ON 3-DIMENSIONAL (\(\varepsilon, \delta\))-TRANS-SASAKIAN STRUCTURE

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Abstract: The object of present paper is to study 3-dimensional \((\varepsilon, \delta)\)-trans-Sasakian manifold admitting Ricci solitons and \(K\)-torse forming vector fields. We prove the conditions for the Ricci solitons to be shrinking, expanding and steady. Further, we have obtained a condition for the vector field \(\xi\) in a generalized recurrent 3-dimensional \((\varepsilon, \delta)\)-trans-Sasakian manifold to be co-symplectic. We have also shown that such a manifold \(M\) is of constant scalar curvature.

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1. Introduction

The concept of \((\varepsilon)\)-Sasakian manifolds was introduced by A. Bejancu and K.L. Duggal [1] and further investigation was taken up by Xufend and Xiaoli[13] and Rakesh kumar et al [7]. De and Sarkar [3] introduced and studied conformally flat, Weyl semisymmetric, \(\varphi\)-recurrent \((\varepsilon)\)-Kenmotsu manifolds. In [1], the authors obtained Riemannian curvature tensor of \((\varepsilon)\)-Sasakian manifolds and established relations among different curvatures. H.G. Nagraja et al [4] have studied \((\varepsilon, \delta)\)-trans-Sasakian structures which generalizes both \((\varepsilon)\)-manifolds and \((\varepsilon)\)-Kenmotsu manifolds.
2. Preliminaries

Let \((M, g)\) be an almost contact metric manifold of dimension \(n\) equipped with an almost contact metric structure \((\varphi, \xi, \eta, g)\) consisting of a \((1,1)\) tensor field \(\varphi\), a vector field \(\xi\), a 1-form \(\eta\) and a Riemannian metric \(g\) satisfying

\[
\varphi^2 = -I + \eta \otimes \xi, \quad (2.1)
\]
\[
\eta(\xi) = 1, \quad (2.2)
\]
\[
\varphi \xi = 0, \quad \eta \circ \varphi = 0. \quad (2.3)
\]

An almost contact metric manifold \(M\) is called an \((\varepsilon)\)-almost contact metric manifold if

\[
g(\xi, \xi) = \varepsilon, \quad (2.4)
\]
\[
\eta(X) = \varepsilon g(X, \xi), \quad (2.5)
\]
\[
g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon \eta(X) \eta(Y), \quad \forall X, Y \in TM, \quad (2.6)
\]

where \(\varepsilon = g(\xi, \xi) = \pm 1\). An \((\varepsilon)\)-almost contact metric manifold is called an \((\varepsilon, \delta)\)-trans-Sasakian manifold if

\[
(\nabla_X \phi)Y = \alpha [g(X, Y)\xi - \varepsilon \eta(Y)X] + \beta [g(\varphi X, Y)\xi - \delta \eta(Y)\varphi X], \quad (2.7)
\]

holds for some smooth functions \(\alpha\) and \(\beta\) on \(M\) and \(\varepsilon = \pm 1, \delta = \pm 1\). For \(\beta = 0, \alpha = 1\), an \((\varepsilon, \delta)\)-trans-Sasakian manifold reduces to an \((\varepsilon)\)-Sasakian and for \(\alpha = 0, \beta = 1\) it reduces to a \((\delta)\)-Kenmotsu manifold.

Let \((M, g)\) be a \((\varepsilon, \delta)\)-trans-Sasakian manifold. Then from (2.7), it is easy to see that

\[
(\nabla_X \xi) = \varepsilon \alpha \phi X - \beta \delta \phi^2 X, \quad (2.8)
\]
\[
(\nabla_X \eta)Y = \alpha g(Y, \phi X) + \varepsilon \delta g(\phi X, \phi Y) + \xi \alpha + 2 \alpha \beta = 0. \quad (2.10)
\]

