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LOWER AND UPPER BOUNDS ON THE BINOMIAL-POISSON ERROR

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Abstract: In this paper, we give lower and upper bounds for the error between the binomial distribution function with parameters n and p and the Poisson distribution function with mean $\lambda = \frac{np}{q}$. Numerical examples have been given to illustrate the result obtained.

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1. Introduction

Let $Y_1, ..., Y_n$ be n independently distributed Bernoulli random variables, each with probability of success $p = P(Y_i = 1) = 1 - P(Y_i = 0)$, and let $X = \sum_{i=1}^{n} Y_i$. Then probability function of variable X is of the form

$$p_X(x) = \binom{n}{x} p^x q^{n-x},\ x = 0, 1, ..., n, \tag{1.1}$$

where $q = 1 - p$. It is called the distribution of X that the binomial random variable with parameters $n \in \mathbb{N}$ and $p \in (0, 1)$, and the mean and variance

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of X are $E(X) = np$ and $Var(X) = npq$, respectively. It is a well-known result that if $n \to \infty$ and $p \to 0$ while np remains fixed, then the binomial distribution with parameters n and p converges to the Poisson distribution with mean np. Therefore the Poisson distribution with mean $\lambda = np$ can be used as an approximation to the binomial distribution with parameters n and p if n is sufficiently large and p is sufficiently small.

Let us consider (1.1), by setting $\lambda = \frac{np}{q}$ or $p = \frac{\lambda}{n+\lambda}$, it follows that

$$\begin{aligned} p_X(x) &= q^n \frac{n!}{x!(n-x)!}\left(\frac{p}{q}\right)^x, \ x = 0, 1, ..., n \\ &= \begin{cases} \left(\frac{1}{1+\frac{\lambda}{n}}\right)^n & \text{if } x = 0, \\ \left(\frac{1}{1+\frac{\lambda}{n}}\right)^n \frac{\lambda^x}{x!}\left(\frac{n}{n} \cdots \frac{n-x+1}{n}\right) & \text{if } x = 1, ..., n. \end{cases} \end{aligned} \tag{1.2}$$

It can be observed that if $n \to \infty$ and $p \to 0$ while $\lambda = \frac{np}{q}$ remains fixed, then $p_X(x) = \binom{n}{x} p^x q^{n-x} \to \frac{e^{-\lambda}\lambda^x}{x!}$ for every $x \in \{0, 1, ..., n\}$, that is, the binomial distribution with parameters n and p also converges to the Poisson distribution with mean $\lambda = \frac{np}{q}$. Therefore, the Poisson distribution with mean $\lambda = \frac{np}{q}$ can also be used as an approximation of the binomial distribution with parameters n and p when n is sufficiently large and p is sufficiently small.

In this paper, we are interested to give lower and upper bounds for the error $\mathbb{B}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0)$, where $\mathbb{B}_{n,p}(x_0)$ is the binomial distribution function with parameters n and p and $\mathbb{P}_\lambda(x_0)$ is the Poisson distribution function with mean $\lambda = \frac{np}{q}$ at $x_0 \in \{0, 1, ..., n\}$.

2. Method

The classical Stein's method was first introduced by Stein [2]. The version appropriate for the Poisson case was first developed by Chen [1], which is referred to as the Stein-Chen method.

Following Teerapabolarn [3], Stein's equation of the Poisson cumulative distribution function with parameter $\lambda > 0$ is of the form

$$h_{x_0}(x) - \mathbb{P}_\lambda(x_0) = \lambda f_{x_0}(x+1) - x f_{x_0}(x) \tag{2.1}$$

for $x_0, x \in \mathbb{N} \cup \{0\}$ and function $h_{x_0} : \mathbb{N} \cup \{0\} \to \mathbb{R}$ is defined by

$$h_{x_0}(x) = \begin{cases} 1 & \text{if } \ x \le x_0, \\ 0 & \text{if } \ x > x_0, \end{cases}$$

and

$$f_{x_0}(x) = \begin{cases} (x-1)!\lambda^{-x}e^{\lambda}[\mathbb{P}_\lambda(x-1)[1-\mathbb{P}_\lambda(x_0)]] & \text{if } x \leq x_0, \\ (x-1)!\lambda^{-x}e^{\lambda}[\mathbb{P}_\lambda(x_0)[1-\mathbb{P}_\lambda(x-1)]] & \text{if } x > x_0, \\ 0 & \text{if } x = 0. \end{cases} \quad (2.2)$$

Lemma 2.1. *For $x_0 \in \{0, 1, ..., n\}$, then the following inequalities holds:*

$$\inf_{1 \leq x \leq n+1} f_{x_0}(x) \geq \min\left\{\lambda^{-1}[1-\mathbb{P}_\lambda(x_0)], n!\lambda^{-(n+1)}e^{\lambda}\mathbb{P}_\lambda(x_0)[1-\mathbb{P}_\lambda(n)]\right\}. \quad (2.3)$$

Proof. It follows from [3] that f_{x_0} is a decreasing function for $x \in \{x_0+1, ..., n+1\}$ and is an increasing function for $x \in \{1, ..., x_0\}$. Thus, for $1 \leq x \leq x_0$, we have $f_{x_0}(x) \geq f_{x_0}(1) = \lambda^{-1}[1-\mathbb{P}_\lambda(x_0)]$. For $x_0+1 \leq x \leq n+1$, we obtain $n!\lambda^{-(n+1)}e^{\lambda}\mathbb{P}_\lambda(x_0)[1-\mathbb{P}_\lambda(n)]$, which yields (2.3) holds. □

Lemma 2.2. *For $x_0 \in \{0, 1, ..., n\}$ and $\lambda = \frac{np}{q}$, then we have the following:*

$$\mathbb{B}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0) \leq (e^{\lambda}q^n - 1)\mathbb{P}_\lambda(x_0). \quad (2.4)$$

Proof. It follows form [4] that $\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1 \leq e^{\lambda}q^n - 1$, which gives (2.4). □

3. Result

The following theorem shows the lower and upper bounds on the absolute error between two such distribution functions.

