QUASI-IDEALS OF
A P-REGULAR NEAR LEFT ALMOST RINGS

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Abstract: S.J. Choi, P. Dheena and S. Manivasan studied property of quasi-ideals of P-regular nearring. In this page we study property of quasi-ideals of P-regular nLA-ring.

Key Words: nLA-ring, P-regular nLA-ring, quasi-ideal

1. Introduction

M.A. Kazim and MD. Naseeruddin [3, Proposition 2.1] asserted that, in every LA-semigroups $G$ a medial law hold

$$(ab)(cd) = (ac)(bd), \quad \forall a, b, c, d \in G.$$ 

Q. Mushtaq and M. Khan [5, p.322] asserted that, in every LA-semigroups $G$ with left identity

$$(ab)(cd) = (db)(ca), \quad \forall a, b, c, d \in G.$$ 

Further M. Khan, Faisal, and V. Amjid [4], asserted that, if a LA-semigroup $G$
with left identity the following law holds

\[ a(bc) = b(ac), \quad \forall a, b, c \in G. \]

M. Sarwar (Kamran) [7, p.112] defined LA-group as the following; a groupoid \( G \) is called a left almost group, abbreviated as LA-group, if (i) there exists \( e \in G \) such that \( ea = a \) for all \( a \in G \), (ii) for every \( a \in G \) there exists \( a' \in G \) such that, \( a'a = e \), (iii) \( (ab)c = (cb)a \) for every \( a, b, c \in G \).

Let \( \langle G, \cdot \rangle \) be an LA-group and \( S \) be a non-empty subset of \( G \) and \( S \) is itself and LA-group under the binary operation induced by \( G \), the \( S \) is called an LA-subgroup of \( G \), then \( S \) is called an LA-subgroup of \( \langle G, \cdot \rangle \).

S.M. Yusuf in [9, p.211] introduces the concept of a left almost ring (LA-ring). That is, a non-empty set \( R \) with two binary operations “+” and “·” is called a left almost ring, if \( \langle R, + \rangle \) is an LA-group, \( \langle R, \cdot \rangle \) is an LA-semigroup and distributive laws of “·” over “+” holds. T. Shah and I. Rehman [9, p.211] asserted that a commutative ring \( \langle R, +, \cdot \rangle \), we can always obtain an LA-ring \( \langle R, +, \cdot \rangle \) by defining, for \( a, b, c \in R \), \( a \oplus b = b - a \) and \( a \cdot b \) is same as in the ring. We can not assume the addition to be commutative in an LA-ring. An LA-ring \( \langle R, +, \cdot \rangle \) is said to be LA-integral domain if \( a \cdot b = 0, a, b \in R \), then \( a = 0 \) or \( b = 0 \). Let \( \langle R, +, \cdot \rangle \) be an LA-ring and \( S \) be a non-empty subset of \( R \) and \( S \) is itself and LA-ring under the binary operation induced by \( R \), the \( S \) is called an LA-subring of \( R \), then \( S \) is called an LA-subring of \( \langle R, +, \cdot \rangle \). If \( S \) is an LA-subring of an LA-ring \( \langle R, +, \cdot \rangle \), then \( S \) is called a left ideal of \( R \) if \( RS \subseteq S \). Right and two-sided ideals are defined in the usual manner.

By [6] a near-ring is a non-empty set \( N \) together with two binary operations “+” and “·” such that \( \langle N, + \rangle \) is a group (not necessarily abelian), \( \langle N, \cdot \rangle \) is a semigroup and one sided distributive (left or right) of “·” over “+” holds.

By [2] If a subgroup \( Q \) of \( \langle N, + \rangle \) has the property \( QN \cap NQ \subseteq Q \), then it is called a quasi-ideal of \( N \).

