**RECURRENCE RELATIONS FOR SINGLE AND PRODUCT
MOMENTS OF LOWER GENERALIZED ORDER STATISTICS
FROM A GENERAL FORM OF DISTRIBUTIONS AND
ITS CHARACTERIZATIONS**

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Abstract: In this paper, we present recurrence relations for moments of lower generalized order statistics within a general form of doubly truncated distributions. Characterizations for the general form of doubly truncated distributions are studied. This the general form of distributions includes distributions such as doubly truncated inverted Weibull, inverted Gompertz, generalized logistic, Burr type X, Burr type XII, logistic, inverted Pareto, inverted compound Weibull, Gumbel and compound Gompertz, among others. Doubly truncated inverted Weibull, log-inverse generalized Weibull and inverted Pareto distribution are given as applications to illustrate the results.

Key Words: generalized order statistics, doubly truncated distribution, or-

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1. Introduction

A random variable (r.v) X is said to have a distribution function out of the following general form of absolutely continuous distributions if its cdf and pdf are given, respectively, by

$$F(x) = \exp(-\theta/\lambda(x)), \quad \alpha < x < \beta, \quad (1)$$

and

$$f(x) = \frac{\theta\lambda(x)}{\lambda^2(x)} \exp(-\theta/\lambda(x)), \quad \alpha < x < \beta, \quad (2)$$

where $\lambda(x)$ is a nonnegative, strictly increasing and differentiable function of x , such that: $\lambda(x) \rightarrow 0$ as $x \rightarrow \alpha$ and $\lambda(x) \rightarrow \infty$ as $x \rightarrow \beta$.

The pdf of the doubly truncated case is given by

$$f_d(x) = \frac{f(x)}{Q - P}, \quad \alpha < P_1 \leq x \leq Q_1 < \beta,$$

where $P = F(P_1)$, $Q = F(Q_1)$ and $P_2 = P/(Q - P)$.

Thus, the cdf of the doubly truncated case (truncated from the left at P_1 and from the right Q_1) of Eqs. (1) and (2) are given by

$$F_d(x) = \frac{1}{Q - P} \exp(-\theta/\lambda(x)) - P_2, \quad \alpha < P_1 \leq x \leq Q_1 < \beta. \quad (3)$$

It is easy too see that

$$f_d(x) = \frac{\theta\lambda(x)}{\lambda^2(x)} \{F_d(x) + P_2\}. \quad (4)$$

The random variables $X'(r, n, \overline{m}, k)$, $1 \leq r \leq n$, are said to be lower generalized order statistics (lgos) from a distribution function F if their joint density function is given by (see Ref. [12])

$$f_{1,2,\dots,n}(x_1, x_2, \dots, x_n) = k \left(\prod_{i=1}^{n-1} \gamma_i \right) \left(\prod_{i=1}^{n-1} (F(x_i))^{m_i} f(x_i) \right) (F(x_n))^{k-1} f(x_n), \quad (5)$$

of $F^{-1}(1) > x_1 > x_2 > \dots > x_n > F^{-1}(0)$, with parameters $n \in \mathbb{N}$, $\overline{m} = (m_1, \dots, m_{n-1}) \in \mathbb{R}^{n-1}$, $k \geq 1$ such that $\gamma_i = k + n - i + M_i > 0$, $M_i = \sum_{j=i}^{n-1} m_j$ for all $1 \leq i \leq n - 1$.

The one and two dimensional marginal pdf of lgos are given, respectively, by (see Ref. [14])

$$f_{X'_r}(x) = C_{r-1} \sum_{i=1}^r a_i(r) (F(x))^{\gamma_i-1} f(x), \quad 1 \leq r \leq n, \quad (6)$$

and

$$\begin{aligned} & f_{X'_r, X'_s}(x, y) \\ &= C_{s-1} \left[\sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \right] \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i} \right] \frac{f(x)f(y)}{F(x)F(y)}, \end{aligned} \quad (7)$$

where $x > y$, $1 \leq r < s \leq n$, $X'_i \equiv X'(i, n, \overline{m}, k)$, $1 \leq i \leq n$,

$$C_{\ell-1} = \prod_{i=1}^{\ell} \gamma_i, \quad a_i^{(r)}(s) = \prod_{j=r+1, j \neq i}^s \frac{1}{\gamma_j - \gamma_i} \quad \text{and} \quad a_i(s) = a_i^{(0)}(s).$$

The conditional pdf of X'_s given X'_r , $1 \leq r < s \leq n$, is given by

$$f_{X'_s|X'_r}(y|x) = \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \frac{f(y)}{F(y)}, \quad y < x. \quad (8)$$

Ref. [14] introduced a new model of generalized order statistics as a unification for several models of ordered random variables (e.g., ordinary order statistics, k-record values and progressively Type II censoring). There is some research regarding gos and lgos, for example: Refs. [4] and [5] obtained recurrence relations for single moments of gos's within a class of doubly truncated distributions. Refs. [6] and [19] characterized the uniform and power function distributions based on distributional properties of lgos's respectively. Ref. [9] defined and proposed that enables a common approach to descendingly ordered random variables like reversed ordered order statistics, lower k-records and lower Pfeifer records. Ref. [16] derived recurrence relations for moments of lower generalized order statistics from exponentiated Weibull distribution. Ref. [17] established some recurrence relations for single and product moments for lgos for the exponentiated Pareto distribution and characterization. Ref. [24] established recurrence relations for single and product moments of lgos from inverse Weibull distribution. For more details on gos, see Refs. [1, 2, 3, 7, 8, 10, 13, 14, 15, 18, 20, 21, 22] and [23].

