QUASI-IDEALS OF A P -REGULAR NEAR LEFT ALMOST RINGS

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Abstract: S.J. Choi, P. Dheena and S. Manivasan studied property of quasi-ideals of P -regular nearring. In this page we study property of quasi-ideals of P -regular nLA-ring.

Key Words: nLA-ring, P -regular nLA-ring, quasi-ideal

1. Introduction

M.A. Kazim and MD. Naseeruddin [3, Proposition 2.1] asserted that, in every LA-semigroups G a *medial law* hold

$$(ab)(cd) = (ac)(bd), \quad \forall a, b, c, d \in G.$$

Q. Mushtaq and M. Khan [5, p.322] asserted that, in every LA-semigroups G with left identity

$$(ab)(cd) = (db)(ca), \quad \forall a, b, c, d \in G.$$

Further M. Khan, Faisal, and V. Amjid [4], asserted that, if a LA-semigroup G

with left identity the following law holds

$$a(bc) = b(ac), \quad \forall a, b, c \in G.$$

M. Sarwar (Kamran) [7, p.112] defined LA-group as the following; a groupoid G is called a left almost group, abbreviated as LA-group, if (i) there exists $e \in G$ such that $ea = a$ for all $a \in G$, (ii) for every $a \in G$ there exists $a' \in G$ such that, $a'a = e$, (iii) $(ab)c = (cb)a$ for every $a, b, c \in G$.

Let $\langle G, \cdot \rangle$ be an LA-group and S be a non-empty subset of G and S is itself and LA-group under the binary operation induced by G , the S is called an LA-subgroup of G , then S is called an LA-subgroup of $\langle G, \cdot \rangle$.

S.M. Yusuf in [9, p.211] introduces the concept of a left almost ring (LA-ring). That is, a non-empty set R with two binary operations “+” and “.” is called a left almost ring, if $\langle R, + \rangle$ is an LA-group, $\langle R, \cdot \rangle$ is an LA-semigroup and distributive laws of “.” over “+” holds. T. Shah and I. Rehman [9, p.211] asserted that a commutative ring $\langle R, +, \cdot \rangle$, we can always obtain an LA-ring $\langle R, \oplus, \cdot \rangle$ by defining, for $a, b, c \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. We can not assume the addition to be commutative in an LA-ring. An LA-ring $\langle R, +, \cdot \rangle$ is said to be LA-integral domain if $a \cdot b = 0$, $a, b \in R$, then $a = 0$ or $b = 0$. Let $\langle R, +, \cdot \rangle$ be an LA-ring and S be a non-empty subset of R and S is itself and LA-ring under the binary operation induced by R , the S is called an LA-subring of R , then S is called an LA-subring of $\langle R, +, \cdot \rangle$. If S is an LA-subring of an LA-ring $\langle R, +, \cdot \rangle$, then S is called a left ideal of R if $RS \subseteq S$. Right and two-sided ideals are defined in the usual manner.

By [6] a near-ring is a non-empty set N together with two binary operations “+” and “.” such that $\langle N, + \rangle$ is a group (not necessarily abelian), $\langle N, \cdot \rangle$ is a semigroup and one sided distributive (left or right) of “.” over “+” holds.

By [2] If a subgroup Q of $\langle N, + \rangle$ has the property $QN \cap NQ \subseteq Q$, then it is called a quasi-ideal of N .

A near-ring N is a regular if for each $x \in N$, there exists $y \in N$ such that $xyx = x$. A regular near-ring was introduced by J.C. Beidleman in 1968 and later S. Leigh and H.E. Heatherly etc. studied the structure of a regular near-ring. Let N be a near-ring with the unity and P be an ideal of N . Then the near-ring N is said to be a P -regular near-ring if for each $a \in N$, there exists $x \in N$ such that $axa - a \in P$. If $P = 0$, then a P -regular near-ring is a regular near-ring. Hence the notion of P -regularity is a generalization of regularity.

2. Near Left Almost Rings

T. Shah, F. Rehman and M. Raees [8, pp.1103-1111] introduces the concept of a near left almost ring (nLA-ring).

Definition 2.1. [8, p.1104]. A non-empty set N with two binary operation “+” and “ $\cdot$ ” is called a near left almost ring (or simply an nLA-ring) if and only if

- (1) $\langle N, + \rangle$ is an LA-group.
- (2) $\langle N, \cdot \rangle$ is an LA-semigroup.
- (3) Left distributive property of $\cdot$ over $+$ holds, that is $a(b + c) = ab + ac$ for all $a, b, c \in N$.