In a 3-dimensional \((\varepsilon, \delta)\)-trans-Sasakian manifold, the curvature tensor \(R\) and Ricci tensor \(S\) are given by[5]

\[
R(X, Y)Z = (2A - \frac{r}{2})(g(Y, Z)X - g(X, Z)Y) + B(g(Y, Z)\eta(X) - g(X, Z)\eta(Y))\xi + B\eta(Z)(\eta(Y)X - \eta(X)Y), \quad (2.11)
\]
\[
R(X, Y)\xi = \varepsilon[(\alpha^2 - \beta^2) + (\frac{2 - \varepsilon}{2})r] \eta(Y)X - \eta(X)Y, \quad (2.12)
\]
\[ S(X,Y) = Ag(X,Y) + B\eta(X)\eta(Y), \] (2.13)

where \( \varepsilon\delta = 1, A = (\frac{\varepsilon}{2} - (\alpha^2 - \beta^2)) \), \( B = (3(\alpha^2 - \beta^2) - \varepsilon \frac{r}{2}) \) and \( r \) is the scalar curvature.

Let \((M, g)\) be a Riemannian manifold with metric \( g \). The metric \( g \) is called a Ricci soliton if [8]
\[ L_V g + 2S + 2\lambda g = 0, \] (2.14)

where \( L \) is the Lie derivative, \( S \) is a Ricci tensor, \( V \) is a complete vector field on \( M \) and \( \lambda \) is a constant. So Ricci soliton is a generalization of Einstein metric. In [9], Ramesh Sharma started the study of the Ricci solitons in contact geometry. Later Mukutmani Tripathi [11], Cornelia Livia Bejan and Mircea Crasmareanu [2] and others extensively studied Ricci solitons in contact metric manifolds. The Ricci soliton is said to be shrinking, steady and expanding according as \( \lambda \) is positive, zero and negative respectively.

We also study \( K \)-torse forming vector fields in 3-dimensional \((\varepsilon, \delta)\)-trans-Sasakian manifold. Torse forming vector fields were introduced by K.Yano [6],
\[ \nabla_X \rho = aX + \pi(X)\rho, \] (2.15)
where \( \rho \) is a vector field and \( \pi \) is a non-zero 1-form. Further, a complex analogue of a torse forming vector field is called \( K \)-torse forming vector field and it was introduced by S.Yamaguchi [10],
\[ \nabla_X \rho = aX + b\varphi X + B(X)\rho + D(X)\varphi \rho, \] (2.16)
where \( \rho \) is a vector field, \( a \) and \( b \) are functions, \( B(X) \) and \( D(X) \) are 1-forms.

A 3-dimensional \((\varepsilon, \delta)\)-trans-Sasakian manifold \((M, g)\) is called generalized recurrent [12], if its curvature tensor \( R \) satisfies the condition
\[ (\nabla_X R)(Y, Z)(W) = A(X)R(Y, Z)W + B(X)[g(Z, W)Y - g(Y, W)Z], \] (2.17)
where \( A \) and \( B \) are two 1-forms and \( B \) is non-zero.

### 3. Ricci Soliton

Let \( M \) be a 3-dimensional \((\varepsilon, \delta)\)-trans-Sasakian manifold with metric \( g \). A Ricci soliton is a generalization of an Einstein metric and defined on a Riemannian manifold \((M, g)\) by (2.14).

Let \( V \) be pointwise collinear with \( \xi \) ie., \( V = b\xi \), then (2.14) implies
\[ g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \] (3.1)
which is reduced to
\[ bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \] (3.2)

Using (2.8) in (3.2), we obtain
\[ 2b\beta \delta g(X, Y) - 2b\beta \eta(X)\eta(Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \] (3.3)

Replace \( Y \) by \( \xi \) in (3.3) and use (2.13) to get
\[ 2b\beta \delta \varepsilon \eta(X) - 2b\beta \eta(X) + (Xb) + (\xi b)\eta(X) + 2A\varepsilon \eta(X) + 2B\eta(X) + 2\lambda \varepsilon \eta(X) = 0. \] (3.4)

Again putting \( X = \xi \) in (3.4), we obtain
\[ \xi b = -b\beta \delta \varepsilon + b\beta - A\varepsilon - B - \lambda \varepsilon. \] (3.5)

By using (3.5) in (3.4), we get
\[ Xb = [-b\beta \delta \varepsilon + b\beta - A\varepsilon - B - \lambda \varepsilon]\eta(X), \] (3.6)

or
\[ db = [-b\beta \delta \varepsilon + b\beta - A\varepsilon - B - \lambda \varepsilon]\eta, \] (3.7)

where we have put \( \nabla_X = d \).