2. Recurrence Relation for Single Moments

Theorem 1. Let X be a r.v. has a distribution function $F(x)$ defined on (P_1, Q_1) . For any measurable function $\psi(x)$, the recurrence relation for the single moments of lgos is

$$\begin{aligned} E[\psi(X'(r, n, \overline{m}, k))] &= E[\psi(X'(r-1, n, \overline{m}, k))] - \frac{1}{\theta \gamma_r} E[\phi(X'(r, n, \overline{m}, k))] \\ &\quad - P_2 d_{r-2} E[\psi(X'(r, n-1, \tilde{m}, k+m_{n-1}))] \\ &\quad + P_2 d_{r-2} E[\psi(X'(r-1, n-1, \tilde{m}, k+m_{n-1}))], \end{aligned} \quad (9)$$

where $\phi(x) = \psi(x)\lambda^2(x)/\lambda(x)$, $\cdot \equiv \frac{d}{dx}$, $d_{r-2} = \frac{C_{r-2}}{C_{r-2}^*}$, $C_{\ell-1}^* = \prod_{i=1}^{\ell} (\gamma_i - 1)$ and $\tilde{m} = (m_1, \dots, m_{n-2}) \in \Re^{n-2}$.

Proof. From Eqs. (4) and (6), we have

$$\begin{aligned} E[\phi(X'(r, n, \overline{m}, k))] &= C_{r-1} \sum_{i=1}^r a_i(r) \int_{P_1}^{Q_1} \frac{\psi(x)\lambda^2(x)}{\lambda(x)} (F(x))^{\gamma_i-1} f(x) dx \\ &= \theta C_{r-1} \sum_{i=1}^r a_i(r) \int_{P_1}^{Q_1} \psi(x) (F(x))^{\gamma_i} dx \\ &\quad + \theta P_2 C_{r-1} \sum_{i=1}^r a_i(r) \int_{P_1}^{Q_1} \psi(x) (F(x))^{\gamma_i-1} dx. \end{aligned} \quad (10)$$

Integrating by parts, yields

$$\begin{aligned} E[\phi(X'(r, n, \overline{m}, k))] &= -\theta C_{r-1} \sum_{i=1}^r a_i(r) \gamma_i \int_{P_1}^{Q_1} \psi(x) (F(x))^{\gamma_i-1} f(x) dx \\ &\quad - \theta P_2 C_{r-1} \sum_{i=1}^r a_i(r) (\gamma_i - 1) \int_{P_1}^{Q_1} \psi(x) (F(x))^{(\gamma_i-1)-1} f(x) dx. \end{aligned} \quad (11)$$

Using $a_i(r-1) = (\gamma_r - \gamma_i)a_i(r)$ and $\gamma_r C_{r-2} = C_{r-1}$, we get

$$\begin{aligned} E[\phi(X'(r, n, \overline{m}, k))] &= \theta C_{r-1} \sum_{i=1}^{r-1} a_i(r-1) \int_{P_1}^{Q_1} \psi(x) (F(x))^{\gamma_i-1} f(x) dx \\ &\quad - \theta \gamma_r C_{r-1} \sum_{i=1}^r a_i(r) \int_{P_1}^{Q_1} \psi(x) (F(x))^{\gamma_i-1} f(x) dx \end{aligned}$$

$$\begin{aligned}
 & + \theta P_2 C_{r-1} \sum_{i=1}^{r-1} a_i(r-1) \int_{P_1}^{Q_1} \psi(x) (F(x))^{(\gamma_i-1)-1} f(x) dx \\
 & - \theta P_2 (\gamma_r - 1) C_{r-1} \sum_{i=1}^r a_i(r) \int_{P_1}^{Q_1} \psi(x) (F(x))^{(\gamma_i-1)-1} f(x) dx.
 \end{aligned} \tag{12}$$

It is easily shown that

$$\begin{aligned}
 E [\psi(X'(r, n, \overline{m}, k))] &= C_{r-2} \sum_{i=1}^{r-1} a_i(r-1) \int_{P_1}^{Q_1} \psi(x) (F(x))^{\gamma_i-1} f(x) dx \\
 &- \frac{1}{\theta \gamma_r} E [\phi(X'(r, n, \overline{m}, k))] \\
 &+ P_2 d_{r-2} C_{r-2}^* \sum_{i=1}^{r-1} a_i(r-1) \int_{P_1}^{Q_1} \psi(x) (F(x))^{(\gamma_i-1)-1} f(x) dx \\
 &- P_2 d_{r-2} C_{r-1}^* \sum_{i=1}^r a_i(r) \int_{P_1}^{Q_1} \psi(x) (F(x))^{(\gamma_i-1)-1} f(x) dx.
 \end{aligned} \tag{13}$$

After simplifying, the proof is completed.