Definition 2.2. [8, p.1105]. An nLA-ring $\langle N, +, \cdot \rangle$ with left identity 1, such that $1a = a$ for all $a \in N$, is called an nLA-ring with left identity.

Definition 2.3. [8, p.1106]. A nonempty subset S of an nLA-ring N is said to be an nLA-subring if and only if S is itself an nLA-ring under the same binary operations as in N .

Theorem 2.4. [8, p.1106]. A non-empty subset S of an nLA-ring $\langle N, + \rangle$ is an nLAsubring if and only if $a - b \in S$ and $ab \in S$ for all $a, b \in S$.

Definition 2.5. [8, p.1107]. An nLA-subring I of an nLA-ring N is called a left ideal of N if $NI \subseteq I$, and I is called a right ideal if for all $n, m \in N$ and $i \in I$ such that $(i + n)m - nm \in I$, and is called two sided ideal or simply ideal if it is both left and right ideal.

Lemma 2.6. Let N be a nLA-ring and Q is a nonempty subset of N . Then

$$QNQ \subseteq QN \cap NQ.$$

Proof. We see that

$$QNQ \subseteq QN \quad \text{and} \quad QNQ \subseteq NQ.$$

Thus $QNQ \subseteq QN \cap NQ$. □

Lemma 2.7. Let N be a nLA-ring, A, B and C are a nonempty subset of N . Then

$$C(A \cap B) \subseteq CA \cap CB,$$

Proof. Let $x \in C(A \cap B)$ then $x = cy$ for some $y \in A \cap B$ and $c \in C$. Then $x = cy \in CA$ and $x = cy \in CB$, since $y \in A \cap B$. Thus $x \in CA \cap CB$. Hence $C(A \cap B) \subseteq CA \cap CB$ $\square$

Lemma 2.8. *Let N be a nLA-ring, $x, y \in N$ and A is a non-empty subset of N then*

$$A(x + y) \subseteq Ax + Ay$$

Proof. Let $c \in A(x + y)$ then $c = a(x + y)$ for some $a \in A$ By 2.1(3) then

$$c = a(x + y) = ax + ay \in Ax + Ay.$$

Thus $A(x + y) \subseteq Ax + Ay$. $\square$

3. Quasi-ideals of a P -Regular Near Left Almost Rings

Next we defines of a regular, quasi-ideal and P -regular in nLA-ring is defines the same as a regular, quasi-ideal and P -regular in near-ring in[2].

Definition 3.1. A nLA-ring N is called a regular nLA-ring if for each $x \in N$ there exists $y \in N$ such that $xyx = x$.

Definition 3.2. If a LA-subgroup Q of $\langle N, + \rangle$ has the property $QN \cap NQ \subseteq Q$, then it is called a quasi-ideal of N .

Lemma 3.3. *Let N be a nLA-ring and Q_1, Q_2 are quasi-ideal of N . Then $Q_1 \cap Q_2$ is a quasi-ideal of N .*

Proof. Since Q_1, Q_2 are LA-subgroup of $\langle N, + \rangle$ we have $Q_1 \cap Q_2$ is a LA-subgroup of $\langle N, + \rangle$. We must show that $(Q_1 \cap Q_2)N \cap N(Q_1 \cap Q_2) \subseteq Q_1 \cap Q_2$, by lemma 2.7. Then

$$\begin{aligned} (Q_1 \cap Q_2)N \cap N(Q_1 \cap Q_2) &\subseteq Q_1N \cap Q_2N \cap NQ_1 \cap NQ_2 \\ &= (Q_1N \cap NQ_1) \cap (Q_2N \cap NQ_2) \\ &\subseteq Q_1 \cap Q_2. \end{aligned}$$

Thus $Q_1 \cap Q_2$ is a quasi-ideal of N . $\square$

Theorem 3.4. *Each quasi-ideal of an nLA-ring N is a nLA-subring.*

Proof. Let Q be a quasi-ideal an nLA-ring N Then Q is a nLA-subring of $\langle N, + \rangle$. Let $a, b \in Q \subseteq N$. Then $ab \in NQ \subseteq NQ$ and $ab \in QN \subseteq QN$. Thus $ab \in NQ \cap QN \subseteq Q$, since Q is a quasi-ideal of N . Hence $ab \in Q$. Therefore Q is a nLA-subring of N . $\square$

Definition 3.5. Let N be a nLA-ring with the unity and P be an ideal of N . Then the nLA-ring N is said to be a P -regular nLA-ring if for each $a \in N$, there exists $x \in N$ such that $axa - a \in P$

Theorem 3.6. Let N be a P -regular nLA-ring. If $P = 0$, then a P -regular nLA-ring is a regular nLA-ring.

