SHORT NOTE ON THE DISTRIBUTIONAL DIFFRACTION FRESNEL SINE (COSINE) TRANSFORM

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Abstract: The diffraction Fresnel sine and diffraction Fresnel cosine transforms are extended to spaces of distributions of compact support. The convolution Theorem of the transforms has been established in a generalized sense. Certain theorems are also discussed.

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1. Introduction

The diffraction Fresnel transform of a function \( f(t) \) is defined by [5]

\[
F_d f (\xi) = \int_{\mathbb{R}} K (\alpha_1, \gamma_1, \gamma_2, \alpha_2; \xi, t) f(t) \, dt.
\]  

(1)

\( K (\alpha_1, \gamma_1, \gamma_2, \alpha_2; \xi, t) \) being the transform kernel function given by

\[
K (\alpha_1, \gamma_1, \gamma_2, \alpha_2; \xi, t) = \frac{1}{\sqrt{2\pi i \gamma_1}} \exp \left( \frac{i}{2\gamma_1} (\alpha_1 t^2 - 2\xi t + \alpha_2 \xi^2) \right),
\]

where the real parameters, \( \alpha_1, \gamma_1, \gamma_2 \) and the parameter \( \alpha_2 \) are defined so that
\[\alpha_1 \alpha_2 - \gamma_1 \gamma_2 = 1.\]

If the parameters \(\alpha_1, \gamma_1, \gamma_2\) and \(\alpha_2\) are related by the matrix
\[
\begin{pmatrix}
\alpha_1 & \gamma_1 \\
\gamma_2 & \alpha_2
\end{pmatrix} = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\]
then the diffraction Fresnel transform becomes a fractional Fourier transform.

Let \(S\) denote the space of all complex valued functions \(\phi(t)\) that are infinitely smooth and satisfies the infinite set of inequalities
\[
\left| t^m \phi^{(k)}(t) \right| \leq C_{m,k}, t \in \mathbb{R},
\]
where \(m\) and \(k\) traverse the set of nonnegative integers \([5, (2.1)]\). Members of \(S\) are testing functions of rapid descent. The strong dual \(S'\) of \(S\) defines a space of distributions of slow growth (the space of tempered distributions). See \([7, 6, 4]\).

The Parseval’s relation of the diffraction Fresnel transform was established in \([5, \text{Theorem 2.2}]\) as
\[
\int_{\mathbb{R}} f(x) F_d g(x) \, dx = \int_{\mathbb{R}} F_d f(x) g(x) \, dx
\]
where \(F_d g\) and \(F_d f\) are the respective diffraction Fresnel transforms of \(f\) and \(g\).

2. Distribution Spaces and \(F_d\) Analysis

When discussing the distribution spaces \(S'\); see \([7]\), the extended transform \(\mapsto F_d\) of a slow growth distribution \(f \in S'\) is expressed as \([5, (2.10)]\)
\[
\left\langle \mapsto F_d f, \phi \right\rangle = \left\langle f, F_d \phi \right\rangle,
\]
for every \(\phi \in S\).

Let \(f \in \mathcal{E}'\) (the space of distributions of compact supports) then the distributional transform of \(f\) is extended in the same citation as
\[
\mapsto F_d f (\xi) = \frac{1}{\sqrt{2\pi i \gamma_1}} \left\langle f(t), \exp \left( i \frac{(\alpha_1 t^2 - 2t\xi + \alpha_2 \xi^2)}{2\gamma_1} \right) \right\rangle;
\]
see \([5, (2.10)]\).

In this note, let (1) be factored into the components:
\[
F_d f (\xi) = F_{c} f (\xi) + i F_{s} f (\xi)
\]
where
\[ \mathcal{F}_s f(\xi) = \frac{e^{i \alpha^2 \xi^2}}{\sqrt{2\pi i \gamma_1}} \int_\mathbb{R} f(t) \sin \left( \frac{\alpha_1 t^2 - 2\xi t}{2\gamma_1} \right) dt \]  
(5)
and
\[ \mathcal{F}_c f(\xi) = \frac{e^{i \alpha^2 \xi^2}}{\sqrt{2\pi i \gamma_1}} \int_\mathbb{R} f(t) \cos \left( \frac{\alpha_1 t^2 - 2\xi t}{2\gamma_1} \right) dt. \]  
(6)

then the integral equations (5) and (6) interpreted to present a diffraction Fresnel sine and diffraction Fresnel cosine transforms, respectively.

