OSCILLATION CRITERIA FOR SECOND ORDER
NEUTRAL DIFFERENCE EQUATIONS WITH
NEGATIVE NEUTRAL TERM

E. Thandapani¹, V. Balasubramanian², John R. Graef³ §

¹Ramanujan Institute For Advanced Study in Mathematics

University of Madras

Chennai, 600005, INDIA

²Department of Mathematics

Periyar University

Salem, 636 011, INDIA

³Department of Mathematics

University of Tennessee at Chattanooga

Chattanooga, TN 37403, USA

Abstract: In this paper the authors obtain some new results on the oscillatory behavior of second order neutral difference equations of the form

$$\Delta \left(a_n (\Delta (x_n - p_n x_{n-\tau}))^\alpha \right) + q_n f(x_{n-\sigma}) = 0,$$

where $0 \leq p_n \leq p < 1$, $q_n > 0$, and α is a ratio of odd positive integers. Examples are provided to illustrate the main results.

AMS Subject Classification: 39A11

Key Words: second order, nonlinear, neutral difference equation, oscillation, negative neutral term

1. Introduction

In this paper, we study the oscillation of the second order nonlinear neutral

Received: April 9, 2013

© 2013 Academic Publications, Ltd.
url: www.acadpubl.eu

§Correspondence author

delay difference equation

$$\Delta \left(a_n (\Delta (x_n - p_n x_{n-\tau}))^\alpha \right) + q_n f(x_{n-\sigma}) = 0, \quad n \in \mathbb{N}(n_0), \quad (1)$$

where $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \dots\}$, and n_0 is a nonnegative integer. We assume that the following conditions hold without further mention:

(H₁) $\{a_n\}$ is a positive real sequence with $R_n = \sum_{s=n_0}^{n-1} \frac{1}{a_s^{1/\alpha}} \rightarrow \infty$ as $n \rightarrow \infty$;

(H₂) $\{p_n\}$ is a real sequence with $0 \leq p_n \leq p < 1$, for all $n \in \mathbb{N}(n_0)$;

(H₃) $\{q_n\}$ is a positive real sequence for all $n \in \mathbb{N}(n_0)$;

(H₄) α is a ratio of odd positive integers, and τ and σ are positive integers;

(H₅) $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function with $uf(u) > 0$ for $u \neq 0$ and there exists a constant $L > 0$ such that $\frac{f(u)}{u^\alpha} \geq L$ for all $u \neq 0$.

Let $\theta = \max\{\tau, \sigma\}$. By a *solution* of equation (1), we mean a real sequence $\{x_n\}$ defined for $n \geq n_0 - \theta$ and satisfying equation (1) for all $n \in \mathbb{N}(n_0)$. A solution of equation (1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative, and it is *nonoscillatory* otherwise. From a review of the literature, it is known that there are many results available on the oscillatory and asymptotic behavior of solutions of equation (1) when the neutral term is nonnegative, i.e., $p_n \leq 0$; see, for example, [1, 2, 3, 4, 6, 7] and the references cited therein. However, there are very few results available on the oscillatory behavior of solutions of equation (1) when the neutral term is negative; see, for example, [1, 2, 5, 8, 9, 10, 11, 12], and the references therein.

Motivated by these observations, in this paper we study the oscillatory and asymptotic behavior of equation (1). In Section 2, we establish some preliminary lemmas, and in Section 3, we obtain sufficient conditions for the oscillation of all solutions of equation (1). In Section 4, we present some examples to illustrate our main results.

2. Some Preliminary Lemmas

To prove our main results, we need the following lemmas. Define

$$z_n = x_n - p_n x_{n-\tau}.$$

Lemma 1. *Let $\{x_n\}$ be an eventually positive solution of equation (1). Then for a sufficiently large $N \in \mathbb{N}(n_0)$, we have that either*

$$z_n > 0, \quad a_n \Delta z_n > 0, \quad \Delta(a_n(\Delta z_n)^\alpha) \leq 0, \quad \text{for } n \geq N, \quad (2)$$

or

$$\lim_{n \rightarrow \infty} x_n = 0.$$

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (1) that does not converge to 0 as $n \rightarrow \infty$. Then there exists $n_1 \in \mathbb{N}(n_0)$ such that $x_n > 0$, $x_{n-\tau} > 0$, and $x_{n-\sigma} > 0$ for all $n \geq n_1$. From equation (1), the definition of z_n , and condition (H_5) , we have

$$\Delta(a_n(\Delta z_n)^\alpha) \leq -Lq_n z_{n-\sigma}^\alpha < 0, \quad n \geq n_1. \quad (3)$$