Remark 1. Putting $\psi(x) = x^\ell$ in Eq. (9) yields the ℓ -th moment,

$$\begin{aligned}
 \mu'^{(\ell)}(r, n, \overline{m}, k) &= \\
 \mu'^{(\ell)}(r-1, n, \overline{m}, k) &- \frac{\ell}{\theta \gamma_r} E \left[X^{\ell-1}(r, n, \overline{m}, k) \frac{\lambda^2(X'(r, n, \overline{m}, k))}{\lambda(X'(r, n, \overline{m}, k))} \right] \\
 &- P_2 d_{r-2} \left(\mu'^{(\ell)}(r, n-1, \tilde{m}, k + m_{n-1}) \right. \\
 &\left. - \mu'^{(\ell)}(r-1, n-1, \tilde{m}, k + m_{n-1}) \right).
 \end{aligned} \tag{14}$$

Similarly, putting $\psi(x) = \exp(tx)$ in Eq. (9) yields the moment generating function.

3. Recurrence Relation For Double Moments

Theorem 2. Let X be a r.v. has a distribution function $F(x)$ defined on (P_1, Q_1) . For $r \leq s < n$ and any measurable function $\psi(x, y)$, the recurrence relation for the product moments of lgos is

$$E [\psi(X'(r, s, n, \overline{m}, k))] = E [\psi(X'(r, s-1, n, \overline{m}, k))] \tag{15}$$

$$\begin{aligned}
& - \frac{1}{\theta \gamma_s} E [\phi(X'(r, s, n, \bar{m}, k))] \\
& - P_2 d_{s-2} E [\psi(X'(r, s, n-1, \tilde{m}, k + m_{n-1}))] \\
& + P_2 d_{s-2} E [\psi(X'(r, s-1, n-1, \tilde{m}, k + m_{n-1}))], \quad (16)
\end{aligned}$$

where $\phi(x, y) = \psi(x, y) \lambda^2(y) / \lambda(y)$, $\lambda \equiv \frac{d}{dy}$.

Proof. From Eqs. (4) and (7), we have

$$\begin{aligned}
E [\phi(X'(r, s, n, \bar{m}, k))] &= C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \frac{\psi(x, y) \lambda^2(y)}{\lambda(y)} \left[\sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \right] \\
&\times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i} \right] \frac{f(x) f(y)}{F(x) F(y)} dy dx \\
&= \theta C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \right] \\
&\times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i} \right] \frac{f(x)}{F(x)} dy dx \\
&+ \theta P_2 C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i-1} \right] \\
&\times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i-1} \right] \frac{f(x)}{F(x)} dy dx, \quad P_1 \leq y < x \leq Q_1. \quad (17)
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
E [\phi(X'(r, s, n, \bar{m}, k))] &= -\theta C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^s a_i^{(r)}(s) \gamma_i \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \right] \\
&\times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i} \right] \frac{f(x) f(y)}{F(x) F(y)} dx dy \\
&- \theta P_2 C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^s a_i^{(r)}(s) (\gamma_i - 1) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i-1} \right] \\
&\times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i-1} \right] \frac{f(x) f(y)}{F(x) F(y)} dy dx. \quad (18)
\end{aligned}$$

Using $a_i^{(r)}(s-1) = (\gamma_s - \gamma_i)a_i^{(r)}(s)$ and $\gamma_s C_{s-2} = C_{s-1}$, we get

$$\begin{aligned}
 E [\phi(X'(r, s, n, \overline{m}, k))] &= \theta C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^{s-1} a_i^{(r)}(s-1) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \right] \\
 &\times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i} \right] \frac{f(x)f(y)}{F(x)F(y)} dy dx \\
 &- \theta \gamma_s C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \right] \\
 &\times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i} \right] \frac{f(x)f(y)}{F(x)F(y)} dy dx \\
 &+ \theta P_2 C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^{s-1} a_i^{(r)}(s-1) \left(\frac{F(y)}{F(x)} \right)^{\gamma_{i-1}} \right] \\
 &\times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_{i-1}} \right] \frac{f(x)f(y)}{F(x)F(y)} dy dx \\
 &- \theta P_2 (\gamma_s - 1) C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_{i-1}} \right] \\
 &\times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_{i-1}} \right] \frac{f(x)f(y)}{F(x)F(y)} dy dx, \tag{19}
 \end{aligned}$$

which reduce to

$$\begin{aligned}
 E [\psi(X'(r, s, n, \overline{m}, k))] &= C_{s-2} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^{s-1} a_i^{(r)}(s-1) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \right] \\
 &\times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i} \right] \frac{f(x)f(y)}{F(x)F(y)} dy dx \\
 &- \frac{1}{\theta \gamma_s} E [\phi(X'(r, s, n, \overline{m}, k))] \\
 &+ P_2 d_{s-2} C_{s-2}^* \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^{s-1} a_i^{(r)}(s-1) \left(\frac{F(y)}{F(x)} \right)^{\gamma_{i-1}} \right] \\
 &\times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_{i-1}} \right] \frac{f(x)f(y)}{F(x)F(y)} dy dx
 \end{aligned}$$

$$\begin{aligned}
& - P_2 d_{s-2} C_{s-1}^* \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i-1} \right] \\
& \times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i-1} \right] \frac{f(x)f(y)}{F(x)F(y)} dy dx. \tag{20}
\end{aligned}$$

After simplifying, the proof is completed.