From (4) we have the following definition:

**Definition 2.1.** Let \( f \in \mathcal{E}'(\mathbb{R}) \) then the extended diffraction Fresnel sine and diffraction Fresnel cosine transforms of \( f \) are defined respectively as

\[ \mathcal{F}_s f(\xi) = e^{i \alpha^2 \xi^2} \left( f(t), \sin \left( \frac{\alpha_1 t^2 - 2\xi t}{2\gamma_1} \right) \right), \]  
(7)

and
\[ \mathcal{F}_c f(\xi) = e^{i \alpha^2 \xi^2} \left( f(t), \cos \left( \frac{\alpha_1 t^2 - 2\xi t}{2\gamma_1} \right) \right). \]  
(8)

Analyticity of \( \mathcal{F}_s \) and \( \mathcal{F}_c \) can be expressed to mean:

**Theorem 2.2.** Let \( f \in \mathcal{E}'(\mathbb{R}) \) then \( \mathcal{F}_s \) and \( \mathcal{F}_c \) are analytic and

\[ \mathcal{D}_\xi \mathcal{F}_s f(\xi) = \left( \mathcal{D}_\xi \mathcal{F}_s f(\xi), \mathcal{D}_\xi e^{i \alpha^2 \xi^2} \sin \left( \frac{\alpha_1 t^2 - 2\xi t}{2\gamma_1} \right) \right), \]  
(9)

and
\[ \mathcal{D}_\xi \mathcal{F}_c f(\xi) = \left( \mathcal{D}_\xi \mathcal{F}_c f(\xi), \mathcal{D}_\xi e^{i \alpha^2 \xi^2} \cos \left( \frac{\alpha_1 t^2 - 2\xi t}{2\gamma_1} \right) \right). \]  
(10)

Proof of this theorem is analogous to that found in the literature; see the corresponding theorem in [5]

**Theorem 2.3.** The transforms (7) and (8) are linear.

The proof of theorem follows from simple computations in \( \mathcal{E}' \).

Let \( f \) and \( g \) be in \( \mathcal{E}' \) then the generalized convolution of \( f \) and \( g \) is defined by [7, P.123, (2)]

\[ \langle f \ast g, \phi \rangle = \langle f(t) \times g(\tau), \phi(t + \tau) \rangle = \langle f(t), \langle g(\tau), \phi(t + \tau) \rangle \rangle. \]
We have the following theorem.

**Theorem 2.4.** (Convolution Theorem) Let \( f \in \mathcal{E}', g \in \mathcal{E}' \) then

\[
\vec{F}_s (f \ast g) (s) = \left\langle f (t) \sin \frac{st}{\gamma_1}, \vec{F}_c(g(x)) (t) \right\rangle + \left\langle f (t) \cos \frac{st}{\gamma_1}, \vec{F}_s(g(x)) (t) \right\rangle.
\]

**Proof.** Since the diffraction Fresnel sine transform is defined by

\[
(F_s f) (s) = \frac{e^{\frac{i}{2} \left( \frac{\alpha_2}{\gamma_1} \right) s^2}}{\sqrt{2\pi i \gamma_1}} \int_{\mathbb{R}} \sin \left( \frac{s}{\gamma_1} t \right) e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) t^2} f (t) \, dt,
\]

it acts on \( \ast \) as

\[
\vec{F}_s (f \ast g) (s) = \frac{e^{\frac{i}{2} \left( \frac{\alpha_2}{\gamma_1} \right) s^2}}{\sqrt{2\pi i \gamma_1}} \left\langle f \ast g (t), e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) t^2} \sin \left( \frac{s}{\gamma_1} t \right) \right\rangle
\]

\[
= \frac{e^{\frac{i}{2} \left( \frac{\alpha_2}{\gamma_1} \right) s^2}}{\sqrt{2\pi i \gamma_1}} \left\langle f (t), \left\langle g (x), e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) t^2} \sin \left( \frac{s}{\gamma_1} (t + x) \right) \right\rangle \right\rangle.
\]

Using the fact that

\[
\sin \left( \frac{st}{\gamma_1} + \frac{sx}{\gamma_1} \right) = \sin \frac{st}{\gamma_1} \cos \frac{sx}{\gamma_1} + \sin \frac{sx}{\gamma_1} \cos \frac{st}{\gamma_1}
\]

we get

\[
\vec{F}_s (f \ast g) (s) = \frac{e^{\frac{i}{2} \left( \frac{\alpha_2}{\gamma_1} \right) s^2}}{\sqrt{2\pi i \gamma_1}} \times
\]

\[
\left\langle f (t), \left\langle g (x), e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) (t+x)^2} \left( \sin \frac{st}{\gamma_1} \cos \frac{sx}{\gamma_1} + \sin \frac{sx}{\gamma_1} \cos \frac{st}{\gamma_1} \right) \right\rangle \right\rangle
\]

\[
= \frac{e^{\frac{i}{2} \left( \frac{\alpha_2}{\gamma_1} \right) s^2}}{\sqrt{2\pi i \gamma_1}} \left\langle f (t), \left\langle g (x), e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) t^2} \sin \frac{st}{\gamma_1} \cos \frac{sx}{\gamma_1} \right\rangle \right\rangle
\]

\[
+ \frac{e^{\frac{i}{2} \left( \frac{\alpha_2}{\gamma_1} \right) s^2}}{\sqrt{2\pi i \gamma_1}} \left\langle f (t), \left\langle g (x), e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) t^2} \sin \frac{st}{\gamma_1} \cos \frac{st}{\gamma_1} \right\rangle \right\rangle
\]

