EXISTENCE RESULTS FOR QUASISTATIC CONTACT PROBLEM WITH INTERNAL STATE VARIABLES

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Abstract: The subject of this work is the study of a value problem describing the quasistatic evolution of semilinear retetype viscoplastic models with internal state variables, and we suppose the problem of Tresca's Friction law at the presence of recal forces. The existence and uniqueness of the solution is proved using results of evolutionary variational inequalities and a fixed point theorem.

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1. Introduction

For the bilateral contact problem studied in this paper, the friction is modelled by tresca's law with internal state variables.

In Section 2 we present some basic notations on the functional spaces and some hypothesis used in the following.

The model is described in Section 3, where the variational formulation is given. In Section 4, using results of evolutionary variational inequalities and a fixed point theorem, an existence and uniqueness result is obtained. Finally,

in Section 5, we establish a study of the dependence of the solution by the parameter.

2. Statement of the Problem. Hypothesis

We assume the contact is bilateral, i.e. there is no loss of the contact during the process. Thus, the normal displacement u_ν vanishes on Γ_3 at any time. We model the friction with the tresca friction law.

Let $\Omega \subset \mathbb{R}^N$ ($N = 1, 2, 3$) a bounded domain and whose boundary Γ assumed to be sufficiently smooth, is partitioned into three disjoint measurable parts Γ_1 , Γ_2 and Γ_3 . Let $\text{meas } \Gamma_1 > 0$, let $T > 0$ be a time.

Let M be a natural number. We consider the following mixed problem

$$\dot{\sigma} = \xi(\varepsilon(\dot{u})) + G(\sigma, \varepsilon(u), k, \theta) \text{ in } \Omega \times [0, T]. \quad (1)$$

$$\dot{k} = \phi(\varepsilon(u), \sigma, k, \theta) \text{ in } \Omega \times [0, T]. \quad (2)$$

$$\text{Div} \sigma + f_0 = 0 \text{ in } \Omega \times [0, T]. \quad (3)$$

$$u = 0 \text{ on } \Gamma_1 \times [0, T]. \quad (4)$$

$$\sigma \nu + \Phi(x)u = f_2 \text{ on } \Gamma_2 \times [0, T]. \quad (5)$$

$$\left. \begin{aligned} u_\nu &= 0, \quad \|\sigma_\tau\| \leq g, \\ \|\sigma_\tau\| < g &\Rightarrow \dot{u}_\tau = 0, \\ \|\sigma_\tau\| = g &\Rightarrow \exists \lambda > 0 \quad \sigma_\tau = -\lambda \dot{u}_\tau \end{aligned} \right\} \text{ on } \Gamma_3 \times [0, T]. \quad (6)$$

$$u(0) = u_0, \quad \sigma(0) = \sigma_0, \quad k(0) = k_0 \text{ in } \Omega. \quad (7)$$

Here the unknowns are the functions $u : \Omega \times (0, T) \rightarrow \mathbb{R}^N$, $\sigma : \Omega \times (0, T) \rightarrow S_N$ and $k : \Omega \times (0, T) \rightarrow \mathbb{R}^M$ in which k may be interpreted as an internal state variable and ξ , k and G are given constitutive functions. Φ represent rappel force.

In (1)-(7) u represents the displacement function, σ represents the stress function, $\varepsilon(u)$ denotes the small strain tensor, f_0 is given body force, f_2 is given boundary data, and u_0 , σ_0 and k_0 are the initial data. In (6), $g \geq 0$ is the friction bound function, i.e. magnitude of the limiting friction traction at which slip begins. The strict inequality in (6) holds in the stick zone and the equality holds in the slip zone. The equation (5) means that the Cauchy vector $\sigma \nu$ is proportional on the displacement.

Viscoplastic models of the form (1), (2) are used in order to model the behavior of real bodies for which the plastic rate of deformation depends also on an internal state variable.

Some of the internal state variables considered by many authors are the plastic strain, a number of tensor variables that takes into account the spatial display of dislocation for the internal state variables. Here we suppose that k is a vector-valued function which satisfies (2) where ϕ is a given function.