Remark 2. Putting $\psi(x, y) = x^\ell y^\tau$ in Eq. (15) yields the ℓ -th moment,

$$\begin{aligned}
& \mu'^{(\ell, \tau)}(r, s, n, \overline{m}, k) = \\
& \mu'^{(\ell, \tau)}(r, s-1, n, \overline{m}, k) \\
& - \frac{\tau}{\theta \gamma_s} E \left[X'^\ell(r, n, \overline{m}, k) X'^{\tau-1}(s, n, \overline{m}, k) \frac{\lambda^2(X'(s, n, \overline{m}, k))}{\lambda(X'(s, n, \overline{m}, k))} \right] \\
& - P_2 d_{s-2} \left(\mu'^{(\ell, \tau)}(r, s, n-1, \tilde{m}, k + m_{n-1}) \right. \\
& \left. - \mu'^{(\ell, \tau)}(r, s-1, n-1, \tilde{m}, k + m_{n-1}) \right). \tag{21}
\end{aligned}$$

Similarly, putting $\psi(x, y) = \exp(tx + ty)$ in Eq. (15) yields the moment generating function.

4. Characterization of lgOSs

We shall require the following definition (see Ref. [11]):

Definition 1. (see [11]) Let $L_\delta(A)$ be the space of δ -integrable functions on a measurable set $A \subset R$. A sequence $(f_n)_{n \in N}$ of functions in $L_\delta(A)$ is called complete on $L_\delta(A)$, if for all functions $g \in L_\delta(A)$ the condition $\int_A g(x) f_n(x) dx = 0$ for all $n \in N$ implies: $g(x) = 0$ a.e on A .

Theorem 3. Let X be a r.v. has a distribution function $F(y)$ defined on $(Q1, P1)$. For any measurable function $\psi(y)$, the conditional expectation of lgos is given by

$$\begin{aligned}
& E [\psi(X'(s, n, \overline{m}, k)) | X'(r, n, \overline{m}, k) = x] = \\
& E [\psi(X'(s-1, n, \overline{m}, k)) | X'(r, n, \overline{m}, k)] - \frac{1}{\theta \gamma_s} E [\phi(X'(s, n, \overline{m}, k)) | X'(r, n, \overline{m}, k)] \\
& + \frac{P_2}{F(x)} d_{r-1, s-2} E [\psi(X'(s-1, n-1, \tilde{m}, k + m_{n-1})) | X'(r, n, \overline{m}, k)] \\
& - \frac{P_2}{F(x)} d_{r-1, s-2} E [\psi(X'(s, n-1, \tilde{m}, k + m_{n-1})) | X'(r, n, \overline{m}, k)], \tag{22}
\end{aligned}$$

where $d_{r-1,s-2} = \frac{d_{s-2}}{d_{r-1}}$.

Proof. From Eqs. (4) and (8), we have

$$\begin{aligned}
 & E [\phi(X'(s, n, \overline{m}, k)) | X'(r, n, \overline{m}, k) = x] \\
 &= \theta \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^s a_i^{(r)}(s) \int_{P_1}^x \psi(y) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \left\{ 1 + \frac{P_2}{F(y)} \right\} dy \\
 &= \theta \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^s a_i^{(r)}(s) \int_{P_1}^x \psi(y) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} dy \\
 &+ \theta P_2 \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^s a_i^{(r)}(s) \int_{P_1}^x \frac{\psi(y)}{F(x)} \left(\frac{F(y)}{F(x)} \right)^{\gamma_i-1} dy. \tag{23}
 \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
 & E [\phi(X'(s, n, \overline{m}, k)) | X'(r, n, \overline{m}, k) = x] \\
 &= -\theta \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^s a_i^{(r)}(s) \gamma_i \int_{P_1}^x \psi(y) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i-1} \frac{f(y)}{F(x)} dy \\
 &- \theta \frac{P_2}{F(x)} \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^s a_i^{(r)}(s) (\gamma_i - 1) \int_{P_1}^x \psi(y) \left(\frac{F(y)}{F(x)} \right)^{(\gamma_i-1)-1} \frac{f(y)}{F(x)} dy \tag{24}
 \end{aligned}$$

Using $a_i^{(r)}(s-1) = (\gamma_s - \gamma_i) a_i^{(r)}(s)$ and $\gamma_s C_{s-2} = C_{s-1}$, we get

$$\begin{aligned}
 & E [\phi(X'(s, n, \overline{m}, k)) | X'(r, n, \overline{m}, k) = x] \\
 &= \theta \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^{s-1} a_i^{(r)}(s-1) \int_{P_1}^x \psi(y) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \frac{f(y)}{F(y)} dy \\
 &- \theta \gamma_s \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^s a_i^{(r)}(s) \int_{P_1}^x \psi(y) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \frac{f(y)}{F(y)} dy \\
 &+ \theta \frac{P_2}{F(x)} \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^{s-1} a_i^{(r)}(s-1) \int_{P_1}^x \psi(y) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i-1} \frac{f(y)}{F(y)} dy \\
 &- \theta (\gamma_s - 1) \frac{P_2}{F(x)} \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^s a_i^{(r)}(s) \int_{P_1}^x \psi(y) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i-1} \frac{f(y)}{F(y)} dy, \tag{25}
 \end{aligned}$$