This implies $\{a_n(\Delta z_n)^\alpha\}$ is a decreasing sequence for all $n \geq n_1$. We claim that $\Delta z_n > 0$ for $n \geq n_1$. If not, then there exists $n_2 \geq n_1$ such that $\Delta z_{n_2} < 0$ and so

$$a_n(\Delta z_n)^\alpha \leq a_{n_2}(\Delta z_{n_2})^\alpha = c < 0 \quad \text{for all } n \geq n_2.$$

Dividing the last inequality by a_n and then summing from n_2 to $n-1$, we obtain

$$z_n \leq z_{n_2} + c^{1/\alpha} \sum_{s=n_2}^{n-1} \frac{1}{a_s^{1/\alpha}}. \quad (4)$$

Letting $n \rightarrow \infty$ in (4), we have $z_n \rightarrow -\infty$ by (H_1) . Therefore, there exists a constant $d > 0$ and $n_3 \geq n_2$ such that $z_n \leq -d$ for all $n \geq n_3$. From the definition of z_n ,

$$x_n \leq -d + p_n x_{n-\tau} \leq -d + p x_{n-\tau}, \quad n \geq n_3,$$

or

$$x_{n_3+\tau} \leq -d + p x_{n_3}.$$

Hence,

$$x_{n_3+2\tau} \leq -d - pd + p^2 x_{n_3},$$

and so

$$x_{n_3+m\tau} \leq -d - pd - p^2 d - \dots - p^{m-1} d + p^m x_{n_3}.$$

Since $p < 1$, we see that $x_{n_3+m\tau} < 0$ for all sufficiently large m , which contradicts the fact that $x_n > 0$ for all $n \geq n_3$. Hence, $\Delta z_n > 0$ eventually. Consequently, there are two possible cases:

(I) $z_n > 0$, eventually; or

(II) $z_n < 0$, eventually.

If case (II) holds, then $z_n < 0$ and $\Delta z_n > 0$ for $n \geq N$ for some sufficiently large N . Hence, $\lim_{n \rightarrow \infty} z_n$ exists, and $z_n \leq \ell \leq 0$ for all n sufficiently large. Then,

$$x_n = p_n x_{n-\tau} + z_n < p x_{n-\tau} + \ell. \quad (5)$$

Next we show that $\{x_n\}$ is bounded. If this is not the case, there is a sequence $\{n_k\}$ with $n_k \rightarrow \infty$ as $k \rightarrow \infty$ such that

$$x_{n_k} = \sup_{n_0 \leq j \leq n_k} x_j \quad \text{and} \quad \lim_{k \rightarrow \infty} x_{n_k} = \infty.$$

From (5) with k sufficiently large, we obtain

$$x_{n_k} \leq p x_{n_k} + \ell$$

or

$$(1-p)x_{n_k} \leq \ell,$$

which, as k tends to infinity, leads to a contradiction. Thus, $\{x_n\}$ is bounded.

Let $\limsup_{n \rightarrow \infty} x_n = M > 0$. Then there exists a sequence $\{n_j\}$ such that $x_{n_j} \rightarrow M$ as $j \rightarrow \infty$. Now

$$z_{n_j} \geq x_{n_j} - p x_{n_j-\tau}$$

and so

$$x_{n_j-\tau} \geq \frac{1}{p}(x_{n_j} - z_{n_j}).$$

Letting $j \rightarrow \infty$, we obtain

$$M \geq \lim_{j \rightarrow \infty} x_{n_j-\tau} \geq \frac{M}{p}.$$

Since $p \in (0, 1)$, it follows that $M = 0$. This contradiction shows that case (II) does not hold and completes the proof of the lemma. $\square$

Lemma 2. *If $\{x_n\}$ is an eventually positive solution of equation (1) such that (2) holds, then*

$$z_n \geq R_n a_n^{1/\alpha} \Delta z_n, \quad n \geq N \in \mathbb{N}(n_0), \quad (6)$$

and $\left\{ \frac{z_n}{R_n} \right\}$ is eventually strictly decreasing.