In order to obtain variational formulations for the problem (1)-(7), let us consider the following notations

$$\begin{aligned} H &= \{v = (v_i) / v_i \in \mathbb{L}^2(\Omega), i = \overline{1, N}\}, \\ H_1 &= \{v = (v_i) / v_i \in H^1(\Omega), i = \overline{1, N}\}, \\ V &= \{v \in H_1 : v = 0 \text{ a.e on } \Gamma_1\}, \\ V_1 &= \{v \in V / v_\nu = 0 \text{ a.e on } \Gamma_3\}, \\ Q &= \{\tau = (\tau_{ij}) : (\tau_{ij}) = (\tau_{ji}) \in \mathbb{L}^2(\Omega), 1 \leq i, j \leq p\}, \\ Q_1 &= \{\tau \in Q, \operatorname{Div} \tau \in H\}. \end{aligned}$$

The spaces H, H_1, V, V_1, Q, Q_1 are real Hilbert spaces endowed with the canonical inner products denoted by $\langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle_{H_1}, \langle \cdot, \cdot \rangle_V, \langle \cdot, \cdot \rangle_{V_1}, \langle \cdot, \cdot \rangle_Q, \langle \cdot, \cdot \rangle_{Q_1}$ respectively.

We assume that the force and traction satisfy

$$f_0 \in W^{1,\infty}(0, T, H), \quad f_2 \in W^{1,1}(0, T, H_{\Gamma_2}), \quad (8)$$

and the friction bound satisfies

$$g \in L^\infty(\Gamma_3), \quad g \geq 0 \text{ on } \Gamma_3, \quad \theta \in L^\infty(0, T, L^2(\Omega)^M). \quad (9)$$

Denote by $f(t)$ the element of V_1 giving by

$$\langle f(t), v \rangle_V = \int_{\Omega} f_0 \cdot v \, ds + \int_2 f_2 \cdot v \, da \quad \forall v \in V_1. \quad (10)$$

For all $t \in [0, T]$ let $j : V_1 \rightarrow \mathbb{R}_+$ be the functional

$$j(v) = \int_2 g \|v_\tau\| \, da.$$

For all $t \in [0, T]$, we note that conditions (8) and (9) imply

$$f \in W^{1,1}(0, T, V_1) \quad (11)$$

and

$$j(v) \leq c \|g\|_{\mathbb{L}(\Gamma_3)} \|v\| \quad \forall v \in V_1 \quad (12)$$

We also introduce the notation $\Sigma(t)$ defined by

$$\Sigma(t) = \left\{ \tau \in Q: \quad \langle \tau, \varepsilon(v) \rangle_Q + \langle \Phi u_0, v \rangle_{L^2(\Gamma_2)} + j(v) \geq \langle f(t), v \rangle_V, \forall v \in V_1 \right\}, \\ t \in \overline{[0, T]}. \quad (13)$$

Finally, assume that the initial data satisfy

$$u_0 \in V_1, \quad \sigma_0 \in \Sigma(0). \quad (14)$$

We have the following result

3. Variational Formulation

Lemma 1. *If u and σ are regular functions satisfying (3)-(6) then for all $t \in [0, T]$,*

$$u(t) \in V_1, \quad \langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_Q + \langle \Phi u(t), v - \dot{u}(t) \rangle_{L^2(\Gamma_2)^N} \\ + j(v) - j(\dot{u}(t)) \geq \langle f(t), v - \dot{u}(t) \rangle_V \quad \forall v \in V_1. \quad (15)$$

Proof. Let $t \in [0, T]$ and $v \in V_1$ be an arbitrary test function. Multiplying the equation (3) by $v - \dot{u}(t)$ and integrating by parts, we obtain

$$\langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_Q = \int_{\Omega} f_0(t) \cdot (v - \dot{u}(t)) ds + \int \sigma \nu \cdot (v - \dot{u}(t)) da.$$

Using the boundary conditions (4) and (5) we can replace the right hand side of the above equality by

$$\langle f(t), v - \dot{u}(t) \rangle_V - \int_2 \Phi u \cdot (v - \dot{u}(t)) da + \int_3 \sigma \nu \cdot (v - \dot{u}(t)) da.$$

From the contact boundary condition (6) we derive the relation

$$\sigma_{\tau}(t) \cdot \dot{u}_{\tau}(t) = -g \|\dot{u}_{\tau}(t)\| \quad \text{on } \Gamma_3.$$