Applying \( d \) on (3.7), we get
\[ [b\beta \delta \varepsilon - b\beta + A\varepsilon + B + \lambda \varepsilon]d\eta = 0. \] (3.8)

Since \( d\eta \neq 0 \) we have
\[ b\beta \delta \varepsilon - b\beta + A\varepsilon + B + \lambda \varepsilon = 0. \] (3.9)

Using (3.9) in (3.7) then \( db = 0 \) and hence \( b \) is a constant. Therefore from (3.3) it follows
\[ S(X, Y) = -(\lambda + b\beta \delta)g(X, Y) + b\beta \eta(X)\eta(Y), \] (3.10)

which implies that \( M \) is of constant scalar curvature provided \( \beta = \text{constant} \).

This leads to the following:

**Theorem 3.1.** Let \( M \) be a 3-dimensional \((\varepsilon, \delta)\)-trans-Sasakian manifold in which the metric tensor \( g \) is Ricci soliton with the vector field \( V \) as pointwise collinear with \( \xi \), then \( M \) is a space of constant scalar curvature provided \( \beta \) is a constant.
Now let $V = \xi$. Then the (2.14) becomes

$$L_\xi g + 2S + 2\lambda g = 0.$$  \hspace{1cm} (3.11)

Using (2.8), we get

$$L_\xi g(X, Y) = 2\beta\delta g(X, Y) - 2\beta\delta\varepsilon \eta(X)\eta(Y),$$  \hspace{1cm} (3.12)

and therefore,

$$(L_\xi g + 2S)(X, Y) = 2\beta\delta[g(X, Y) - \varepsilon\eta(X)\eta(Y)] - 2[(\frac{r}{2} - (\alpha^2 - \beta^2))g(X, Y)$$

$$+ (3\alpha^2 - \beta^2) - \varepsilon\frac{r}{2})\eta(X)\eta(Y)].$$  \hspace{1cm} (3.13)

Using (3.13) in (3.11), we obtain

$$2\left[-\frac{r}{2} + \beta\delta + (\alpha^2 - \beta^2) + \lambda\right]g(X, Y) - 2\left[\beta\delta\varepsilon + 3(\alpha^2 - \beta^2) - \varepsilon\frac{r}{2}\right]\eta(X)\eta(Y) = 0.$$  \hspace{1cm} (3.14)

Take $X = Y = \xi$ in (3.14). We get

$$\lambda = \frac{(\beta^2 - \alpha^2)(\varepsilon + 3) - r(\varepsilon - 1)}{2\varepsilon}.$$  \hspace{1cm} (3.15)

Hence we state the following

**Theorem 3.2.** Let $M$ be a 3-dimensional $(\varepsilon, \delta)$-trans-Sasakian manifold. Then a Ricci soliton $(g, \xi, \lambda)$ in $(M, g)$ is:

(i) shrinking for $(\beta^2 - \alpha^2) (\varepsilon + 3) > r(\varepsilon - 1)$,

(ii) expanding for $(\beta^2 - \alpha^2) (\varepsilon + 3) < r(\varepsilon - 1)$.

**Corollary 3.1.** In a 3-dimensional $(\varepsilon, \delta)$-trans-Sasakian manifold $M$ Ricci soliton is:

(i) shrinking if $\beta > \alpha$ and $\varepsilon = 1$,

(ii) expanding if $\beta < \alpha$ and $\varepsilon = 1$,

(iii) steady if $\alpha = \beta$ and $\varepsilon = 1$.