then

$$E [\psi(X'(s, n, \overline{m}, k)) | X'(r, n, \overline{m}, k) = x]$$

$$\begin{aligned}
&= \frac{C_{s-2}}{C_{r-1}} \sum_{i=r+1}^{s-1} a_i^{(r)}(s-1) \int_{P_1}^x \psi(y) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \frac{f(y)}{F(y)} dy \\
&- \frac{1}{\theta \gamma_s} E [\phi(X'(s, n, \overline{m}, k)) | X'(r, n, \overline{m}, k) = x] \\
&+ \frac{P_2}{F(x)} d_{r-1, s-2} \frac{C_{s-2}^*}{C_{r-1}^*} \sum_{i=r+1}^{s-1} a_i^{(r)}(s-1) \int_{P_1}^x \psi(y) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i-1} \frac{f(y)}{F(y)} dy \\
&- \frac{P_2}{F(x)} d_{r-1, s-2} \frac{C_{s-1}^*}{C_{r-1}^*} \sum_{i=r+1}^s a_i^{(r)}(s) \int_{P_1}^x \psi(y) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i-1} \frac{f(y)}{F(y)} dy.
\end{aligned}$$

After simplifying, the proof is completed.

Theorem 4. For $1 \leq r \leq n-1$ and for any measurable function $\psi(x)$, a random variable X has a distribution function with cdf given by Eq. (4) iff Eq. (9) holds.

Proof. The necessary part is proved in Theorem (1). Now, to prove the sufficient part, consider Eq. (9) holds, then we obtain

$$\begin{aligned}
E [\phi(X'(r, n, \overline{m}, k))] &= \theta C_{r-1} \sum_{i=1}^{r-1} a_i(r-1) \int_{P_1}^{Q_1} \psi(x) (F(x))^{\gamma_i-1} f(x) dx \\
&- \theta \gamma_r C_{r-1} \sum_{i=1}^r a_i(r) \int_{P_1}^{Q_1} \psi(x) (F(x))^{\gamma_i-1} f(x) dx \\
&+ \theta P_2 C_{r-1} \sum_{i=1}^{r-1} a_i(r-1) \int_{P_1}^{Q_1} \psi(x) (F(x))^{(\gamma_i-1)-1} f(x) dx \\
&- \theta P_2 (\gamma_r - 1) C_{r-1} \sum_{i=1}^r a_i(r) \int_{P_1}^{Q_1} \psi(x) (F(x))^{(\gamma_i-1)-1} f(x) dx. \quad (26)
\end{aligned}$$

Since $a_i(r-1) = (\gamma_r - \gamma_i) a_i(r)$, then we obtain

$$\begin{aligned}
E [\phi(X'(r, n, \overline{m}, k))] &= -\theta C_{r-1} \sum_{i=1}^r a_i(r) \gamma_i \int_{P_1}^{Q_1} \psi(x) (F(x))^{\gamma_i-1} f(x) dx \\
&- \theta P_2 C_{r-1} \sum_{i=1}^r a_i(r) (\gamma_i - 1) \int_{P_1}^{Q_1} \psi(x) (F(x))^{(\gamma_i-1)-1} f(x) dx. \quad (27)
\end{aligned}$$

Integrating by parts, we get

$$C_{r-1} \sum_{i=1}^r a_i(r) \int_{P_1}^{Q_1} \frac{\psi(x) \lambda^2(x)}{\lambda(x)} (F(x))^{\gamma_i-1} f(x) dx$$

$$\begin{aligned}
 &= \theta C_{r-1} \sum_{i=1}^r a_i(r) \int_{P_1}^{Q_1} \psi(x) (F(x))^{\gamma_i} dx \\
 &+ \theta P_2 C_{r-1} \sum_{i=1}^r a_i(r) \int_{P_1}^{Q_1} \psi(x) (F(x))^{\gamma_i-1} dx.
 \end{aligned} \tag{28}$$

After simplifying we obtain

$$C_{r-1} \sum_{i=1}^r a_i(r) \left[\int_{P_1}^{Q_1} \psi(x) (F(x))^{\gamma_i-1} \left(\frac{\lambda^2(x)f(x)}{\lambda(x)} - \theta(P_2 + F(x)) \right) dx \right] = 0. \tag{29}$$

Using Definition 1, we get

$$\frac{\lambda^2(x)f(x)}{\lambda(x)} - \theta(P_2 + F(x)) = 0, \Rightarrow f(x) = \frac{\theta\lambda(x)}{\lambda^2(x)} \{F(x) + P_2\}.$$

Thus, the theorem.

Theorem 5. For $1 \leq r < s \leq n-1$ and for any measurable function $\psi(x, y)$, a random variable X has a distribution function with cdf given by Eq. (4) iff Eq. (15) holds.

Proof. The necessary part is proved in Theorem (2). Now, to prove the sufficient part, consider Eq. (15) holds, then we obtain

$$\begin{aligned}
 E [\phi(X'(r, s, n, \overline{m}, k))] &= \theta C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^{s-1} a_i^{(r)}(s-1) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \right] \\
 &\times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i} \right] \frac{f(x)f(y)}{F(x)F(y)} dy dx \\
 &- \theta \gamma_s C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \right] \\
 &\times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i} \right] \frac{f(x)f(y)}{F(x)F(y)} dy dx \\
 &+ \theta P_2 C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^{s-1} a_i^{(r)}(s-1) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i-1} \right] \\
 &\times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i-1} \right] \frac{f(x)f(y)}{F(x)F(y)} dy dx
 \end{aligned}$$

$$\begin{aligned}
& - \theta P_2 (\gamma_s - 1) C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i - 1} \right] \\
& \times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i - 1} \right] \frac{f(x)f(y)}{F(x)F(y)} dy dx. \tag{30}
\end{aligned}$$