Thus, on Γ_3 :

$$\begin{aligned} \sigma \nu \cdot (v - \dot{u}(t)) &= \sigma_{\nu}(t) \cdot (v_{\nu} - \dot{u}_{\nu}(t)) + \sigma_{\tau}(t) \cdot (v_{\tau} - \dot{u}_{\tau}(t)) \\ &= \sigma_{\tau}(t) \cdot v_{\tau} - \sigma_{\tau}(t) \cdot \dot{u}_{\tau}(t) \end{aligned}$$

$$\begin{aligned}
&= \sigma_\tau(t) \cdot v_\tau + g \|\dot{u}_\tau(t)\| \\
&\geq g(\|\dot{u}_\tau(t)\| - \|v_\tau\|).
\end{aligned}$$

Therefore

$$\int_3 \sigma \nu \cdot (v - \dot{u}(t)) da \geq j(\dot{u}(t)) - j(v),$$

and we have

$$\langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_Q + \langle \Phi w, v - \dot{u}(t) \rangle_{\mathbb{L}^2(\Gamma_2)^N} \geq \langle f(t), v - \dot{u}(t) \rangle_V + j(\dot{u}(t)) - j(v),$$

i.e. the inequality (12) holds true. $\square$

The previous lemma leads us to consider the following formulations of the frictional problem (1)-(7).

Problem 2. Find a displacement field $u : [0, T] \rightarrow \mathbb{V}_1$, a stress field $\sigma : [0, T] \rightarrow Q_1$, and $k : [0, T] \rightarrow \mathbb{R}^M$:

$$\dot{\sigma}(t) = \xi(\varepsilon(\dot{u})) + G(\sigma, \varepsilon(u), k, \theta) \text{ a.e. } t \in (0, T), \quad (16)$$

$$\dot{k} = \phi(\varepsilon(u), \sigma, k, \theta), \quad (17)$$

$$\begin{aligned}
&\langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_Q + \langle \Phi u(t), v - \dot{u}(t) \rangle_{\mathbb{L}^2(\Gamma_2)^N} \\
&\quad + j(v) - j(\dot{u}(t)) \geq \langle f(t), v - \dot{u}(t) \rangle_V \quad \forall v \in V_1, \quad (18) \\
&u(0) = u_0, \quad \sigma(0) = \sigma_0, \quad k(0) = k_0.
\end{aligned}$$

4. Existence and Uniqueness Results

In the study of the problem (1)-(7), we consider the following assumptions:

$$\left\{ \begin{array}{l} \xi : \Omega \times S_N \rightarrow S_N \\ \text{(a)} \quad \xi_{ijkh} \in L^\infty(\Omega) \text{ for all } i, j, k, h = \overline{0, N} \\ \text{(b)} \quad \xi \sigma \cdot \tau = \xi \cdot \sigma \tau \quad \forall \sigma, \tau \in S_N. \\ \text{(c)} \quad \exists \alpha > 0 / \xi \sigma \cdot \sigma \geq \alpha |\sigma|^2 \quad \forall \sigma \in S_N. \end{array} \right. \quad (19)$$

$$\left\{ \begin{array}{l} G : \Omega \times S_N \times S_N \times \mathbb{R}^M \times \mathbb{R}^M \rightarrow S_N \\ \text{(a)} \quad \exists \tilde{L} > 0 \text{ that } \forall \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in S_N, k_1, k_2, \theta_1, \theta_2 \in \mathbb{R}^M \text{ a.e in } \Omega \\ |G(x, \sigma_1, \varepsilon_1, k_1, \theta_1) - G(x, \sigma_2, \varepsilon_2, k_2, \theta_1)| \\ \leq \tilde{L}(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |k_1 - k_2| + |\theta_1 - \theta_2|) \\ \text{(b)} \quad x \rightarrow G(x, \sigma, \varepsilon, k, \theta) \text{ is measurable function with respect} \\ \text{to the Lebesgue measure on } \Omega \\ \text{for all } \sigma, \varepsilon \in S_N, k, \theta \in \mathbb{R}^M \\ \text{(c)} \quad x \rightarrow G(x, 0, 0, 0, 0) \in \mathbb{L}^2(\Omega)^{N \times N} \end{array} \right. \quad (20)$$

$$\left\{ \begin{array}{l} \phi : \Omega \times S_N \times S_N \times \mathbb{R}^M \times \mathbb{R}^M \rightarrow \mathbb{R}^M \text{ and} \\ \text{(a)} \exists C > 0 \text{ that } \forall \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in S_N, k_1, k_2, \theta_1, \theta_2 \in \mathbb{R}^M \text{ a.e in } \Omega \\ |\phi(x, \sigma_1, \varepsilon_1, k_1, \theta_1) - \phi(x, \sigma_2, \varepsilon_2, k_2, \theta_2)| \\ \leq C(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |k_1 - k_2| + |\theta_1 - \theta_2|) \\ \text{(b)} x \rightarrow \phi(x, \sigma, \varepsilon, k, \theta) \text{ is measurable function with respect} \\ \text{to the Lebesgue measure on } \Omega \\ \text{for all } \sigma, \varepsilon \in S_N, k, \theta \in \mathbb{R}^M \\ \text{(c)} x \rightarrow \phi(x, 0, 0, 0, 0) \in \mathbb{L}^2(\Omega)^M. \end{array} \right. \quad (21)$$

