta} \end{pmatrix}.$$

Thus $\alpha|_{X'}$ can have $m!$ forms. Since $1 \leq m \leq n - r$, the number of the maps $\alpha|_{X'}$ when α is left regular is $\sum_{m=1}^{n-r} \binom{n-r}{m} m!$. $\square$

Theorem 4.3. *The number of left regular elements in $S(X, Y)$ is*

$$\sum_{m=0}^{n-r} \sum_{k=1}^r \binom{r}{k} k! k^{r-k} \binom{n-r}{m} m! (k+m)^{n-r-m}$$

where $|X| = n, |Y| = r$.

Proof. Let $\alpha \in S(X, Y)$ be a left regular element. Suppose that Y' has k elements and X' has m elements. Let $Y' = \{y'_1, \dots, y'_k\}$ and $X' = \{x'_1, \dots, x'_m\}$. Then by Lemma 3.3, we can write

$$\alpha = \begin{pmatrix} A_1 & \dots & A_k & C_1 & \dots & C_m \\ y'_1 & \dots & y'_k & x'_1 & \dots & x'_m \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$, $C_j \subseteq X \setminus Y$ and $|A_i \cap Y'| = 1$, $|C_j \cap X'| = 1$ for all $i = 1, \dots, k$ and $j = 1, \dots, m$.

If $X' = \emptyset$, then $\bigcup_{i=1}^m C_i = \emptyset$, that is $\alpha = \begin{pmatrix} A_1 & \dots & A_k \\ y'_1 & \dots & y'_k \end{pmatrix}$. By Lemma 4.1, $\alpha|_Y$ has $\sum_{k=1}^r \binom{r}{k} k! k^{r-k}$ forms. Since the number of ways of placing $n - r$ elements of the set $X \setminus Y$ into k distinct places is k^{n-r} , in this case the number of left regular elements is $\sum_{k=1}^r \binom{r}{k} k! k^{n-k}$.

If $X' \neq \emptyset$, then by Lemma 4.1 we have the number of the maps $\alpha|_Y$ when α is left regular is $\sum_{k=1}^r \binom{r}{k} k!k^{r-k}$. And by Lemma 4.2, the number of $\alpha|_{X'}$ is $\sum_{m=1}^{n-r} \binom{n-r}{m} m!$. So, the number of $\alpha|_{Y \cup X'}$ is $\sum_{m=1}^{n-r} \sum_{k=1}^r \binom{r}{k} k!k^{r-k} \binom{n-r}{m} m!$. Now, $X \setminus (Y \cup X')$ has $n-r-m$ elements, and there are $(k+m)^{n-r-m}$ ways to placing $n-r-m$ elements into $k+m$ places. Therefore, the number of left regular elements in this case is

$$\sum_{m=1}^{n-r} \sum_{k=1}^r \binom{r}{k} k!k^{r-k} \binom{n-r}{m} m! (k+m)^{n-r-m}.$$

Observe that if we put $m = 0$ in the above expression, then we obtain $\sum_{k=1}^r \binom{r}{k} k!k^{n-k}$. Therefore, we conclude that the number of left regular elements in $S(X, Y)$ is

$$\sum_{m=0}^{n-r} \sum_{k=1}^r \binom{r}{k} k!k^{r-k} \binom{n-r}{m} m! (k+m)^{n-r-m}.$$

□

Remark. Since left regular, right regular and intra-regular elements of $S(X, Y)$ are the same when X is a finite set, we obtain the numbers of left regular, right regular and intra-regular elements are equal.

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