Suppose $(M, g)$ is a 3-dimensional $(\varepsilon, \delta)$-trans-Sasakian manifold and $(g, V, \lambda)$ is a Ricci soliton in $(M, g)$. If $V$ is a conformal killing vector field, then

$$LV g = \rho g,$$  \hspace{1cm} (3.16)
for some scalar function $\rho$. From (2.14)

$$S(X, Y) = -[\lambda g(X, Y) + \frac{1}{2} L_V g(X, Y)]. \quad (3.17)$$

Now from (3.16) it follows that

$$S(X, Y) = -(\lambda + \frac{\rho}{2}) g(X, Y), \quad (3.18)$$

$$QX = -(\lambda + \frac{\rho}{2}) X, \quad (3.19)$$

$$r = -3(\lambda + \frac{\rho}{2}). \quad (3.20)$$

Put $X = Z = \xi$ and use (3.20) in (2.11) to obtain

$$R(\xi, Y)\xi = (\alpha^2 - \beta^2)(3 - 2\varepsilon)[\eta(Y)\xi - Y]. \quad (3.21)$$

Put $X = \xi$ in (2.12) to obtain

$$R(\xi, Y)\xi = \varepsilon \left[(\alpha^2 - \beta^2) + \left(\frac{2-\varepsilon}{2}\right) r\right][\eta(Y)\xi - Y]. \quad (3.22)$$

Comparing (3.21) and (3.22), we obtain

$$r = \frac{6(1-\varepsilon)(\alpha^2 - \beta^2)}{(2\varepsilon - 1)}. \quad (3.23)$$

By using (3.20) and (3.23), we get

$$\lambda = \frac{2(\varepsilon - 1)(\alpha^2 - \beta^2)}{(2\varepsilon - 1)} - \frac{\rho}{2}. \quad (3.24)$$

**Theorem 3.3.** Let $M$ be a 3-dimensional $$(\varepsilon, \delta)$$-trans-Sasakian manifold admitting Ricci soliton $(g, V, \lambda)$, where $V$ is conformal killing vector field. Then $(g, V, \lambda)$ is:

i) expanding for $\rho > \frac{4(\varepsilon - 1)(\alpha^2 - \beta^2)}{(2\varepsilon - 1)}$,

ii) shrinking for $\rho < \frac{4(\varepsilon - 1)(\alpha^2 - \beta^2)}{(2\varepsilon - 1)}$,

iii) steady for $\rho = \frac{4(\varepsilon - 1)(\alpha^2 - \beta^2)}{(2\varepsilon - 1)}$. 

4. K-Torse Forming Vector Field

In 3-dimensional \((\epsilon, \delta)\)-trans-Sasakian manifold, \(\xi\) is always a \(K\)-torse forming vector field. Take \(\rho = \xi\) in (2.16) and compare the result with (2.8) to obtain
\[
a = \beta \delta, \quad b = -\epsilon \alpha, \quad B(X) = -\beta \eta(X)\text{ and } D(X) = 0.
\]
Then it implies
\[
\nabla_X \xi = \beta \delta X - \epsilon \alpha \varphi X - \beta \eta(X) \xi, \quad (4.1)
\]
\[
(\nabla_X \eta)Y = \beta \delta g(X, Y) - \epsilon \alpha g(\varphi X, Y) - \epsilon \beta \eta(X) \eta(Y), \quad (4.2)
\]
\[
R(X, Y) \xi = (a \beta + b \alpha) [\eta(X) Y - \eta(Y) X], \quad (4.3)
\]
\[
R(\xi, X)Y = (a \beta + b \alpha) [\eta(Y) X - \epsilon g(X, Y) \xi], \quad (4.4)
\]
and
\[
S(Y, \xi) = -2(a \beta + b \alpha) \eta(Y), \quad (4.5)
\]
where \(\alpha\) and \(\beta\) are constants.