The mapping $\Phi : \Gamma_2 \rightarrow S_N$ is such that:

$$\left\{ \begin{array}{l} \Phi(x)u.u \geq 0 \text{ on } \Gamma_2. \\ \Phi_{km} \in L^\infty(\overline{\Gamma}_2), \quad \Phi_{km} = \Phi_{mk} \text{ for all } k, m = 1, \dots, N \end{array} \right. \quad (22)$$

$$k_0 \in \mathbb{L}^2(\Omega)^M. \quad (23)$$

The main result in this section is:

Theorem 1. Assume (8), (9), (14), (19) and (20)-(23). Then the problem (2) has a unique solution

$$u \in W^{1, \cdot}(0, T, V_1), \quad \sigma \in W^{1, \cdot}(0, T, Q_1), \quad k \in W^{1, \cdot}(0, T, \mathbb{L}^2(\Omega)^M). \quad (24)$$

Proof. We start by the existence part.

Let X be the product Hilbert space $X = Q \times \mathbb{L}^2(\Omega)^M$ and let $\eta = (\eta^1, \eta^2)$ in $C^0(0, T, X)$.

We define the function $Z_\eta = Z_\eta^1 + Z_\eta^2 \in C^0(0, T, X)$ by

$$Z_\eta(t) = \int_0^t \eta(s) ds + Z_0 \quad (25)$$

where

$$Z_0 = (\sigma_0 - \xi\varepsilon(u_0), k_0). \quad (26)$$

Consider the following auxiliary variational problem.

Problem 2. Find a displacement field $u_\eta : [0, T] \rightarrow \mathbb{V}_1$, a stress field $\sigma_\eta : [0, T] \rightarrow Q_1$, and $k \in W^{1, \cdot}(0, T, \mathbb{L}^2(\Omega)^M)$:

$$\sigma_\eta(t) = \xi(\varepsilon(u_\eta)) + Z_\eta^1(t) \text{ in } \Omega \times (0, T), \quad (27)$$

$$\begin{aligned} \langle \sigma_\eta(t), \varepsilon(v) - \varepsilon(\dot{u}_\eta(t)) \rangle_Q + \langle \Phi u(t), v - \dot{u}(t) \rangle_{\mathbb{L}^2(\Gamma_2)^N} + j(v) - j(\dot{u}_\eta(t)) \\ \geq \langle f(t), v - \dot{u}_\eta(t) \rangle_V \quad \forall v \in V_1, \end{aligned} \quad (28)$$

$$u_\eta(0) = u_0, \sigma_\eta(0) = \sigma_0, k(0) = k_0. \quad (29)$$

Lemma 2. *The variational problem (2) has a unique solution $u_\eta \in W^{1,1}(0, T, V_1)$, $\sigma_\eta \in W^{1,1}(0, T, Q_1)$.*

Proof. Define a bilinear form $a : V_1 \times V_1 \rightarrow \mathbb{R}$ by

$$a(u, v) = \langle \xi(\varepsilon(u)), \varepsilon(v) \rangle_Q + \langle \Phi u(t), v - \dot{u}(t) \rangle_{\mathbb{L}^2(\Gamma_2)^N} \quad \forall u, v \in V_1.$$

It follows from the assumptions (19) on ξ that $a(u, v)$ is continuous, symmetric and V_1 -elliptic. Also by (9) we see j is a continuous seminorm on V_1 .

Let $f_\eta : [0, T] \rightarrow V_1$ defined by

$$\langle f_\eta, v \rangle_V = \langle f(t), v \rangle_V - \langle Z_\eta^1(t), \varepsilon(v) \rangle_Q \quad \forall v \in V_1.$$

Since $f \in W^{1,1}([0, T], V_1)$ and $Z_\eta^1 \in W^{1,1}([0, T], Q)$, we have $f_\eta \in W^{1,1}([0, T], V_1)$.