A near-ring \( N \) is a regular if for each \( x \in N \), there exists \( y \in N \) such that \( xyx = x \). A regular near-ring was introduced by J.C. Beidlemann in 1968 and later S. Leigh and H.E. Heatherly etc. studied the structure of a regular near-ring. Let \( N \) be a near-ring with the unity and \( P \) be an ideal of \( N \). Then the near-ring \( N \) is said to be a \( P \)-regular near-ring if for each \( a \in N \), there exists \( x \in N \) such that \( axa - a \in P \). If \( P = 0 \), then a \( P \)-regular near-ring is a regular near-ring. Hence the notion of \( P \)-regularity is a generalization of regularity.
2. Near Left Almost Rings

T. Shah, F. Rehman and M. Raees [8, pp.1103-1111] introduces the concept of a near left almost ring (nLA-ring).

**Definition 2.1.** [8, p.1104]. A non-empty set $N$ with two binary operation “$+$” and “$\cdot$” is called a near left almost ring (or simply an nLA-ring) if and only if

1. $\langle N, + \rangle$ is an LA-group.
2. $\langle N, \cdot \rangle$ is an LA-semigroup.
3. Left distributive property of $\cdot$ over $+$ holds, that is $a(b + c) = ab + ac$ for all $a, b, c \in N$.

**Definition 2.2.** [8, p.1105]. An nLA-ring $\langle N, +, \cdot \rangle$ with left identity $1$, such that $1a = a$ for all $a \in N$, is called an nLA-ring with left identity.

**Definition 2.3.** [8, p.1106]. A nonempty subset $S$ of an nLA-ring $N$ is said to be an nLA-subring if and only if $S$ is itself an nLA-ring under the same binary operations as in $N$.

**Theorem 2.4.** [8, p.1106]. A non-empty subset $S$ of an nLA-ring $\langle N, + \rangle$ is an nLA-subring if and only if $a - b \in S$ and $ab \in S$ for all $a, b \in S$.

**Definition 2.5.** [8, p.1107]. An nLA-subring $I$ of an nLA-ring $N$ is called a left ideal of $N$ if $NI \subseteq I$, and it is called a right ideal if for all $n, m \in N$ and $i \in I$ such that $(i + n)m - nm \in I$, and is called two sided ideal or simply ideal if it is both left and right ideal.

**Lemma 2.6.** Let $N$ be a nLA-ring and $Q$ is a nonempty subset of $N$. Then

$$QNQ \subseteq QN \cap NQ.$$  

*Proof.* We see that

$$QNQ \subseteq QN \quad \text{and} \quad QNQ \subseteq NQ.$$  

Thus $QNQ \subseteq QN \cap NQ$.  

**Lemma 2.7.** Let $N$ be a nLA-ring, $A$, $B$ and $C$ are a nonempty subset of $N$. Then

$$C(A \cap B) \subseteq CA \cap CB,$$
Proof. Let \( x \in C(A \cap B) \) then \( x = cy \) for some \( y \in A \cap B \) and \( c \in C \). Then \( x = cy \in CA \) and \( x = cy \in CB \), since \( y \in A \cap B \). Thus \( x \in CA \cap CB \). Hence \( C(A \cap B) \subseteq CA \cap CB \). □

**Lemma 2.8.** Let \( N \) be a nLA-ring, \( x, y \in N \) and \( A \) is a non-empty subset of \( N \) then
\[
A(x + y) \subseteq Ax + Ay
\]

Proof. Let \( c \in A(x + y) \) then \( c = a(x + y) \) for some \( a \in A \) By 2.1(3) then
\[
c = a(x + y) = ax + ay \in Ax + Ay.
\]
Thus \( A(x + y) \subseteq Ax + Ay \). □

3. Quasi-ideals of a \( P \)-Regular Near Left Almost Rings

Next we defines of a regular, quasi-ideal and \( P \)-regular in nLA-ring is defines the same as a regular, quasi-ideal and \( P \)-regular in near-ring in[2].

**Definition 3.1.** A nLA-ring \( N \) is called a regular nLA-ring if for each \( x \in N \) there exists \( y \in N \) such that \( xyx = x \).