Proof. Let N be a P -regular nLA-ring, then for each $n \in N$, there exists $x \in N$ such that $nxn - n \in P$ that is $nxn - n = P$, where P is ideal of N .

If $P = 0$ then $nxn - n = 0$ implies that $nxn = n$. Thus P is a regular nLA-ring. $\square$

The following theorems with proved is analogous as in [2, pp.1006-1010].

Theorem 3.7. Let N be a P -regular nLA-ring. Then for each $n \in N$, there exists $n' \in N$ such that $n'n \in P$.

Proof. Let N be a P -regular nLA-ring, then for each $n \in N$, there exists $x \in N$ such that $nxn - n \in P$, where P is ideal of N . So $(nx - 1)n \in P$ and then put $n' = nx - 1$. Thus we obtain $n'n \in P$. $\square$

Theorem 3.8. Let N be a P -regular nLA-ring. Then for every left ideal L and every right ideal R of N , $(P + L) \cap (P + R) = P + LR$.

Proof. Suppose that N is a P -regular nLA-ring, L is a left ideal and R is a right ideal of N . If $n \in (P + L) \cap (P + R)$ then element n can be written as $n = p_1 + l$ and $n = p_2 + r$ for some $p_1, p_2 \in P, l \in L$ and $r \in R$. By definition P -regularity of N , $nxn - n \in P$ for some $x \in N$, which means that the element n can also be expressed in the form $n = -p + nxn$ for some $p \in P$. From these one the obtains.

$$\begin{aligned}
 n &= -p + nxn \\
 &= -p + (p_1 + l)x(p_2 + r) \\
 &= -p + (p_1 + l)(xp_2 + xr), && \text{by (3)} \\
 &= -p + [(p_1 + l)xp_2] + [(p_1 + l)xr], && \text{by (3)} \\
 &= -p + [(p_1 + l)xp_2] \\
 &\quad + [(p_1 + l)xr - lxr] + lxr \\
 &= -p + [(p_1 + l)xp_2] + p_3 + lxr, && p_3 = [(p_1 + l)xr - lxr] \in P \\
 &= p_4 + lxr \in P + LR, && p_4 = -p + [(p_1 + l)xp_2] + p_3 \in P.
 \end{aligned}$$

Hence $(P + L) \cap (P + R) \subseteq P + LR$.

For the converse, if $n \in P + LR$, then the element n can be written as $n = p + lr$ for some $p \in P, l \in L$ and $r \in R$. Since L is a left ideal and R is a

right ideal of N , it is obvious that $n = p + lr$ belongs to $(P + L) \cap (P + R)$. Thus $P + LR \subseteq (P + L) \cap (P + R)$. Hence $(P + L) \cap (P + R) = P + LR$. $\square$

Now the question to be raised is what relationship is between a quasi-ideal and the ideal P of a P -regular nLA-ring. It leads at once to the representation of elements of quasi-ideals of a P -regular nLA-ring in connection with the ideal P . So the coming theorems present several representations of elements of quasi-ideals of a P -regular nLA-ring.

Theorem 3.9. *If N is a P -regular nLA-ring, then every element of a quasi-ideal Q of N can be represented as the sum of two elements of P and Q .*

Proof. Let N be a P -regular nLA-ring and Q be a quasi-ideal of N . If $q \in Q$, then there exists $x \in N$ such that $qxq - q \in P$, where P is ideal of N . So it has a representation $q = -p + qxq$ for some $p \in P$. By definition of a quasi-ideal, $qxq \in QNQ \subseteq QN \cap NQ \subseteq Q$ and therefore, we obtain $q = -p + qxq \in P + Q$. $\square$

Theorem 3.10. *Let N be an P -regular nLA-ring, Q_1 and Q_2 are quasi-ideals of N . If $q \in Q_1 \cap Q_2$, then the element q can be represented as*

$$q = p + q_1 x q_2$$

for some $p \in P, x \in N, q_1 \in Q_1$ and $q_2 \in Q_2$.

Proof. Suppose that N be a P -regular nLA-ring, Q_1 and Q_2 are quasi-ideals of N . Then for each $q \in N$ there is $x \in N$ such that

$$qxq - q \in P,$$

where P is ideal of N .

