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ON A NONDIFFERENTIABLE VECTOR OPTIMIZATION INVOLVING SEMI E-TYPE-I MAPS IN BANACH SPACES

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Abstract: This paper deals with a vector optimization problem with restrictions of inequalities involving nondifferentiable maps in Banach spaces. Some new concepts of η-E-semidifferentiablity, semi E-invexity and semi E-type-I maps in Banach spaces are introduced. A necessary optimality condition and a few sufficient optimality conditions are obtained by generalizing alternative theorem of Gordan type and using semi E-type-I maps. Moreover, weak, strong and converse duality results are proved under various types of semi E-type-I maps assumptions.

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Key Words: η-E-semidifferentiablity, semi E-invexity, semi E-type-I maps, vector optimization, optimality, duality

1. Introduction

Convexity plays a key role in optimality and duality of mathematical programming problems. See, e.g., [1, 2]. Many attempts have been made during the past several decades to weaken convexity hypothesis [3-21]. In this endeavor, Hanson and Mond[6] introduced a new class of functions called type-I function for a scalar optimization problem, which was further generalized to pseudo-type-I and quasi-type-I by Rueda and Hanson[7]. Later, various generalized

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type-I functions have been introduced and optimality and duality have been established involving these functions[8-15].

Recently, Anurag Jayswal[16] introduced new classes of generalized α-univex type-I vector-valued functions and obtained several K-T type sufficient optimality conditions and Mond-Weir type duality results for a multiobjective programming problem with inequality constraints. More recently, Suneja et al.[17] defined generalized type-I functions over cones and established sufficient optimality conditions and duality results for a vector minimization problem using Clarkes generalized gradients. Especially, Yu and Liu[18] obtained some sufficient optimality conditions and duality results for a differentiable vector problem with inequality constraint involving the generalized type-I maps in Banach spaces.

On the other hand, Cristian Niculescu[19] studied optimality and duality for a nonlinear fractional multiobjective programming problem under η-semidifferentiablity and generalized ρ-semilocally type-I preinvexity assumptions. Additionally, Luo and Jian[20] presented semi E-preinvex maps in Banach spaces and discussed their properties.

Motivated by work of [18-20], in the present paper, by generalizing η-semidifferentiablity proposed by Cristian Niculescu[19] to η-E-semidifferentiablity, we define a nondifferentiable semi E-invex map in Banach spaces and thus extend type-I maps presented by Yu and Liu[18] to semi E-type-I maps. We obtain a necessary optimality condition by extending alternative theorem of Gordan type and some sufficient optimization conditions by using semi E-type-I maps for a nondifferentiable vector optimization problem with restrictions of inequalities. Moreover, we prove weak, strong and converse duality results under various types of semi E-type-I maps assumptions. Our results generalize and improve some results obtained in the literatures on this topic.

2. Preliminaries and Definitions

Throughout this paper, let X, Y and Z_j, $j \in M = \{1, 2, \ldots, m\}$ be real Banach spaces with topological duals X^*, Y^* and Z_j^*, respectively, $E : X \to X$ and $\eta : X \times X \to X$ be two fixed mappings.

Consider the following optimization problem:

$$(P) \quad \begin{cases} min\ f(x) \\ s.t. \quad -g(x) = -(g_1(x), \ldots, g_m(x)) \in (D_1\times, \ldots, \times D_m), \\ x \in K \subset X, \end{cases} \tag{2.1}$$

where $f : X \to Y$ and $g_j : X \to Z_j$ are maps, K and D_j are subsets of X and Z_j. Denote the feasible set of (P) by $F = \{x \in K : -g_j(x) \in D_j,\ j \in M\}$. We assume that the spaces Y and Z_j are ordered by cones $C \subset Y$, $D_j \subset Z_j$ and these cones are pointed, closed, convex, and with nonempty interior. The dual cone of C is denoted by

$C^* = \{\mu^* \in Y^* : \langle \mu^*, x\rangle \geq 0, \forall x \in C\}$.

The cone C induces a partial order $\leq_C$ on Y given by:

$x, y \in Y, \quad x \leqq_C y \iff y - x \in C;$

$x, y \in Y, \quad x \leq_C y \iff y - x \in C\backslash\{0_Y\};$

$x, y \in Y, \quad x <_C y \iff y - x \in intC.$

Similarly, D_j induces a partial order on $Z_j, j \in M$.

Recall some definitions and results that will be used in the sequel.

Definition 2.1. ([21]) We say that problem (P) satisfies the Slater regularity condition if there exists $\tilde{x} \in F$ such that $g_j(\tilde{x}) <_{D_j} 0, \quad j \in M$.