**Definition 3.2.** If a LA-subgroup \( Q \) of \( \langle N, + \rangle \) has the property \( QN \cap NQ \subseteq Q \), then it is called a quasi-ideal of \( N \).

**Lemma 3.3.** Let \( N \) be a nLA-ring and \( Q_1, Q_2 \) are quasi-ideal of \( N \). Then \( Q_1 \cap Q_2 \) is a quasi-ideal of \( N \).

Proof. Since \( Q_1, Q_2 \) are LA-subgroup of \( \langle N, + \rangle \) we have \( Q_1 \cap Q_2 \) is a LA-subgroup of \( \langle N, + \rangle \). We must show that \( (Q_1 \cap Q_2)N \cap N(Q_1 \cap Q_2) \subseteq Q_1 \cap Q_2 \), by lemma 2.7. Then
\[
(Q_1 \cap Q_2)N \cap N(Q_1 \cap Q_2) \subseteq Q_1N \cap Q_1NQ_1 \cap NQ_2 = (Q_1N \cap NQ_1) \cap (Q_2N \cap NQ_2) \subseteq Q_1 \cap Q_2.
\]
Thus \( Q_1 \cap Q_2 \) is a quasi-ideal of \( N \). □

**Theorem 3.4.** Each quasi-ideal of an nLA-ring \( N \) is a nLA-subring.

Proof. Let \( Q \) be a quasi-ideal an nLA-ring \( N \) Then \( Q \) is a nLA-subring of \( \langle N, + \rangle \). Let \( a, b \in Q \subseteq N \). Then \( ab \in NQ \subseteq NQ \) and \( ab \in QN \subseteq QN \). Thus \( ab \in NQ \cap QN \subseteq Q \), since \( Q \) is a quasi-ideal of \( N \). Hence \( ab \in Q \). Therefore \( Q \) is a nLA-subring of \( N \). □
**Definition 3.5.** Let $N$ be a nLA-ring with the unity and $P$ be an ideal of $N$. Then the nLA-ring $N$ is said to be a $P$-regular nLA-ring if for each $a \in N$, there exists $x \in N$ such that $axa - a \in P$.

**Theorem 3.6.** Let $N$ be a $P$-regular nLA-ring. If $P = 0$, then a $P$-regular nLA-ring is a regular nLA-ring.

**Proof.** Let $N$ be a $P$-regular nLA-ring, then for each $n \in N$, there exists $x \in N$ such that $nxn - n \in P$ that is $nxn - n = P$, where $P$ is ideal of $N$.

If $P = 0$ then $nxn - n = 0$ implies that $nxn = n$. Thus $P$ is a regular nLA-ring.

The following theorems with proved is analogous as in [2, pp.1006-1010].

**Theorem 3.7.** Let $N$ be a $P$-regular nLA-ring. Then for each $n \in N$, there exists $n' \in N$ such that $n'n \in P$.

**Proof.** Let $N$ be a $P$-regular nLA-ring, then for each $n \in N$, there exists $x \in N$ such that $nxn - n \in P$, where $P$ is ideal of $N$. So $(nx - 1)n \in P$ and then put $n' = nx - 1$. Thus we obtain $n' \in N$.

**Theorem 3.8.** Let $N$ be a $P$-regular nLA-ring. Then for every left ideal $L$ and every right ideal $R$ of $N$, $(P + L) \cap (P + R) = P + LR$.