Moreover, since $\sigma_0 = \xi\varepsilon(u_0) + Z_0^1 = \xi\varepsilon(u_0) + Z_\eta^1(0)$ by (26) and (25), the assumption $\sigma_0 \in \Sigma(0)$ from (14) can be written as

$$\langle \xi\varepsilon(u_0), \varepsilon(v) \rangle_Q + \langle Z_\eta^1(0), \varepsilon(v) \rangle_Q + j(v) \geq \langle f(0), v \rangle_V \quad \forall v \in V_1.$$

Applying a theorem in ([3] page70) we see that there exists a unique solution $u_\eta \in W^{1,1}(0, T, V_1)$ to the problem

$$a(u_\eta(t), v - \dot{u}_\eta(t)) + j(v) - j(\dot{u}_\eta(t)) \geq \langle f_\eta(t), v - \dot{u}_\eta(t) \rangle_V \quad \forall v \in V_1, \quad (30)$$

$$u_\eta(0) = u_0. \quad (31)$$

Let $\sigma_\eta \in W^{1,1}(0, T, Q)$ be given by (26). Then (u_η, σ_η) is a solution for (27)-(29). From inequality (28) we obtain $\text{Div}\sigma_\eta + f_0 = 0$ a. e in $\Omega \times (0, T)$. Therefore, $\sigma_\eta \in W^{1,1}(0, T, Q_1)$.

Let $k_\eta \in L^\infty(0, T, \mathbb{L}^2(\Omega)^M)$ be the function defined by

$$k_\eta = Z_\eta^2. \quad (32)$$

Finally, the uniqueness part of the lemma follows from the unique solvability of the variational inequality problem (30)-(31).

We consider next an operator $\Lambda : L^\infty(0, T, X) \rightarrow L^\infty(0, T, X)$ defined by

$$\Lambda\eta(t) = (G(\sigma_\eta(t), \varepsilon(u_\eta), k_\eta(t), \theta), \phi(\sigma_\eta(t), \varepsilon(u_\eta), k_\eta(t), \theta)) \quad \text{for all } t \in [0, T]. \quad (33)$$

Here $(u_\eta, \sigma_\eta, k_\eta)$ is solution of problem (3).

Lemma 3. *The operator Λ has a unique fixed point $\eta_* \in L^\infty(0, T, X)$.*

Proof. Let $\eta_1, \eta_2 \in L^\infty(0, T, X)$ and denote $u_i = u_{\eta_i}$, $\sigma_{i_i} = \sigma_{\eta_i}$, $Z_{i_i} = Z_{\eta_i}$, $Z_i = Z_i^1 + Z_i^2$ and $\eta_i = (\eta_i^1, \eta_i^2)$ for all $i = 1, 2$. Rewrite (27) and (28) for u_1 and u_2 as

$$a(u_1(t), v - \dot{u}_1(t)) + \langle Z_1^1, \varepsilon(v) - \varepsilon(\dot{u}_1(t)) \rangle_Q + j(v) - j(\dot{u}_1(t)) \geq \langle f(t), v - \dot{u}_1(t) \rangle_V,$$

$$a(u_2(t), v - \dot{u}_2(t)) + \langle Z_2^1, \varepsilon(v) - \varepsilon(\dot{u}_2(t)) \rangle_Q + j(v) - j(\dot{u}_2(t)) \geq \langle f(t), v - \dot{u}_2(t) \rangle_V.$$

For all $v \in V_1$, a. e on $(0, T)$. we take $v = \dot{u}_1$ in the first inequality, $v = \dot{u}_2$ in the second inequality, and add the two inequalities to obtain

$$a(u_2 - u_1, \dot{u}_2 - \dot{u}_1) \leq - \langle Z_1^1 - Z_2^1, \varepsilon(\dot{u}_1(t)) - \varepsilon(\dot{u}_2(t)) \rangle_Q \text{ a.e. on } (0, T).$$

Let $t \in \overline{[0, T]}$. Integrate the inequality from 0 to t

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} a(u_2 - u_1, u_2 - u_1) &\leq - \langle Z_1^1 - Z_2^1, \varepsilon(u_1(t)) - \varepsilon(u_2(t)) \rangle_Q \\ &\quad + \int_0^t \langle \eta_1^1(s) - \eta_2^1(s), \varepsilon(u_1(s)) - \varepsilon(u_2(s)) \rangle_Q ds. \end{aligned}$$