Using $a_i^{(r)}(s-1) = (\gamma_s - \gamma_i) a_i^{(r)}(s)$, we get

$$\begin{aligned}
E [\phi(X'(r, s, n, \overline{m}, k))] &= -\theta C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^s a_i^{(r)}(s) \gamma_i \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \right] \\
& \times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i} \right] \frac{f(x)f(y)}{F(x)F(y)} dx dy \\
& - \theta P_2 C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^s a_i^{(r)}(s) (\gamma_i - 1) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i - 1} \right] \\
& \times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i - 1} \right] \frac{f(x)f(y)}{F(x)F(y)} dy dx. \tag{31}
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
E [\phi(X'(r, s, n, \overline{m}, k))] &= C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \frac{\psi(x, y) \lambda^2(y)}{\lambda(y)} \left[\sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \right] \\
& \times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i} \right] \frac{f(x)f(y)}{F(x)F(y)} dy dx \\
& = \theta C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \right] \\
& \times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i} \right] \frac{f(x)}{F(x)} dy dx \\
& + \theta P_2 C_{s-1} \int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \left[\sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i - 1} \right] \\
& \times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i - 1} \right] \frac{f(x)}{F(x)} dy dx. \tag{32}
\end{aligned}$$

After simplifying we obtain

$$\int_{P_1}^{Q_1} \int_{P_1}^x \psi(x, y) \frac{f(x)}{F(x)} \left[\sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i-1} \right] \times \left[\sum_{i=1}^r a_i(r) (F(x))^{\gamma_i-1} \right] \left\{ \frac{\lambda^2(y)f(y)}{\lambda(y)} - \theta(F(y) + P_2) \right\} dy dx = 0. \quad (33)$$

By using Definition 1, Eq. (33) reduce to

$$\frac{\lambda^2(y)f(y)}{\lambda(y)} - \theta(F(y) + P_2) = 0, \Rightarrow f(y) = \frac{\theta\lambda(y)}{\lambda^2(y)} \{F(y) + P_2\}.$$

Thus, the theorem.

Remark 3. In the non-truncated case ($Q = 1, P = 0 \Rightarrow P_2 = 0$), Eqs.(9), (15) and (22) reduce to

$$E [\psi(X'(r, n, \overline{m}, k))] = E [\psi(X'(r-1, n, \overline{m}, k))] - \frac{1}{\theta\gamma_r} E [\phi(X'(r, n, \overline{m}, k))],$$

$$E [\psi(X'(r, s, n, \overline{m}, k))] = E [\psi(X'(r, s-1, n, \overline{m}, k))] - \frac{1}{\theta\gamma_s} E [\phi(X'(r, s, n, \overline{m}, k))],$$

and

$$E [\psi(X'(s, n, \overline{m}, k)) | X'(r, n, \overline{m}, k) = x] = E [\psi(X'(s-1, n, \overline{m}, k)) | X'(r, n, \overline{m}, k)] - \frac{1}{\theta\gamma_s} E [\phi(X'(s, n, \overline{m}, k)) | X'(r, n, \overline{m}, k)].$$

Remark 4. If we put $P = 0$ (i.e $P_2 = 0$), then we obtain the right truncated case. If we put $Q = 1$ (i.e $P_2 = P/(1 - P)$), then we obtain the left truncated case.

Remark 5. In case of ordinary record values ($k = 1$), the relations (14) and (21) reduce, respectively, to

$$\mu_r'^{(\ell)} = \mu_{r-1}'^{(\ell)} - \frac{\ell}{\theta} E \left[X_r'^{\ell-1} \frac{\lambda^2(X_r')}{\lambda'(X_r')} \right] + P_2 E \left[X_r'^{\ell} e^{\theta/\lambda(X_r')} \right], \quad (34)$$

and

$$\mu_{r,s}'^{(\ell,\tau)} = \mu_{r,s-1}'^{(\ell,\tau)} - \frac{\tau}{\theta} E \left[X_r'^{\ell} X_s'^{\tau-1} \frac{\lambda^2(X_s')}{\lambda'(X_s')} \right] + P_2 E \left[X_r'^{\ell} X_s'^{\tau} e^{\theta/\lambda(X_s')} \right]. \quad (35)$$

5. Applications

In this section, we suggest some applications based on recurrence relations for the expectations and conditional expectations of functions can easily be discussed arise as special cases of our results.

1. Doubly truncated inverted Weibull distribution.

Choosing $\lambda(x) = x^{\beta}$ and $\theta = \alpha > 0$. Then the relations (14) and (21) reduce, respectively, to

$$\begin{aligned} \mu'^{(j)}(r, n, \overline{m}, k) &= \mu'^{(j)}(r-1, n, \overline{m}, k) - \frac{j}{\alpha\beta\gamma_r} \mu'^{(j+\beta)}(r, n, \overline{m}, k) \\ &- P_2 d_{r-2}(\mu'^{(j)}(r, n-1, \tilde{m}, k + m_{n-1}) \\ &- \mu'^{(j)}(r-1, n-1, \tilde{m}, k + m_{n-1})), \end{aligned} \quad (36)$$

and

$$\begin{aligned} \mu'^{(j,\ell)}(r, s, n, \overline{m}, k) &= \mu'^{(j,\ell)}(r, s-1, n, \overline{m}, k) \\ &- \frac{\ell}{\alpha\beta\gamma_s} \mu'^{(j,\ell+\beta)}(r, s, n, \overline{m}, k) \\ &- P_2 d_{s-2}(\mu'^{(j,\ell)}(r, s, n-1, \tilde{m}, k + m_{n-1}) \\ &- \mu'^{(j,\ell)}(r, s-1, n-1, \tilde{m}, k + m_{n-1})). \end{aligned} \quad (37)$$