Proof. From (2), for $n \geq N$, we have

$$z_n - z_N = \sum_{s=N}^{n-1} \frac{a_s^{1/\alpha} \Delta z_s}{a_s^{1/\alpha}} \geq a_n^{1/\alpha} \Delta z_n R_n$$

or

$$z_n > R_n a_n^{1/\alpha} \Delta z_n \quad \text{for } n \geq N. \quad (7)$$

Furthermore,

$$\Delta \left(\frac{z_n}{R_n} \right) = \frac{a_n^{1/\alpha} R_n \Delta z_n - z_n}{a_n^{1/\alpha} R_n R_{n+1}} \quad \text{for } n \geq N.$$

By (7), we have $\Delta \left(\frac{z_n}{R_n} \right) < 0$ for all $n \geq N$. Thus, $\left\{ \frac{z_n}{R_n} \right\}$ is strictly decreasing for all $n \geq N$. This completes the proof. $\square$

3. Oscillation Results

In this section, we obtain some new sufficient conditions for the oscillation and asymptotic behavior of solutions of equation (1).

Theorem 3. *If there exists a positive nondecreasing sequence $\{\rho_n\}$ such that*

$$\sum_{n=N}^{\infty} \left[q_n \rho_n \left(\frac{R_{n-\sigma}}{R_{n+1}} \right)^\alpha - \frac{1}{L(\alpha+1)^{\alpha+1}} a_n \frac{(\Delta \rho_n)^{\alpha+1}}{\rho_n^\alpha} \right] = \infty, \quad (8)$$

then any solution $\{x_n\}$ of equation (1) is either oscillatory or satisfies $\lim_{n \rightarrow \infty} x_n = 0$.

Proof. Assume that there exists a nonoscillatory solution $\{x_n\}$ of equation (1), say $x_n > 0$ and $x_{n-\theta} > 0$ for all $n \geq n_1$ for some $n_1 \in \mathbb{N}(n_0)$. By Lemma 2.1, there exists $n_2 \geq n_1$ such that condition (2) holds or $\lim_{n \rightarrow \infty} x_n = 0$.

Suppose (2) holds. Define

$$w_n = \rho_n \frac{a_n (\Delta z_n)^\alpha}{z_n^\alpha}, \quad n \geq n_2. \quad (9)$$

Then $w_n > 0$ and from (9) and (3), we have

$$\Delta w_n \leq -L \rho_n q_n \frac{z_{n-\sigma}^\alpha}{z_{n+1}^\alpha} + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n}{\rho_{n+1}} w_{n+1} \frac{\Delta z_n^\alpha}{z_n^\alpha} \quad \text{for all } n \geq n_2. \quad (10)$$

By the Mean Value Theorem and the fact that z_n is nondecreasing, we have

$$\Delta z_n^\alpha \geq \begin{cases} \alpha z_n^{\alpha-1} \Delta z_n, & \text{if } \alpha \geq 1 \\ \alpha z_{n+1}^{\alpha-1} \Delta z_n, & \text{if } \alpha < 1. \end{cases}$$

Since $a^{1/\alpha} \Delta z_n$ is positive and nonincreasing, and z_n is nondecreasing, from (9), we obtain

$$\frac{w_{n+1}^{1/\alpha}}{\rho_{n+1}^{1/\alpha}} = a_{n+1}^{1/\alpha} \frac{\Delta z_{n+1}}{z_{n+1}} \leq \frac{a_n^{1/\alpha} \Delta z_n}{z_{n+1}} \leq \frac{a_n^{1/\alpha} \Delta z_n}{z_n}.$$

If $\alpha \geq 1$,

$$\frac{w_{n+1}^{1/\alpha}}{\rho_{n+1}^{1/\alpha}} \leq \frac{a_n^{1/\alpha} \Delta z_n}{z_n},$$

so

$$\frac{\alpha w_{n+1}^{1/\alpha}}{\rho_{n+1}^{1/\alpha} a_n^{1/\alpha}} \leq \frac{\alpha \Delta z_n}{z_n} \leq \frac{\Delta z_n^\alpha}{z_n^\alpha}.$$