Then

$$\begin{aligned} c \|u_1(t) - u_2(t)\|_V^2 &\leq \|Z_1^1(t) - Z_2^1(t)\|_Q \|u_1(t) - u_2(t)\|_V \\ &\quad + \int_0^t \|\eta_1^1(s) - \eta_2^1(s)\|_Q \|u_1(s) - u_2(s)\|_V ds. \end{aligned}$$

Now

$$Z_1(t) - Z_2(t) = \int_0^t (\eta_1^1(s) - \eta_2^1(s)) ds.$$

So

$$\|Z_1^1(t) - Z_2^1(t)\|_Q \leq \int_0^t \|\eta_1^1(s) - \eta_2^1(s)\|_Q ds.$$

Therefore, we have

$$\|u_1(t) - u_2(t)\|_V^2 \leq c \int_0^t \|\eta_1^1(s) - \eta_2^1(s)\|_Q^2 ds + \int_0^t \|u_1(s) - u_2(s)\|_V^2 ds.$$

Applying the Gronwal inequality, we obtain

$$\|u_1(t) - u_2(t)\|_V^2 \leq c \int_0^t \|\eta_1^1(s) - \eta_2^1(s)\|_Q^2 ds. \quad (34)$$

By the definition (27) and (33) we have

$$\|\sigma_1(t) - \sigma_2(t)\|_Q^2 \leq c \int_0^t \|\eta_1^1(s) - \eta_2^1(s)\|_Q^2 ds. \quad (35)$$

Using (32) we have

$$\begin{aligned} \|k_1 - k_2\|_{\mathbb{L}^2(\Omega)^M} &= \|Z_1^2 - Z_2^2\|_{\mathbb{L}^2(\Omega)^M} \leq \|Z_1 - Z_2\|_{\mathbb{L}^2(\Omega)^M} \\ &\leq \int_0^t \|\eta_1^1(s) - \eta_2^1(s)\|_Q^2 ds. \end{aligned}$$

Finally,

$$\begin{aligned} \Lambda\eta_1(t) - \Lambda\eta_2(t) &= (G(\sigma_{\eta_1}(t), \varepsilon(u_{\eta_1}), K_{\eta_1}(t), \theta), \phi(\sigma_{\eta_1}(t), \varepsilon(u_{\eta_1}), K_{\eta_1}(t), \theta)) \\ &\quad - (G(\sigma_{\eta_2}(t), \varepsilon(u_{\eta_2}), K_{\eta_2}(t), \theta), \phi(\sigma_{\eta_2}(t), \varepsilon(u_{\eta_2}), K_{\eta_2}(t), \theta))). \end{aligned}$$

Using now (20), (21), (34) and (33) we get

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_X \leq \tilde{C} \int_0^t |\eta_1(s) - \eta_2(s)|_X ds \text{ for all } t \in (0, T). \quad (36)$$

Lemma (3) follows from (36) and an application of Banach's fixed point theorem.

Now we are ready to prove Theorem (1).

Proof of Theorem 1

Existence. Let $\eta^* \in L^\infty(0, T, X)$ be the fixed point of Λ and let $(u_\eta, \sigma_\eta, K_\eta)$ be the solution of problem (3) for $\eta = \eta^*$. Using (27) and (25) we have

$$\dot{\sigma}_\eta(t) = \xi(\varepsilon(\dot{u}_\eta(t))) + \eta^*(t) \text{ a.e. } t \in (0, T).$$

Since

$$\eta^*(t) = \Lambda\eta^*(t) = (G(\sigma_\eta(t), \varepsilon(u_\eta), K_\eta(t), \theta), \phi(\sigma_\eta(t), \varepsilon(u_\eta), K_\eta(t), \theta))$$

a.e. $t \in (0, T)$.

it follows that $(u_\eta, \sigma_\eta, K_\eta)$ satisfies (16). Using now (28) and (29) we conclude that is a solution of problem (2) with regularity (24).

Uniqueness. The uniqueness part follows from the uniqueness of the fixed point of the operator Λ .