In the case of oOSs ($m_1 = \dots = m_{n-1} = 0$, $k = 1$, $\gamma_r = n - r + 1$ and $d_{r-2} = \frac{n}{n-r+1}$), the relations (36) and (37) reduce, respectively, to

$$\mu_{r:n}'^{(j)} = \mu_{r-1:n}'^{(j)} - \frac{j}{\alpha\beta(n-r+1)} \mu_{r:n}'^{(j+\beta)} - \frac{P_2 n}{n-r+1} \left(\mu_{r:n-1}'^{(j)} - \mu_{r-1:n-1}'^{(j)} \right),$$

and

$$\mu_{r,s:n}'^{(j,\ell)} = \mu_{r,s-1:n}'^{(j,\ell)} - \frac{\ell}{\alpha\beta(n-s+1)} \mu_{r,s:n}'^{(j,\ell+\beta)}$$

$$- \frac{P_2 n}{n - s + 1} \left(\mu'_{r,s;n-1}^{(j,\ell)} - \mu'_{r,s-1;n-1}^{(j,\ell)} \right).$$

In the case of k -th record values ($m = -1$, $k \geq 2$, $\gamma_r = k$ and $d_{r-2} = (\frac{k}{k-1})^{r-1}$), the relations (36) and (37) reduce, respectively, to

$$\mu'_{r:k}^{(j)} = \mu'_{r-1:k}^{(j)} - \frac{j}{\alpha\beta k} \mu'_{r:k}^{(j+\beta)} - P_2 \left(\frac{k}{k-1} \right)^{r-1} \left(\mu'_{r:k-1}^{(j)} - \mu'_{r-1:k-1}^{(j)} \right),$$

and

$$\mu'_{r,s:k}^{(j,\ell)} = \mu'_{r,s-1:k}^{(j,\ell)} - \frac{\ell}{\alpha\beta k} \mu'_{r,s:k}^{(j,\ell+\beta)} - P_2 \left(\frac{k}{k-1} \right)^{s-1} \left(\mu'_{r,s:k-1}^{(j,\ell)} - \mu'_{r,s-1:k-1}^{(j,\ell)} \right).$$

2. Doubly truncated inverted pareto distribution.

Choosing $\lambda(x) = -(\ln x)^{-1}$ and $\theta = \eta$. The relations (14) and (21) reduce, respectively, to

$$\begin{aligned} \mu'^{(j)}(r, n, \overline{m}, k) &= \mu'^{(j)}(r-1, n, \overline{m}, k) - \frac{j}{\eta\gamma_r} \mu'^{(j)}(r, n, \overline{m}, k) \\ &- P_2 d_{r-2}(\mu'^{(j)}(r, n-1, \tilde{m}, k + m_{n-1})) \\ &- \mu'^{(j)}(r-1, n-1, \tilde{m}, k + m_{n-1}), \end{aligned} \quad (38)$$

and

$$\begin{aligned} \mu'^{(j,\ell)}(r, s, n, \overline{m}, k) &= \mu'^{(j,\ell)}(r, s-1, n, \overline{m}, k) - \frac{\ell}{\eta\gamma_s} \mu'^{(j,\ell)}(r, s, n, \overline{m}, k) \\ &- P_2 d_{s-2}(\mu'^{(j,\ell)}(r, s, n-1, \tilde{m}, k + m_{n-1})) \\ &- \mu'^{(j,\ell)}(r, s-1, n-1, \tilde{m}, k + m_{n-1}). \end{aligned} \quad (39)$$

In the case of oOSs, the relations (38) and (39) reduce, respectively, to

$$\mu'_{r:n}^{(j)} = \mu'_{r-1:n}^{(j)} - \frac{j}{\eta(n-r+1)} \mu'_{r:n}^{(j)} - \frac{P_2 n}{n-r+1} \left(\mu'_{r:n-1}^{(j)} - \mu'_{r-1:n-1}^{(j)} \right),$$

and

$$\mu'_{r,s:n}^{(j,\ell)} = \mu'_{r,s-1:n}^{(j,\ell)} - \frac{\ell}{\eta(n-s+1)} \mu'_{r,s:n}^{(j,\ell)} - \frac{P_2 n}{n-s+1} \left(\mu'_{r,s:n-1}^{(j,\ell)} - \mu'_{r,s-1:n-1}^{(j,\ell)} \right).$$