If $q \in Q_1 \cap Q_2$ then by lemma 3.3 thus $Q_1 \cap Q_2$ is a quasi-ideal of N . By theorem 3.9, the element q of $Q_1 \cap Q_2$ we can be written as both $q = p_1 + q_1$ and $q = p_2 + q_2$ for some $p_1, p_2 \in P, q_1 \in Q_1$ and $q_2 \in Q_2$. By P -regularity of N , the element q also has the form $q = -p_3 + qxq$ for some $p_3 \in P$ and then it follows that

$$\begin{aligned} q &= -p_3 + qxq = p_3 + (p_1 + q_1)x(p_2 + q_2) \\ &= -p_3 + (p_1 + q_1)(xp_2 + xq_2), && \text{by (3)} \\ &= -p_3 + [(p_1 + q_1)xp_2] + [(p_1 + q_1)xq_2], && \text{by (3)} \\ &= -p_3 + [(p_1 + q_1)xp_2] \\ &\quad + [(p_1 + q_1)xq_2 - q_1xq_2] + q_1xq_2 \\ &= -p_3 + [(p_1 + q_1)xp_2] + p_4 + q_1xq_2, && p_4 = (p_1 + q_1)xq_2 - q_1xq_2 \in P \\ &= p + q_1xq_2, && p = -p_3 + [(p_1 + q_1)xp_2] + p_4 \in P. \end{aligned}$$

Hence $q = p + q_1 x q_2$. □

Next, by induction, Theorem 3.9 can be extended to the case of the intersection of finitely many quasi-ideals of a P -regular nLA-ring as follows.

Theorem 3.11. *Let N be a P -regular nLA-ring, Q_i be a quasi-ideals of N for $1 \leq i \leq n$. If $q \in \bigcap_{i=1}^n Q_i$, then the element q can be represented as*

$$q = p + q_1 x q_2 x q_3 x \cdots x q_{n-1} x q_n$$

for some $p \in P, x \in N, q_i \in Q_i, i = 1, 2, \dots, n$.

Proof. Let N be a P -regular nLA-ring, Q_i be a quasi-ideals of N for $1 \leq i \leq n$. By induction on i , if $q \in Q_i$, then by Theorem 3.9, the element q can be represented as $q = p + q_1$ thus $q_1 = -p + q$ for some $p \in P$ and $q_1 \in Q_1$.

Assume that an element q of $\bigcap_{i=1}^{n-1} Q_i$ can be represented as

$$q = p_1 + q_1 x q_2 x q_3 x \cdots x q_{n-2} x q_{n-1}$$

for some $p_1 \in P, x \in N, q_i \in Q_i$ ($1 \leq i \leq n-1$). If $q \in \bigcap_{i=1}^{n-1} Q_i$, then by the trivial inclusion $\bigcap_{i=1}^n Q_i \subseteq \bigcap_{i=1}^{n-1} Q_i$ and the inductive assumption, the element q can be represented as $q = p_1 + q_1 x q_2 x q_3 x \cdots x q_{n-1}$ for some $p_1 \in P, x \in N$ and $q_i \in Q_i$ ($1 \leq i \leq n-1$) by Theorem 3.9, it also has a representation $q = p_2 + q_n$ for some $p_2 \in P$ and $q_n \in Q_n$.

Hence $q x q = (p_1 + q_1 x q_2 x \cdots x q_{n-1}) x (p_2 + q_n)$ and by P -regularity of N , the element q of $\bigcap_{i=1}^{n-1} Q_i$ has another representation $q = -p_3 + q x q$ for some $p_3 \in P$ and $x \in N$. So we have the following,

$$\begin{aligned} q &= -p_3 + q x q \\ &= -p_3 + (p_1 + q_1 x q_2 x \cdots x q_{n-1}) x (p_2 + q_n) \\ &= -p_3 + (p_1 + q_1 x q_2 x \cdots x q_{n-1}) (x p_2 + x q_n) && \text{by (3)} \\ &= -p_3 + [(p_1 + q_1 x q_2 x \cdots x q_{n-1}) x p_2] + [(p_1 + q_1 x q_2 x \cdots x q_{n-1}) x q_n] && \text{by (3)} \\ &= -p_3 + (p_1 + q_1 x q_2 x \cdots x q_{n-1}) x p_2 + (p_1 + q_1 x q_2 x \cdots x q_{n-1}) x q_n \\ &\quad - (q_1 x q_2 x \cdots x q_{n-1} x q_n) + (q_1 x q_2 x \cdots x q_{n-1} x q_n) \\ &= -p_3 + (p_1 + q_1 x q_2 x \cdots x q_{n-1}) x p_2 + [(p_1 + q_1 x q_2 x \cdots x q_{n-1}) x q_n \\ &\quad - (q_1 x q_2 x \cdots x q_{n-1} x q_n)] + (q_1 x q_2 x \cdots x q_{n-1} x q_n) \\ &= p + q_1 x q_2 x \cdots x q_{n-1} x q_n, \end{aligned}$$

where $p = -p_3 + (p_1 + q_1 x q_2 x \cdots x q_{n-1}) x p_2 + [(p_1 + q_1 x q_2 x \cdots x q_{n-1}) x q_n - (q_1 x q_2 x \cdots x q_{n-1} x q_n)] \in P$.

