IMPROVEMENT OF POISSON APPROXIMATION FOR INDEPENDENT GEOMETRIC TRIALS

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Abstract: We use the Stein-Chen method to improve the bound in [5] to be more appropriate for the Poisson approximation to a sum of independently distributed geometric random variables.

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1. Introduction

Let W be a sum $\sum_{i=1}^n X_i$ of independently distributed geometric random variables, each with probability $P(X_i = k) = (1 - p_i)^k p_i$ for $k = 0, 1, \dots$. It is well-known that if all $q_i = (1 - p_i)$ are small, the distribution of W can be approximated by the Poisson distribution with mean $\lambda = E(W) = \sum_{i=1}^n q_i p_i^{-1}$. Correspondingly, the distribution function of W can also be approximated by the Poisson distribution function with mean λ . Let $\mathbb{P}W(w_0) = P(W \leq w_0)$ and $\mathbb{P}_\lambda(w_0) = \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!}$ be the distribution function of W and the Poisson distribution function at $w_0 \in \mathbb{N} \cup \{0\}$, respectively. In this case, Teerapabolarn and Wongkasem [5] used the Stein-Chen method to give a non-uniform bound

for the difference of $\mathbb{PW}(w_0)$ and $\mathbb{P}_\lambda(w_0)$ as follows:

$$|\mathbb{PW}(w_0) - \mathbb{P}_\lambda(w_0)| \leq \lambda^{-1}(e^\lambda - 1) \sum_{i=1}^n \min \left\{ 1, \frac{1}{p_i(w_0 + 1)} \right\} q_i^2 p_i^{-1}, \quad (1.1)$$

where $w_0 \in \mathbb{N} \cup \{0\}$. We observe that the result in (1.1) gives a good Poisson approximation whenever e^λ is small; however, when e^λ is large, this result may be inappropriate for approximating the distribution function of W . In this paper, our goal is to improve the results in (1.1) by eliminating the influence of the factor e^λ .

2. Method

Stein's method was originally formulated for normal approximation by Stein [2]. It was adapted and applied to the Poisson case by Chen [1], which is referred to as the Stein-Chen method. Following Teerapabolarn [3], Stein's equation of the Poisson cumulative distribution function with parameter $\lambda > 0$ is of the form

$$h_{w_0}(w) - \mathbb{P}_\lambda(w_0) = \lambda f_{w_0}(w + 1) - w f_{w_0}(w), \quad (2.1)$$

where $w_0, w \in \mathbb{N} \cup \{0\}$, and for $h_{w_0}(w) = 1$ if $w \leq w_0$ and $h_{w_0}(w) = 0$ if $w > w_0$, the solution f_{w_0} is

$$f_{w_0}(w) = \begin{cases} (w-1)! \lambda^{-w} e^\lambda [\mathbb{P}_\lambda(w-1)[1 - \mathbb{P}_\lambda(w_0)]] & \text{if } w \leq w_0, \\ (w-1)! \lambda^{-w} e^\lambda [\mathbb{P}_\lambda(w_0)[1 - \mathbb{P}_\lambda(w-1)]] & \text{if } w > w_0, \\ 0 & \text{if } w = 0. \end{cases} \quad (2.2)$$

The following lemma gives a non-uniform bound of (2.2), which is used to improve the bound in (1.1).

Lemma 2.1. *For $w_0, k \in \mathbb{N}$, let $\Delta f_{w_0}(w) = f_{w_0}(w+1) - f_{w_0}(w)$. Then the following inequality holds:*

$$\sup_{w \geq k} |\Delta f_{w_0}(w)| \leq \min \left\{ \lambda^{-1}(1 - e^{-\lambda}), \frac{1}{w_0}, \frac{1}{k} \right\}. \quad (2.3)$$

Proof. The first and second bounds follow from [4]. Next, we shall show that $\sup_{w \geq k} |\Delta f_{w_0}(w)| \leq \frac{1}{k}$. It is clear that $\sup_{w \geq k} |\Delta f_{w_0}(w)| \leq \frac{1}{w_0} \leq \frac{1}{k}$ for $k \leq w \leq w_0$.

For $w \geq k$ and $w > w_0$, it follows from [4] that $\sup_{w \geq k} |\Delta f_{w_0}(w)| \leq \frac{1}{w} \leq \frac{1}{k}$.

Therefore, $\sup_{w \geq k} |\Delta f_{w_0}(w)| \leq \frac{1}{k}$, hence (2.3) holds. $\square$

3. Result

The following theorem presents a new non-uniform bound on $|\mathbb{P}\mathbb{W}(w_0) - \mathbb{P}_\lambda(w_0)|$.

Theorem 3.1. For $w_0 \in \mathbb{N}$, if $\lambda = \sum_{i=1}^n q_i p_i^{-1}$, then we have the following:

$$|\mathbb{P}\mathbb{W}(w_0) - \mathbb{P}_\lambda(w_0)| \leq \sum_{i=1}^n \min \left\{ \frac{\lambda^{-1}(1 - e^{-\lambda})}{p_i}, \frac{1}{p_i w_0}, 1 \right\} q_i^2 p_i^{-1}, \quad (3.1)$$

where $\mathbb{P}\mathbb{W}(0) = \prod_{i=1}^n p_i$.

Proof. Teerapabolarn and Wongkasem [5] showed that

$$\begin{aligned} |\mathbb{P}\mathbb{W}(w_0) - \mathbb{P}_\lambda(w_0)| &\leq \sum_{i=1}^n \sum_{k=1}^{\infty} k q_i^{k+1} \sup_{w \geq k} |\Delta f_{w_0}(w)| \\ &\leq \sum_{i=1}^n \sum_{k=1}^{\infty} k q_i^{k+1} \min \left\{ \lambda^{-1}(1 - e^{-\lambda}), \frac{1}{w_0}, \frac{1}{k} \right\} \quad (\text{by (2.3)}) \\ &= \sum_{i=1}^n \min \left\{ \frac{\lambda^{-1}(1 - e^{-\lambda})}{p_i}, \frac{1}{p_i w_0}, 1 \right\} q_i^2 p_i^{-1}. \end{aligned}$$

Hence, the theorem is proved. $\square$

Remark. Because $\min \left\{ \frac{1-e^{-\lambda}}{\lambda p_i}, \frac{1}{p_i w_0}, 1 \right\} < \frac{e^{\lambda}-1}{\lambda} \min \left\{ 1, \frac{1}{p_i(w_0+1)} \right\}$ for every $w_0 \in \mathbb{N}$. Therefore, the bound in (3.1) is sharper than that in (1.1).

4. Conclusion

The non-uniform bound in the Theorem 3.1, which was improved by the Stein-Chen method, provides an appropriate criteria for measuring the accuracy of approximating the distribution function of W by the Poisson distribution function with mean $\lambda = \sum_{i=1}^n q_i p_i^{-1}$. In addition, the influence of factor e^λ in the old bound is eliminated from or reduced in the new bound. So, the bound in this study is sharper than that reported in [5].

References

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