In the case of k -th records, the relations (38) and (39) reduce, respectively, to

$$\mu'_{r:k}^{(j)} = \mu'_{r-1:k}^{(j)} - \frac{j}{\eta k} \mu'_{r:k}^{(j)} - P_2 \left(\frac{k}{k-1} \right)^{r-1} \left(\mu'_{r:k-1}^{(j)} - \mu'_{r-1:k-1}^{(j)} \right),$$

and

$$\mu_{r,s:k}^{(j,\ell)} = \mu_{r,s-1:k}^{(j,\ell)} - \frac{\ell}{\eta k} \mu_{r,s:k}^{(j,\ell)} - P_2 \left(\frac{k}{k-1} \right)^{s-1} \left(\mu_{r,s:k-1}^{(j,\ell)} - \mu_{r,s-1:k-1}^{(j,\ell)} \right).$$

3. Doubly truncated log-inverse generalized Weibull distribution.

Choosing $\lambda(x) = \exp\{(x - \nu)/\beta\}$ and $\theta = \alpha$. The relations (14) and (21), reduce, respectively, to

$$\begin{aligned} \mu^{(j)}(r, n, \overline{m}, k) &= \mu^{(j)}(r-1, n, \overline{m}, k) - \frac{j\beta}{\alpha\gamma_r} E \left[\frac{X'^{j-1}(r, n, \overline{m}, k)}{e^{-(X'(r, n, \overline{m}, k) - \nu)/\beta}} \right] \\ &- P_2 d_{r-2}(\mu^{(j)}(r, n-1, \tilde{m}, k + m_{n-1})) \\ &- \mu^{(j)}(r-1, n-1, \tilde{m}, k + m_{n-1}), \end{aligned} \quad (40)$$

and

$$\begin{aligned} \mu^{(j,\ell)}(r, s, n, \overline{m}, k) &= \mu^{(j,\ell)}(r, s-1, n, \overline{m}, k) \\ &- \frac{\ell\beta}{\alpha\gamma_s} E \left[\frac{X'^j(r, n, \overline{m}, k) X'^{\ell-1}(s, n, \overline{m}, k)}{e^{-(X'(s, n, \overline{m}, k) - \nu)/\beta}} \right] \\ &- P_2 d_{s-2}(\mu^{(j,\ell)}(r, s, n-1, \tilde{m}, k + m_{n-1})) \\ &- \mu^{(j,\ell)}(r, s-1, n-1, \tilde{m}, k + m_{n-1}). \end{aligned} \quad (41)$$

In the case of oOSs, the relations (40) and (41) reduce, respectively, to

$$\begin{aligned} \mu_{r:n}^{(j)} &= \mu_{r-1:n}^{(j)} - \frac{j\beta}{\alpha(n-r+1)} E \left[\frac{X'^{j-1}(r:n)}{e^{-(X'(r:n) - \nu)/\beta}} \right] \\ &- \frac{P_2 n}{n-r+1} \left(\mu_{r:n-1}^{(j)} - \mu_{r-1:n-1}^{(j)} \right), \end{aligned}$$

and

$$\begin{aligned} \mu_{r,s:n}^{(j,\ell)} &= \mu_{r,s-1:n}^{(j,\ell)} - \frac{\ell\beta}{\alpha(n-s+1)} E \left[\frac{X'^j(r:n) X'^{\ell-1}(s:n)}{e^{-(X'(s:n) - \nu)/\beta}} \right] \\ &- \frac{P_2 n}{n-s+1} \left(\mu_{r,s:n-1}^{(j,\ell)} - \mu_{r,s-1:n-1}^{(j,\ell)} \right). \end{aligned}$$

In the case of k -th records, the relations (40) and (41), reduce, respectively, to

$$\mu_{r:k}^{\prime(j)} = \mu_{r-1:k}^{\prime(j)} - \frac{j\beta}{\alpha k} E \left[\frac{X^{\prime j-1}(r:k)}{e^{-(X'(r:k)-\nu)/\beta}} \right] - P_2 \left(\frac{k}{k-1} \right)^{r-1} \left(\mu_{r:k-1}^{\prime(j)} - \mu_{r-1:k-1}^{\prime(j)} \right), \quad (42)$$

and

$$\begin{aligned} \mu_{r,s:k}^{\prime(j,\ell)} &= \mu_{r,s-1:k}^{\prime(j,\ell)} - \frac{\ell\beta}{\alpha k} E \left[\frac{X^{\prime j}(r:k) X^{\prime \ell-1}(s:k)}{e^{-(X'(s:k)-\nu)/\beta}} \right] \\ &- P_2 \left(\frac{k}{k-1} \right)^{s-1} \left(\mu_{r,s:k-1}^{\prime(j,\ell)} - \mu_{r,s-1:k-1}^{\prime(j,\ell)} \right). \end{aligned} \quad (43)$$

In the case of ordinary record values ($k = 1$), the relations (14) and (21), reduce, respectively, to

$$\mu_r^{\prime(j)} = \mu_{r-1}^{\prime(j)} - \frac{j}{\theta} E \left[\frac{X_r^{\prime j-1}}{e^{-(X'_r-\nu)/\beta}} \right] + P_2 E \left[X_r^{\prime j} e^{\theta/\lambda(X'_r)} \right], \quad (44)$$

and

$$\mu_{r,s}^{\prime(j,\ell)} = \mu_{r,s-1}^{\prime(j,\ell)} - \frac{\ell}{\theta} E \left[\frac{X_r^{\prime j} X_s^{\prime \ell-1}}{e^{-(X'_s-\nu)/\beta}} \right] + P_2 E \left[X_r^{\prime j} X_s^{\prime \ell} e^{\theta/\lambda(X'_s)} \right]. \quad (45)$$

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