Hence $q = p + q_1 x q_2 x \cdots x q_{n-1} x q_n$, for $p \in P$. □

Theorem 3.12. *If N is a P -regular nLA-ring, then every quasi-ideal Q of N has the form*

$$P + Q = P + QNQ = P + (QN \cap NQ).$$

Proof. Assume that N is a P -regular nLA-ring and let Q be a quasi-ideal of N . By lemma 2.6 then $Q \cap NQ \subseteq Q$ holds and it leads to $P + QNQ \subseteq P + (Q \cap NQ) \subseteq P + Q$.

For the opposite direction, let $n \in P + Q$, then the element n can be expressed as $n = p' + q'$ for some $p' \in P$ and $q' \in Q$. By P -regularity of N , then there exists $x \in N$ such that $q'xq' - q' \in P$. Thus $q'xq' - q' = p''$ for some $p'' \in P$ so $q' = -p'' + q'xq'$. Hence

$$n = p' + q' = p' + (-p'' + q'xq') = (p' - p'') + q'xq' \in P + QNQ.$$

Thus $P + Q \subseteq P + QNQ$, so $P + Q = P + QNQ$. Clearly $QN \cap NQ \subseteq QNQ$. Hence $P + Q = P + QNQ = P + (QN \cap NQ)$. $\square$

Theorem 3.13. *Let N be a P -regular nLA-ring. If Q_1 and Q_2 are quasi-ideals of N , then*

$$P + (Q_1 \cap Q_2) = P + (Q_1NQ_2 \cap Q_2NQ_1).$$

Proof. Suppose that N is a P -regular nLA-ring. Let Q_1 and Q_2 be quasi-ideals of N . If $q \in P + (Q_1 \cap Q_2)$, then by Theorem 3.9, the element q can be written as $q = p + q'$ for some $p \in P$ and $q' \in Q_1 \cap Q_2$. Also, by P -regularity of N , then there exists $x \in N$ such that $q'xq' - q' \in P$. Thus $q' = -p' + q'xq'$. Hence we have

$$\begin{aligned} q &= p + q' = p + (-p' + q'xq') \\ &= p - p' + q'xq' = p'' + q'xq' \in P + (Q_1NQ_2 \cap Q_2NQ_1), \end{aligned}$$

where $p'' = p - p'$. Thus $P + (Q_1 \cap Q_2) \subseteq P + (Q_1NQ_2 \cap Q_2NQ_1)$.

Conversely, by the inclusion

$$Q_1NQ_2 \cap Q_2NQ_1 \subseteq (Q_1N \cap NQ_1) \cap (NQ_2 \cap Q_2N) \subseteq Q_1 \cap Q_2,$$

it is clear to have $P + (Q_1NQ_2 \cap Q_2NQ_1) \subseteq P + (Q_1 \cap Q_2)$. Thus $P + (Q_1 \cap Q_2) = P + (Q_1NQ_2 \cap Q_2NQ_1)$. $\square$

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References

- [1] S.J. Choi, Quasiideals of a P -regular near-rings, *International Journal of Algebra*, **4** (2010), 501-506.
- [2] P. Dheena, S. Manivasan, Quasiideals of a P -regular near-rings, *International Journal of Algebra*, **5** (2011), 1005-1010.
- [3] M.A. Kazim, M.D. Naseeruddin, On almost semigroup, *Portugaliae Mathematica*, **36** (1977), 41-47.
- [4] M. Khan, Faisal, V. Amjid, On some classes of Abel-Grassmann's groupoids, *arXiv*: 1010.5965v2, **2** (2010), 1-6.
- [5] Q. Mushtaq, M. Khan, M-System in LA-semigroups, *Southeast Asian Bulletin of Mathematics*, **33** (2009), 321-327.
- [6] G. Pilz, *Near-rings*, Noth Holland, Amesterdam-New York (1997).
- [7] M. Sarwar (Kamran), *Conditions for LA-Semigroup to Resemble Associative Structures*, PhD Thesis, Quaid-i-Azam University (1993).
- [8] T. Shah, F. Rehman, M. Raees, On near left alomst rings, *International Mathematical Forum*, **6** (2011), 1103-1111.
- [9] T. Shah, I. Rehman, On LA-rings of finitely nonzero function, *Int. J. Contemp. Math. Sciences*, **5** (2010), 209-222.