\[
= \frac{e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) t^2}}{\gamma_1} \frac{e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) 2tx}}{\gamma_1} \frac{e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) 2tx}}{\gamma_1} \sin \frac{sx}{\gamma_1}
\]

\[
+ \frac{e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) t^2}}{\gamma_1} \frac{e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) 2tx}}{\gamma_1} \frac{e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) 2tx}}{\gamma_1} \sin \frac{sx}{\gamma_1}
\]
\[
= \left\langle e^{\frac{1}{2} \left( \frac{\alpha_1}{\gamma_1} \right) t^2} f(t) \sin \frac{st}{\gamma_1}, \mathcal{F}_c \left( e^{\frac{1}{2} \left( \frac{\alpha_1}{\gamma_1} \right) (2tx) g(x) \right)(t) \right\rangle \\
+ \left\langle e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) t^2} f(t) \cos \frac{st}{\gamma_1}, \mathcal{F}_s \left( e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) (2tx) g(x) \right)(t) \right\rangle.
\]

This completes the proof of the theorem.

**Theorem 2.5.** Let \( f \in \mathcal{E}'(\mathbb{R}) \) then we have

\[
\left( \mathcal{F}_s \sin (ut) f(t) \right)(s) = \frac{e^{-\frac{i}{2} (\alpha_2 u^2 \gamma_1)}}{2} e^{-i \alpha_2 s u} \left( \mathcal{F}_c f \right)(s + \gamma_1 u).
\]

**Proof.** Let \( f \in \mathcal{E}'(\mathbb{R}) \) then

\[
\left( \mathcal{F}_s \sin (ut) f(t) \right)(s) = e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) s^2} \frac{1}{\sqrt{2 \pi i \gamma_1}} \left\langle \sin (ut) f(t), e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) t^2} \sin \left( \frac{s}{\gamma_1} t \right) \right\rangle
\]

\[
= e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) s^2} \frac{1}{\sqrt{2 \pi i \gamma_1}} \left\langle f(t), e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) t^2} \sin (ut) f(t) \sin \left( \frac{s}{\gamma_1} t \right) \right\rangle. \quad (13)
\]

Using the identity

\[
2 \sin x \sin y = \cos (x - y) - \cos (x + y),
\]

(13) becomes

\[
\left( \mathcal{F}_s \sin (ut) f(t) \right)(s)
= e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) s^2} \frac{1}{\sqrt{2 \pi i \gamma_1}} \left\langle f(t), e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) t^2} \cos \left( \frac{s}{\gamma_1} t - ut \right) - \cos \left( \frac{s}{\gamma_1} t + ut \right) \right\rangle. \quad (14)
\]

Calculations on (14) together with the equation

\[
e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) s^2} = e^{-\frac{i}{2} (\alpha_2 u^2 \gamma_1)} e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) (s-u \gamma_1^2)} e^{i \alpha_2 s u}
\]

and that

\[
e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) s^2} = e^{-\frac{i}{2} (\alpha_2 u^2 \gamma_1)} e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) (s+u \gamma_1^2)} e^{-i \alpha_2 s u},
\]

when employed to (14), give

\[
\left( \mathcal{F}_s \sin (ut) f(t) \right)(s) = \left( \frac{1}{2} e^{-\frac{i}{2} (\alpha_2 u^2 \gamma_1)} e^{i \alpha_2 s u} \right) \frac{e^{\frac{i}{2} \left( \frac{\alpha_1}{\gamma_1} \right) (s-u \gamma_1^2)}}{\sqrt{2 \pi i \gamma_1}}
\]
\[
\times \left\langle f(t), e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)t^2} \cos\left(\frac{s-u\gamma_1}{\gamma_1}\right)t \right\rangle \\
- \left(\frac{1}{2}e^{-\frac{i}{2}(\alpha_2 u^2 \gamma_1)} e^{-i\alpha_2 su}\right) \\
\times \frac{e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)(s-u\gamma_1)^2}}{\sqrt{2\pi i \gamma_1}} \left\langle f(t), e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)t^2} \cos\left(\frac{s+u\gamma_1}{\gamma_1}\right)t \right\rangle.
\]

which produces our desired result.

**Theorem 2.6.** Let \( f \in \mathcal{E}'(\mathbb{R}) \) then

\[
\left(\mathcal{F}_c \cos(ut)f(t)\right)(s) = \frac{e^{-\frac{i}{2}(\alpha_2 u^2 \gamma_1)}}{2} \left( e^{-i\alpha_2 su} \mathcal{F}_s f(s + \gamma_1 u) + e^{i\alpha_2 su} \mathcal{F}_s f(s - \gamma_1 u) \right).
\]

Proof of this theorem is quite analogous to that of Theorem 2.5, thus avoided.

**References**