5. The Dependence of the Solution on the Given Parameter

Theorem 2. *Under the assumptions, let (u_i, σ_i) , $(i = 1, 2)$ be the variational solution of the Problem 2 associated to parameter θ_i such that:*

Under the assumptions (19)-(21), let (u_i, σ_i) $(i = 1, 2)$ be the variational solution of the problem 2 associated to the parameter $\theta_i \in \mathbb{L}^2(\Omega)^M$ is satisfied. Then there exists a positive constant $C > 0$ depending to Ω , Γ_1 and Γ such that

$$|u_1 - u_2|_V + |\sigma_1 - \sigma_2|_Q + |k_1 - k_2|_{\mathbb{L}^2(\Omega)^M} \leq C |\theta_1 - \theta_2|_{\mathbb{L}^2(\Omega)^M}.$$

Proof. Let (u_i, σ_i) , $(i = 1, 2)$, be a variational solution of the Problem 2 then

$$\begin{aligned} \langle \sigma_i(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_Q + \langle \Phi u_i(t), v - \dot{u}(t) \rangle_{\mathbb{L}^2(\Gamma_2)^N} \\ + j(v) - j(\dot{u}(t)) \geq \langle f(t), v - \dot{u}(t) \rangle_V \quad \forall v \in V_1. \end{aligned} \quad (37)$$

where

$$\sigma_i(t) = \xi(\varepsilon(u_i)) + Z_i(t) \text{ a.e. } t \in (0, T), \quad (38)$$

$$Z_i(t) = \int_0^t G(\sigma_i(s), \varepsilon(u_i(s)), \theta_i(s)) ds + \sigma_0 - \xi \varepsilon(u_0). \quad (39)$$

Rewriting (37) and (38) for u_1 and u_2 , we obtain:

$$a(u_1, v - \dot{u}_1) + \langle Z_1(t), \varepsilon(v) - \varepsilon(\dot{u}_1(t)) \rangle_Q + j(v) - j(\dot{u}_1(t)) \geq \langle f(t), v - \dot{u}_1(t) \rangle_V,$$

$$a(u_2, v - \dot{u}_2) + \langle Z_2(t), \varepsilon(v) - \varepsilon(\dot{u}_2(t)) \rangle_Q + j(v) - j(\dot{u}_2(t)) \geq \langle f(t), v - \dot{u}_2(t) \rangle_V.$$

For all $v \in V_1$, a. e on $(0, T)$, we take $v = \dot{u}_1$ in the first inequality, $v = \dot{u}_2$ in the second inequality, and add the two inequalities to obtain

$$\frac{1}{2} \frac{d}{dt} a(u_1 - u_2, u_1 - u_2) \leq - \langle Z_1 - Z_2, \varepsilon(\dot{u}_1(t)) - \varepsilon(\dot{u}_2(t)) \rangle_Q \text{ a.e. on } (0, T).$$

Let $t \in \overline{[0, T]}$. Integrate the inequality from 0 to t

$$\begin{aligned} \frac{1}{2}a(u_2 - u_1, u_2 - u_1) &\leq -\langle Z_1 - Z_2, \varepsilon(u_1(t)) - \varepsilon(u_2(t)) \rangle_Q \\ &\quad + \int_0^t \langle \eta_1(s) - \eta_2(s), \varepsilon(u_1(s)) - \varepsilon(u_2(s)) \rangle_Q ds. \end{aligned}$$

Then

$$\begin{aligned} c \|u_1(t) - u_2(t)\|_V^2 &\leq \|Z_1(t) - Z_2(t)\|_Q \|u_1(t) - u_2(t)\|_V \\ &+ \int_0^t \|G(\sigma_1(s), \varepsilon(u_1(s)), \theta_1(s)) - G(\sigma_2(s), \varepsilon(u_2(s)), \theta_2(s))\|_Q \|u_1(s) - u_2(s)\|_V ds. \end{aligned}$$

Now

$$Z_1(t) - Z_2(t) = \int_0^t G(\sigma_1(s), \varepsilon(u_1(s)), \theta_1(s)) - G(\sigma_2(s), \varepsilon(u_2(s)), \theta_2(s)) ds.$$

So

$$\|Z_1(t) - Z_2(t)\|_Q \leq \int_0^t \|G(\sigma_1(s), \varepsilon(u_1(s)), \theta_1(s)) - G(\sigma_2(s), \varepsilon(u_2(s)), \theta_2(s))\|_Q ds.$$

Therefore, we have

$$\begin{aligned} \|u_1(t) - u_2(t)\|_V^2 &\leq c \int_0^t \|G(\sigma_1(s), \varepsilon(u_1(s)), \theta_1(s)) - G(\sigma_2(s), \varepsilon(u_2(s)), \theta_2(s))\|_Q^2 ds \\ &\quad + \int_0^t \|u_1(s) - u_2(s)\|_V^2 ds. \end{aligned}$$