**Proof.** Suppose that $N$ is a $P$-regular nLA-ring, $L$ is a left ideal and $R$ is a right ideal of $N$. If $n \in (P + L) \cap (P + R)$ then element $n$ can be written as $n = p_1 + l$ and $n = p_2 + r$ for some $p_1, p_2 \in P, l \in L$ and $r \in R$. By definition $P$-regularity of $N$, $nxn - n \in P$ for some $x \in N$, which means that the element $n$ can also be expressed in the form $n = -p + nxn$ for some $p \in P$. From these one the obtains.

\[
\begin{align*}
n &= -p + nxn \\
&= -p + (p_1 + l)x(p_2 + r) \\
&= -p + (p_1 + l)(xp_2 + xr), \quad \text{by (3)} \\
&= -p + [(p_1 + l)xp_2] + [(p_1 + l)xr], \quad \text{by (3)} \\
&= -p + [(p_1 + l)xp_2] \\
&\quad + [(p_1 + l)xr - lxr] + lxr \\
&= -p + [(p_1 + l)xp_2] + p_3 + lxr, \quad p_3 = [(p_1 + l)xr - lxr] \in P \\
&= p_4 + lxr \in P + LR, \quad p_4 = -p + [(p_1 + l)xp_2] + p_3 \in P.
\end{align*}
\]

Hence $(P + L) \cap (P + R) \subseteq P + LR$.

For the converse, if $n \in P + LR$, then the element $n$ can be written as $n = p + lr$ for some $p \in P, l \in L$ and $r \in R$. Since $L$ is a left ideal and $R$ is a
right ideal of $N$, it is obvious that $n = p + lr$ belongs to $(P + L) \cap (P + R)$. Thus $P + LR \subseteq (P + L) \cap (P + R)$. Hence $(P + L) \cap (P + R) = P + LR$.

Now the question to be raised is what relationship is between a quasi-ideal and the ideal $P$ of a $P$-regular nLA-ring. It leads at once to the representation of elements of quasi-ideals of a $P$-regular nLA-ring in connection with the ideal $P$. So the coming theorems present several representations of elements of quasi-ideals of a $P$-regular nLA-ring.

**Theorem 3.9.** If $N$ is a $P$-regular nLA-ring, then every element of a quasi-ideal $Q$ of $N$ can be represented as the sum of two elements of $P$ and $Q$.

**Proof.** Let $N$ be a $P$-regular nLA-ring and $Q$ be a quasi-ideal of $N$. If $q \in Q$, then there exists $x \in N$ such that $qxq - q \in P$, where $P$ is ideal of $N$. So it has a representation $q = -p + qxq$ for some $p \in P$. By definition of a quasi-ideal, $qxq \in QNQ \subseteq QN \cap NQ \subseteq Q$ and therefore, we obtain $q = -p + qxq \in P + Q$.

**Theorem 3.10.** Let $N$ be an $P$-regular nLA-ring, $Q_1$ and $Q_2$ are quasi-ideals of $N$. If $q \in Q_1 \cap Q_2$, then the element $q$ can be represented as

$$q = p + q_1 x q_2$$

for some $p \in P, x \in N, q_1 \in Q_1$ and $q_2 \in Q_2$.

**Proof.** Suppose that $N$ be a $P$-regular nLA-ring, $Q_1$ and $Q_2$ are quasi-ideals of $N$. Then for each $q \in N$ there is $x \in N$ such that

$$qxq - q \in P,$$

where $P$ is ideal of $N$.

If $q \in Q_1 \cap Q_2$ then by lemma 3.3 thus $Q_1 \cap Q_2$ is a quasi-ideal of of $N$. By theorem 3.9, the element $q$ of $Q_1 \cap Q_2$ we can be written as both $q = p_1 + q_1$ and $q = p_2 + q_2$ for some $p_1, p_2 \in P, q_1 \in Q_1$ and $q_2 \in Q_2$. By $P$-regularity of $N$, the element $q$ also has the form $q = -p_3 + qxq$ for some $p_3 \in P$ and then it follows that