If $\alpha < 1$,

$$\frac{\alpha w_{n+1}^{1/\alpha}}{\rho_{n+1}^{1/\alpha} a_n^{1/\alpha}} \leq \frac{\alpha \Delta z_n}{z_{n+1}} \leq \frac{\Delta z_n^\alpha}{z_{n+1}^\alpha} \leq \frac{\Delta z_n^\alpha}{z_n^\alpha}.$$

Hence, for any $\alpha > 0$, we have

$$-\frac{\Delta z_n^\alpha}{z_n^\alpha} \leq -\frac{\alpha w_{n+1}^{1/\alpha}}{a_n^{1/\alpha} \rho_{n+1}^{1/\alpha}} \quad \text{for } n \geq n_2. \quad (11)$$

Using (11) and the decreasing nature of $\left\{ \frac{z_n}{R_n} \right\}$, (10) implies

$$\Delta w_n \leq -L \rho_n q_n \left(\frac{R_{n-\sigma}}{R_{n+1}} \right)^\alpha + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\alpha \rho_n}{\rho_{n+1}^{1+1/\alpha} a_n^{1/\alpha}} w_{n+1}^{1+1/\alpha}, \quad \text{for } n \geq n_2. \quad (12)$$

If we define the function F by

$$F(u) = \frac{\Delta \rho_n}{\rho_{n+1}} u - \frac{\alpha \rho_n}{\rho_{n+1}^{1+1/\alpha} a_n^{1/\alpha}} u^{1+1/\alpha},$$

then it is easy to show that $F(u) \leq \frac{(\Delta \rho_n)^{\alpha+1} a_n}{(\alpha+1)^{\alpha+1} \rho_n^\alpha}$. Therefore, from (12), we have

$$\Delta w_n \leq -L \rho_n q_n \left(\frac{R_{n-\sigma}}{R_{n+1}} \right)^\alpha + \frac{(\Delta \rho_n)^{\alpha+1} a_n}{\rho_n^\alpha (\alpha+1)^{\alpha+1}} \quad \text{for } n \geq n_2.$$

Summing the last inequality from $N \geq n_2$ to $n - 1$, we obtain

$$0 < w_n \leq w_N - L \sum_{s=N}^{n-1} \left[\rho_s q_s \left(\frac{R_{s-\sigma}}{R_{s+1}} \right)^\alpha - \frac{1}{L(\alpha+1)^{\alpha+1}} a_s \frac{(\Delta \rho_s)^{\alpha+1}}{\rho_s^\alpha} \right].$$

Letting $n \rightarrow \infty$ in the above inequality yields a contradiction to (8). This completes the proof of the theorem. $\square$

If $a_n = 1$, then $R_n = n$. Choosing $\rho_n = n^{(\alpha)} = n(n-1) \cdots (n-\alpha+1)$ in Theorem 8, we obtain the following corollary.

Corollary 4. *If*

$$\sum_{n=n_0}^{\infty} \left[n^{(\alpha)} q_n \left(\frac{n-\sigma}{n+1} \right)^\alpha - \frac{\alpha}{L(\alpha+1)^{\alpha+1}} \frac{n^{(\alpha-1)}}{(n-\alpha+1)^\alpha} \right] = \infty, \quad (13)$$

where $\alpha \geq 1$ is an odd integer, then any solution $\{x_n\}$ of equation (1) is either oscillatory or satisfies $\lim_{n \rightarrow \infty} x_n = 0$.

Our next oscillation result employs a somewhat different type of condition on the coefficients in (1).

Theorem 5. *If there exists a nonnegative real sequence $\{h_n\}$ such that*

$$\sum_{n=N}^{\infty} \left[\left(L q_n \left(\frac{R_{n-\sigma}}{R_{n+1}} \right)^\alpha - \frac{1}{a_n^{1/\alpha}} h_n^{\alpha+1} \right) \prod_{s=N}^{n-1} \left(1 + \frac{(\alpha+1)h_s}{a_s^{1/\alpha}} \right) \right] = \infty, \quad (14)$$

then any solution $\{x_n\}$ of equation (1) is either oscillatory or satisfies $\lim_{n \rightarrow \infty} x_n = 0$.