Applying the Gronwal inequality, we obtain

$$\begin{aligned} \|u_1(t) - u_2(t)\|_V^2 &\leq c \int_0^t \|G(\sigma_1(s), \varepsilon(u_1(s)), \theta_1(s)) - G(\sigma_2(s), \varepsilon(u_2(s)), \theta_2(s))\|_Q^2 ds. \quad (40) \end{aligned}$$

By the definition (36) and (37) we have

$$\begin{aligned} \|\sigma_1(t) - \sigma_2(t)\|_Q^2 &\leq c \int_0^t \|G(\sigma_1(s), \varepsilon(u_1(s)), \theta_1(s)) - G(\sigma_2(s), \varepsilon(u_2(s)), \theta_2(s))\|_Q^2 ds. \quad (41) \end{aligned}$$

Finally, Using (20), and applying the Gronwal inequality, we obtain

$$|u_1 - u_2|_V + |\sigma_1 - \sigma_2|_Q + |k_1 - k_2|_{\mathbb{L}^2(\Omega)^M} \leq C |\theta_1 - \theta_2|_{\mathbb{L}^2(\Omega)^M}.$$

This theorem proves the dependence of the solution to the parameter θ and this result is very important from the mechanical point of view because it prove that small perturbations on the parameter θ gives small perturbations on the solution (u, σ) of the mechanical problem (1) – (7).

References

- [1] A. Bassam, Analyse variationnelle de quelques problèmes viscoélastiques et viscoplastiques avec frottement. Thèse de Doctorat d'Etat, Université de Perpignan (2001).
- [2] S. Boutechebak & N. Lebri, Frictionless contact problem non linear elasticity. Anal. Oradea. Fasc. Math. Tom XV (2008). 129-140
- [3] H. Brezis, Analyse fonctionnelle. Théorie et applications. Maison. Paris (1987).
- [4] O. Chau, Analyse variationnelle et numérique de quelques problèmes aux limites en mécanique du contact. Ph.D. Thesis, University of Perpignan, France (2000).
- [5] S. Djabi & M. Sofonea, A monotony method in quasi-static material with internal state variables. Univ Blaise Pascal (Clermont II) Série math. (1989).
- [6] S. Drabla, Analyse variationnelle de quelques problèmes aux limites en élasticité et en viscoplasticité. Ph.D. Thesis, University of Setif, Algeria (1999).
- [7] W. Han, M. Sofonea & J. M. Viaño, Variational and numerical analysis of a frictionless contact problem for elastic-viscoplastic materials with internal state variables. Quarterly Journal of Mathematics and Applied Mathematics 54, 501-522 (2001).
- [8] W. Han & M. Sofonea, Quasistatic Contact Problem in Viscoelasticity and viscoplasticity. December 14 (2000).
- [9] D. Klarbring, A. Mikelic & Shillor, Frictional contact problems with normal compliance. Int. J. Engng. Sci. 26 (1988), 811-832.

- [10] N. Lebri, S. Djabi & S. Boutechbek, Bilateral contact with Tresca's Friction law and with internal state variables. *Applied Mathematical Sciences*, Vol. 2 (2008), no. 10, 479-488.
- [11] N. Lebri & S. Djabi, Frictional bilateral contact in viscoplasticity. *Far East J. Math. Sci. (FJMS)* 29 (2), 467-479 (2008).
- [12] D. Motreanu & M. Sofonea, Quasivariational inequalities and applications in friction contact problem with normal compliance, *Advance in Mathematical Sciences and Applications* (2000).
- [13] M. Rochdi, Analyse variationnelle de quelques problèmes aux limites en viscoplasticité. Thèse de Doctorat d'Etat, Université de Perpignan (1997).
- [14] M. Sofonea, Problèmes non-linéaires dans la théorie de l'élasticité. Cours de Magister en Mathématiques Appliquées, Université F. Abbes, Sétif, Algérie (1993).
- [15] M. Sofonea & J. M. Viaño, A frictionless contact problem for elastic-viscoplastic materials with normal compliance. *Numerische Mathematik* 90, 689-919 (2002).
- [16] B. Teniou, Etude fonctionnelle des problèmes élasto-visco-plastiques. Ph.D. Thesis, University of Constantine, Algeria, (2000).