$$q = -p_3 + qxq = p_3 + (p_1 + q_1)(xp_2 + xq_2)$$

$$= -p_3 + (p_1 + q_1)(xp_2 + xq_2), \quad \text{by (3)}$$

$$= -p_3 + [(p_1 + q_1)xp_2] + [(p_1 + q_1)xq_2], \quad \text{by (3)}$$

$$= -p_3 + [(p_1 + q_1)xp_2] + [(p_1 + q_1)xq_2 - q_1 x q_2] + q_1 x q_2$$

$$= -p_3 + [(p_1 + q_1)xp_2] + p_4 + q_1 x q_2, \quad p_4 = (p_1 + q_1)xq_2 - q_1 x q_2 \in P$$

$$= p + q_1 x q_2, \quad p = -p_3 + [(p_1 + q_1)xp_2] + p_4 \in P.$$
Hence \( q = p + q_1 x q_2 \). \( \square \)

Next, by induction, Theorem 3.9 can be extended to the case of the intersection of finitely many quasi-ideals of a \( P \)-regular nLA-ring as follows.

**Theorem 3.11.** Let \( N \) be a \( P \)-regular nLA-ring, \( Q_i \) be a quasi-ideals of \( N \) for \( 1 \leq i \leq n \). If \( q \in \bigcap_{i=1}^{n} Q_i \), then the element \( q \) can be represented as

\[
q = p + q_1 x q_2 x q_3 x \cdots x q_{n-1} x q_n
\]

for some \( p \in P, x \in N, q_i \in Q_i, i = 1, 2, \ldots, n \).

**Proof.** Let \( N \) be a \( P \)-regular nLA-ring, \( Q_i \) be a quasi-ideals of \( N \) for \( 1 \leq i \leq n \). By induction on \( i \), if \( q \in Q_i \), then by Theorem 3.9, the element \( q \) can be represented as \( q = p + q_1 \) thus \( q_1 = -p + q \) for some \( p \in P \) and \( q_1 \in Q_1 \).

Assume that an element \( q \) of \( \bigcap_{i=1}^{n-1} Q_i \), can be represented as

\[
q = p_1 + q_1 x q_2 x q_3 x \cdots x q_{n-2} x q_{n-1}
\]

for some \( p_1 \in P, x \in N, q_i \in Q_i \) \((1 \leq i \leq n - 1)\). If \( q \in \bigcap_{i=1}^{n-1} Q_i \), then by the trivial inclusion \( \bigcap_{i=1}^{n} Q_i \subseteq \bigcap_{i=1}^{n-1} Q_i \) and the inductive assumption, the element \( q \) can be represented as \( q = p_1 + q_1 x q_2 x q_3 x \cdots x q_{n-1} \) for some \( p_1 \in P, x \in N \) and \( q_i \in Q_i \) \((1 \leq i \leq n - 1)\) by Theorem 3.9, it also has a representation \( q = p_2 + q_n \) for some \( p_2 \in P \) and \( q_n \in Q_n \).

Hence \( qxq = (p_1 + q_1 x q_2 x \cdots x q_{n-1}) x (p_2 + q_n) \) and by \( P \)-regularity of \( N \), the element \( q \) of \( \bigcap_{i=1}^{n-1} Q_i \) has another representation \( q = -p_3 + qxq \) for some \( p_3 \in P \) and \( x \in N \). So we have the following,

\[
q = -p_3 + qxq
= -p_3 + (p_1 + q_1 x q_2 x \cdots x q_{n-1}) x (p_2 + q_n)
= -p_3 + (p_1 + q_1 x q_2 x \cdots x q_{n-1}) (xp_2 + xq_n)
\]

by \((3)\)

\[
= -p_3 + [(p_1 + q_1 x q_2 x \cdots x q_{n-1})xp_2] + [(p_1 + q_1 x q_2 x \cdots x q_{n-1})xq_n]
\]

by \((3)\)

\[
= -p_3 + (p_1 + q_1 x q_2 x \cdots x q_{n-1})xp_2 + (p_1 + q_1 x q_2 x \cdots x q_{n-1})xq_n
- (q_1 x q_2 x \cdots x q_{n-1} x q_n) + (q_1 x q_2 x \cdots x q_{n-1} x q_n)
\]