Taking \(Y = W = \xi\) in (2.17), we obtain
\[
(\nabla_X R)(\xi, Z)(\xi) = A(X) R(\xi, Z) \xi + B(X) [\epsilon \eta(Z) \xi - \epsilon Z]. \quad (4.6)
\]

By the definition of covariant derivative, we have
\[
(\nabla_X R)(\xi, Z)(\xi) = \nabla_X R(\xi, Z) \xi - R(\nabla_X \xi, Z) \xi - R(\xi, \nabla_X Z) \xi - R(\xi, Z) \nabla_X \xi. \quad (4.7)
\]

Using (4.1), (4.2) and (4.4) in (4.7), we get
\[
(\nabla_X R)(\xi, Z)(\xi) = d(a \beta + b \alpha)(X)[Z - \eta(Z) \xi] + (a \beta + b \alpha)[(1 - \delta) \beta g(X, Z) \xi
+ (\epsilon - 1) \alpha g(\varphi X, Z) \xi + (\epsilon - 1) \beta \eta(X) \eta(Z) \xi
+ 2 \eta(\nabla_X Z) \xi - 2 \beta \eta(X) Z + 2 \epsilon \beta \eta(X) Z]. \quad (4.8)
\]

From (4.6) and (4.8), we have
\[
d(a \beta + b \alpha)(X)[Z - \eta(Z) \xi] + (a \beta + b \alpha)[(1 - \delta) \beta g(X, Z) \xi + (\epsilon - 1) \alpha g(\varphi X, Z) \xi
+ (\epsilon - 1) \beta \eta(X) \eta(Z) \xi
+ 2 \eta(\nabla_X Z) \xi - 2 \beta \eta(X) Z + 2 \epsilon \beta \eta(X) Z]
= A(X) [(a \beta + b \alpha)(Z - \eta(Z) \xi)] + B(X) [\epsilon \eta(Z) \xi - \epsilon Z]. \quad (4.9)
\]

Put \(Z = \xi\) and use (2.2),(2.4),(2.5) in (4.9) to obtain
\[
(a \beta + b \alpha)[2(\epsilon - 1) \beta \eta(X) \xi + \eta(\nabla_X \xi) \xi] = 0. \quad (4.10)
\]

If \((a \beta + b \alpha) \neq 0\) and \(\epsilon = 1\), then (4.10) is
\[
\nabla_X \xi = 0. \quad (4.11)
\]
Thus we have
**Theorem 4.4.** In a generalized recurrent 3-dimensional \((\varepsilon, \delta)\)-trans-Sasakian manifold \(M\) the vector field \(\xi\) is co-symplectic provided \((a\beta + b\alpha) \neq 0\) and \(\varepsilon = 1\).

Suppose the Ricci tensor \(S\) is semi conjugate in 3-dimensional \((\varepsilon, \delta)\)-trans-Sasakian manifold \(M\) with respect to the vector field \(\xi\), which is \(K\)-torse forming vector field in \(M\). So \(R(X, \xi).S(Y, Z) = 0\).

Then we have

\[
S(R(X, \xi)Y, Z) + S(Y, R(X, \xi)Z) = 0. \tag{4.12}
\]

Using (4.4) and (4.5) in (4.12), we get

\[
(a\beta + b\alpha)[S(X, Z)\eta(Y) + 2\varepsilon(a\beta + b\alpha)g(Y, X)\eta(Z) + S(Y, X)\eta(Z) + 2\varepsilon(a\beta + b\alpha)g(X, Z)\eta(Y)] = 0. \tag{4.13}
\]

Put \(Z = \xi\) and use (2.5),(4.5) in (4.13) to obtain

\[
S(X, Y) = -2\varepsilon(a\beta + b\alpha)g(X, Y), \tag{4.14}
\]

which gives

\[
r = -6\varepsilon(a\beta + b\alpha), \tag{4.15}
\]

where \(a\) and \(b\) are constants. Hence \(r\) is constant.

**Theorem 4.5.** Suppose the ricci tensor \(S\) in a 3-dimensional \((\varepsilon, \delta)\)-trans-Sasakian manifold \(M\) is semi-conjugate with respect the vector field \(\xi\).Then \(M\) is a space of constant scalar curvature.

**References**


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