Theorem 3.1. *For $x_0 \in \{0, ..., n\}$ and $\lambda = \frac{np}{q}$, then we have the following:*

$$\min\left\{1-\mathbb{P}_\lambda(x_0), n!\lambda^{-n}e^{\lambda}\mathbb{P}_\lambda(x_0)[1-\mathbb{P}_\lambda(n)]\right\}p \leq \mathbb{B}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0) \leq (e^{\lambda}q^n - 1)\mathbb{P}_\lambda(x_0). \quad (3.1)$$

Proof. Taking expectation in (2.1), we have

$$\mathbb{B}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0) = E[\lambda f(X+1) - Xf(X)]$$

$$= E\left[\sum_{i=1}^{n}\frac{p}{q}f(X+1) - \sum_{i=1}^{n}Y_i f(X)\right]$$

$$= \sum_{i=1}^{n}E\left[\frac{p}{q}f(X+1) - Y_i f(X)\right], \tag{3.2}$$

where f is defined in (2.2). Let $X_i = X - Y_i$ and $\delta = E\left[\frac{p}{q}f(X+1) - Y_i f(X)\right]$, then for each i,

$$\begin{aligned}
\delta &= E\left[\frac{p}{q}f(X_i+Y_i+1) - Y_i f(X_i+Y_i)\right] \\
&= E\left\{E\left[\left(\frac{p}{q}f(X_i+Y_i+1) - Y_i f(X_i+Y_i)\right)|Y_i\right]\right\} \\
&= E\left[\left(\frac{p}{q}f(X_i+Y_i+1) - Y_i f(X_i+Y_i)\right)|Y_i=0\right]P(Y_i=0) \\
&\quad + E\left[\left(\frac{p}{q}f(X_i+Y_i+1) - Y_i f(X_i+Y_i)\right)|Y_i=1\right]P(Y_i=1) \\
&= E\left[\left(\frac{p}{q}f(X_i+Y_i+1) - Y_i f(X_i+Y_i)\right)|Y_i=0\right]q \\
&\quad + E\left[\left(\frac{p}{q}f(X_i+Y_i+1) - Y_i f(X_i+Y_i)\right)|Y_i=1\right]p \\
&= E[pf(X_i+1)] + E\left[\frac{p^2}{q}f(X_i+2) - pf(X_i+1)\right] \\
&= \frac{p^2}{q}E[f(X_i+2)].
\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$\begin{aligned}
\mathbb{B}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0) &= \frac{p^2}{q}\sum_{i=1}^{n}\sum_{k=0}^{n-1}f(k+2)p_{X_i}(k) \\
&\geq \frac{p^2}{q}\sum_{i=1}^{n}\inf_{1\leq x\leq n+1}f(x)\sum_{k=0}^{n-1}p_{X_i}(k) \\
&\geq \min\left\{1-\mathbb{P}_\lambda(x_0), n!\lambda^{-n}e^{\lambda}\mathbb{P}_\lambda(x_0)[1-\mathbb{P}_\lambda(n)]\right\}p.
\end{aligned}$$

From Lemma 2.2, it follows that

$$\min\left\{1-\mathbb{P}_\lambda(x_0), n!\lambda^{-n}e^{\lambda}\mathbb{P}_\lambda(x_0)[1-\mathbb{P}_\lambda(n)]\right\}p \leq \mathbb{B}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0)$$

$$\leq (e^{\lambda}q^{n} - 1)\mathbb{P}_{\lambda}(x_0).$$

Hence, (3.1) is obtained. □

4. Numerical Examples

Lower and upper bounds for the error of binomial and Poisson cumulative distribution functions with the Poisson mean $\lambda = \frac{np}{q}$ are given in the Table 4.1.

n	p	λ	x_0	Lower Bound	Error	Upper Bound
10	0.15	$\frac{1.5}{0.85}$	0	0.00482066	0.02563726	0.02563726
			1	0.01332771	0.07087949	0.07087949
			2	0.02083393	0.08014391	0.11079910
			3	0.01546574	0.05313515	0.13428122
			4	0.00508646	0.02403564	0.14464098
			5	0.00142318	0.00810466	0.14829737
15	0.10	$\frac{1.5}{0.90}$	0	0.00217992	0.01701553	0.01701553
			1	0.00581311	0.04537474	0.04537474
			2	0.00884078	0.04994343	0.06900742
			3	0.00882672	0.03271152	0.08213669
			4	0.00275433	0.01482277	0.08760722
			5	0.00073020	0.00505229	0.08943073
20	0.10	$\frac{2.0}{0.90}$	0	0.00127775	0.01320863	0.01320863
			1	0.00411719	0.04256115	0.04256115
			2	0.00727212	0.06016559	0.07517505
			3	0.00960911	0.05208148	0.09933350
			4	0.00749215	0.03174699	0.11275486
			5	0.00259822	0.01472910	0.11871991

Table 4.1: Lower and upper bounds on $\mathbb{B}_{n,p}(x_0) - \mathbb{P}_{\lambda}(x_0)$

5. Conclusion

In this study, lower and upper bounds for the error of the binomial distribution function with parameters n and p and the Poisson distribution function with mean $\lambda = \frac{np}{q} = \frac{np}{1-p}$ were obtained by using the Stein-Chen method. It can be indicated that how well these bounds cover this error.

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