\[
= -p_3 + (p_1 + q_1 x q_2 x \cdots x q_{n-1})xp_2 + [(p_1 + q_1 x q_2 x \cdots x q_{n-1})xq_n
- (q_1 x q_2 x \cdots x q_{n-1} x q_n)] + (q_1 x q_2 x \cdots x q_{n-1} x q_n)
\]

\[
= p + q_1 x q_2 x \cdots x q_{n-1} x q_n,
\]

where \( p = -p_3 + (p_1 + q_1 x q_2 x \cdots x q_{n-1})xp_2 + [(p_1 + q_1 x q_2 x \cdots x q_{n-1})xq_n
- (q_1 x q_2 x \cdots x q_{n-1} x q_n)] \in P. \)

Hence \( q = p + q_1 x q_2 x \cdots x q_{n-1} x q_n \), for \( p \in P \). \( \square \)
Theorem 3.12. If $N$ is a $P$-regular nLA-ring, then every quasi-ideal $Q$ of $N$ has the form

$$P + Q = P + QNQ = P + (QN \cap NQ).$$

Proof. Assume that $N$ is a $P$-regular nLA-ring and let $Q$ be a quasi-ideal of $N$. By lemma 2.6 then $QNQ \subseteq QN \cap NQ \subseteq Q$ holds and it leads to $P + QNQ \subseteq P + (QN \cap NQ) \subseteq P + Q$.

For the opposite direction, let $n \in P + Q$, then the element $n$ can be expressed as $n = p' + q'$ for some $p' \in P$ and $q' \in Q$. By $P$-regularity of $N$, then there exists $x \in N$ such that $q'xq' - q' \in P$. Thus $q'xq' - q' = p''$ for some $p'' \in P$ so $q' = -p'' + q'xq'$. Hence

$$n = p' + q' = p' + (-p'' + q'xq') = (p' - p'') + q'xq' \in P + QNQ.$$

Thus $P + Q \subseteq P + QNQ$, so $P + Q = P + QNQ$. Clearly $QN \cap NQ \subseteq QNQ$. Hence $P + Q = P + QNQ = P + (QN \cap NQ)$. \hfill \Box

Theorem 3.13. Let $N$ be a $P$-regular nLA-ring. If $Q_1$ and $Q_2$ are quasi-ideals of $N$, then

$$P + (Q_1 \cap Q_2) = P + (Q_1NQ_2 \cap Q_2NQ_1).$$

Proof. Suppose that $N$ is a $P$-regular nLA-ring. Let $Q_1$ and $Q_2$ be quasi-ideals of $N$. If $q \in P + (Q_1 \cap Q_2)$, then by Theorem 3.9, the element $q$ can be written as $q = p + q'$ for some $p \in P$ and $q' \in Q_1 \cap Q_2$. Also, by $P$-regularity of $N$, then there exists $x \in N$ such that $q'xq' - q' \in P$. Thus $q' = -p' + q'xq'$. Hence we have

$$q = p' + q' = p + (-p' + q'xq') = p - p' + q'xq' = p'' + q'xq' \in P + (Q_1NQ_2 \cap Q_2NQ_1),$$

where $p'' = p - p'$. Thus $P + (Q_1NQ_2 \cap Q_2NQ_1) \subseteq P + (Q_1NQ_2 \cap Q_2NQ_1)$.

Conversely, by the inclusion

$$Q_1NQ_2 \cap Q_2NQ_1 \subseteq (Q_1N \cap NQ_1) \cap (NQ_2 \cap Q_2N) \subseteq Q_1 \cap Q_2,$$

it is clear to have $P + (Q_1NQ_2 \cap Q_2NQ_1) \subseteq P + (Q_1 \cap Q_2)$. Thus $P + (Q_1 \cap Q_2) = P + (Q_1NQ_2 \cap Q_2NQ_1)$. \hfill \Box
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