Proof. Proceeding as in the proof of Theorem 8 and taking $\rho_n \equiv 1$, (12) becomes

$$\Delta w_n \leq -L q_n \left(\frac{R_{n-\sigma}}{R_{n+1}} \right)^\alpha - \frac{\alpha}{a_n^{1/\alpha}} w_{n+1}^{1+1/\alpha}, \quad n \geq N.$$

Then,

$$\Delta w_n \leq - \left[L q_n \left(\frac{R_{n-\sigma}}{R_{n+1}} \right)^\alpha - \frac{h_n^{\alpha+1}}{a_n^{1/\alpha}} \right] - \frac{1}{a_n^{1/\alpha}} \left(\alpha w_{n+1}^{1+1/\alpha} + h_n^{\alpha+1} \right) \quad (15)$$

for $n \geq N$. From Holder's inequality, we have

$$\alpha w_{n+1}^{1+1/\alpha} + h_n^{\alpha+1} \geq (\alpha+1) w_{n+1} h_n, \quad n \geq N. \quad (16)$$

so using this in (15), we obtain

$$\Delta w_n \leq - \left[Lq_n \left(\frac{R_{n-\sigma}}{R_{n+1}} \right)^\alpha - \frac{h_n^{\alpha+1}}{a_n^{1/\alpha}} \right] - \frac{(\alpha+1)}{a_n^{1/\alpha}} h_n w_{n+1}, \quad n \geq N,$$

or

$$\Delta \left(w_n \prod_{s=N}^{n-1} B_s \right) \leq -A_n \prod_{s=N}^{n-1} B_s,$$

where $A_n = Lq_n \left(\frac{R_{n-\sigma}}{R_{n+1}} \right)^\alpha - \frac{h_n^{\alpha+1}}{a_n^{1/\alpha}}$ and $B_n = 1 + \frac{(\alpha+1)h_n}{a_n^{1/\alpha}}$. A summation of the last inequality from N to $n-1$ gives

$$\sum_{s=N}^{n-1} \left(A_s \prod_{t=N}^{s-1} B_t \right) \leq w_N - w_n \prod_{s=N}^{n-1} B_s \leq w_N.$$

Letting $n \rightarrow \infty$, we obtain

$$\sum_{n=N}^{\infty} \left[\left(Lq_n \left(\frac{R_{n-\sigma}}{R_{n+1}} \right)^\alpha - \frac{h_n^{\alpha+1}}{a_n^{1/\alpha}} \right) \prod_{s=N}^{n-1} \left(1 + \frac{(\alpha+1)h_s}{a_s^{1/\alpha}} \right) \right] < \infty,$$

which contradicts (14). This proves the theorem. $\square$

Taking $h_n \equiv 0$ in Theorem 5 gives the following corollary.

Corollary 6. *If*

$$\sum_{n=N}^{\infty} q_n \left(\frac{R_{n-\sigma}}{R_{n+1}} \right)^\alpha = \infty, \quad (17)$$

then any solution $\{x_n\}$ of equation (1) is either oscillatory or satisfies $\lim_{n \rightarrow \infty} x_n = 0$.

4. Examples

In this section, we present some examples to illustrate our main results.

Example 7. Consider the difference equation

$$\Delta^2 \left(x_n - \frac{1}{2} x_{n-2} \right) + \frac{1}{2^6} \left(\frac{4^{n-4}}{1 + 4^{n-4}} \right) x_{n-4} (1 + x_{n-4}^2) = 0, \quad n \geq 1. \quad (18)$$

Here, $a_n = 1$, $p_n = \frac{1}{2}$, $\tau = 2$, $\sigma = 4$, $\alpha = 1$, $q_n = \frac{1}{2^6} \left(\frac{4^{n-4}}{1+4^{n-4}} \right)$, and $L = 1$. By taking $\rho_n = 1$, we see that all the conditions of Theorem 3 are satisfied and so any solution of (18) is either oscillatory or tends to zero as $n \rightarrow \infty$. In fact, $\{x_n\} = \{\frac{1}{2^n}\}$ is one such solution of the latter type.

Example 8. Consider the difference equation

$$\Delta \left(n \left(\Delta \left(x_n - \frac{1}{2} x_{n-2} \right) \right)^3 \right) + \frac{(2n+1)}{2} x_{n-1}^3 (1 + x_{n-1}^2) = 0, \quad n \geq 1. \quad (19)$$

We have $a_n = n$, $p_n = \frac{1}{2}$, $\tau = 2$, $\sigma = 1$, $\alpha = 3$, $q_n = \frac{2n+1}{2}$, and $L = 1$. Taking $\rho_n = 1$, we see that all the conditions of Theorem 3 are satisfied and hence any solution of (19) is either oscillatory or tends to zero as $n \rightarrow \infty$. Here $\{x_n\} = \{(-1)^n\}$ is an oscillatory solution.

Example 9. Consider the equation

$$\Delta \left(n \Delta \left(x_n - \frac{1}{3} x_{n-2} \right) \right) + \frac{4}{27} (2n-1) x_{n-1} = 0, \quad n \geq 1. \quad (20)$$

Here, $a_n = n$, $p_n = \frac{1}{3}$, $\tau = 2$, $\sigma = 1$, $\alpha = 1$, $q_n = \frac{4}{9} (2n-1)$, and $L = 1$. Choosing $h_n = 1$, we see that all the conditions of Theorem 5 are satisfied and any solution of (20) is either oscillatory or tends to zero as $n \rightarrow \infty$. We have that $\{x_n\} = \{\frac{1}{3^n}\}$ is one such solution.

Example 10. Consider the difference

$$\Delta^2 \left(x_n - \frac{1}{2} x_{n-2} \right) + \left(\frac{2(n+4)}{n} \right) x_{n-1} = 0, \quad n \geq 1. \quad (21)$$

In this case $a_n = 1$, $p_n = \frac{1}{2}$, $\tau = 2$, $\sigma = 1$, $\alpha = 1$, and $q_n = \frac{2(n+4)}{n}$. If we take $h_n = 0$, all the conditions of Corollary 6 are satisfied and so any solution of (21) is either oscillatory or tends to zero as $n \rightarrow \infty$. Here $\{x_n\} = \{(-1)^n (n+1)\}$ is an oscillatory solution.

We conclude this paper with the following remark.

Remark 11. It would be interesting to obtain results similar to those in this paper but with the condition $\sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} < \infty$ rather than (H_1) .

References

- [1] R. P. Agarwal, *Difference Equations and Inequalities, Theory, Methods and Applications*, Second Edition, Marcel Dekker, New York, 2000.

- [2] R. P. Agarwal, M. Bohner, S. R. Grace, and D. O'Regan, *Discrete Oscillation Theory*, Hindawi Publ., New York, 2005.
- [3] R. P. Agarwal, M. M. S. Manuel, and E. Thandapani, Oscillatory and nonoscillatory behavior of second order neutral delay difference equations, *Math. Comput. Model.* **24** (1996), 5–11.
- [4] R. P. Agarwal, M. M. S. Manuel, and E. Thandapani, Oscillatory and nonoscillatory behavior of second order neutral delay difference equations II, *Appl. Math. Lett.* **10** (1997), 103–109.
- [5] S. R. Grace and J. R. Graef, Oscillation criteria for fourth order nonlinear neutral delay dynamic equations on time scales, *Global J. Pure Appl. Math.* **7** (2011), 439–447.
- [6] H. J. Li and C. C. Yeh, Oscillation criteria for second order neutral delay difference equations, *Comput. Math. Appl.* **36** (1998), 123–132.
- [7] S. H. Saker, New oscillation criteria for second order nonlinear neutral delay difference equations, *Appl. Math. Comput.* **142** (2003), 99–111.
- [8] Y. G. Sun and S. H. Saker, Oscillation for second order nonlinear neutral delay difference equations, *Appl. Math. Comput.* **163** (2005), 909–918.
- [9] E. Thandapani and K. Mahalingam, Necessary and sufficient conditions for oscillation of second order neutral difference equations, *Tamkang J. Math.* **34** (2003), 137–145.
- [10] E. Thandapani and P. Mohankumar, Oscillation and nonoscillation of nonlinear neutral delay difference equations, *Tamkang J. Math.* **38** (2007), 323–333.
- [11] E. Thandapani, P. Sundaram, J. R. Graef, and P. W. Spikes, Asymptotic properties of solutions of nonlinear second order neutral difference equations, *Dyn. Sys. Appl.* **4** (1995), 125–136.
- [12] A. Zafer and R. S. Dahiya, Oscillation of a neutral difference equations, *Appl. Math. Lett.* **6** (1993), 71–74.