Definition 2.2. ([18]) We say that $\bar{x} \in F$ is a weakly efficient solution [or, an efficient solution] of problem (P), if there exists no $x \in F$ such that

$$f(x) <_C f(\bar{x}) \quad [or, f(x) \leq_C f(\bar{x})].$$

Definition 2.3. ([20]) A set $K \subset X$ is said to be E-invex with respect to η if

$$E(y) + \lambda\eta(E(x),\ E(y)) \in K, \quad \forall x,\ y \in K,\ \lambda \in [0, 1].$$

Definition 2.4. ([20]) Let $K \subset X$ be an E-invex set with respect to η. A map $f : X \to Y$ is said to be semi E-preinvex on K with respect to η if

$$f(E(y) + \lambda\eta(E(x),\ E(y))) \leqq_C \lambda f(x) + (1 - \lambda)f(y), \quad \forall x,\ y \in K,\ \lambda \in [0, 1].$$

Lemma 2.1. *([22]) Let $C \subset Y$ be a convex cone with $intC \neq \varnothing$ and C^* the dual cone of C. Then, (a) $\forall\ \mu^* \in C^* \setminus \{0_{Y^*}\}, \quad x \in intC \Rightarrow \langle \mu^*, x\rangle > 0$; (b) $\forall\ \mu^* \in intC^*, \quad x \in C \setminus \{0_Y\} \Rightarrow \langle \mu^*, x\rangle > 0$.*

Now, we introduce some new concepts.

Definition 2.5. Let $f : K \to Y$ be a map, where $K \subset X$ is an E-invex set with respect to η. We say that f is η-E-semidifferentiable at $E(\bar{x}) \in K$ if $f'(E(\bar{x});\ \eta(E(x), E(\bar{x})))$ exists for each $x \in K$, where

$$f'(E(\bar{x});\ \eta(E(x), E(\bar{x}))) = \lim_{\lambda \to 0^+} \frac{1}{\lambda}[f(E(\bar{x}) + \lambda\eta(E(x), E(\bar{x}))) - f(E(\bar{x}))],$$

(the right derivative at $E(\bar{x})$ along the direction $\eta(E(x), E(\bar{x}))$).

Remark 2.1. If E is an identity map and $Y = R^n$, the η-E-semidifferentiability is the η-semidiffer -entiability notion[19]. If $\eta(x, \bar{x}) = x - \bar{x}, Y = R^n$ and E is an identity map, the η-E-semidifferentia- bility is the semidifferentiability notion. If a function is directionally differentiable, then it is semidifferentiable , but the converse is not true.

Definition 2.6. Let $K \subset X$ be an E-invex set with respect to η. A map $f : X \to Y$ is called semi E-invex at $\bar{x}$ on K with respect to η, if f is η-E-semidifferentiable at $\bar{x} \in K$, where $E(\bar{x}) = \bar{x}$ and

$$f(x) - f(\bar{x}) \geqq_C f'(\bar{x};\ \eta(E(x), \bar{x})), \qquad \forall x \in K,$$

which is equivalent to the following relation:

$$\langle \mu^*, f(x) - f(\bar{x}) \rangle \geq (\mu^* \circ f)'(\bar{x};\ \eta(E(x), \bar{x})), \qquad \forall x \in K,\ \mu^* \in C^*,$$

(see [21, Lemma 2.3]).

Remark 2.2. If a semi E-preinvex map f is η-E-semidifferentiable at $\bar{x}$, where $E(\bar{x}) = \bar{x}$, then f is a semi E-invex map at $\bar{x}$.

Next, we extend the generalized type-I maps in [18] as follows.

Definition 2.7. (f, g) is called semi E-type-I at $\bar{x} \in K$ with respect to η, if for each $x \in K$, there exist two maps E and η such that $E(\bar{x}) = \bar{x}$ and for all $\mu^* \in C^*,\ v_j^* \in D_j^*,\ j \in M$

$$\langle \mu^*, f(x) - f(\bar{x}) \rangle \geq (\mu^* \circ f)'(\bar{x};\ \eta(E(x), \bar{x})); \tag{2.2}$$

$$-\sum_{j=1}^{m} \langle v_j^*,\ g_j(\bar{x}) \rangle \geq \sum_{j=1}^{m} (v_j^* \circ g_j)'(\bar{x};\ \eta(E(x), \bar{x})). \tag{2.3}$$

Definition 2.8. (f, g) is called quasi semi E-type-I at $\bar{x} \in K$ with respect to η, if for each $x \in K$, there exist two maps E and η such that $E(\bar{x}) = \bar{x}$ and for all $\mu^* \in C^*,\ v_j^* \in D_j^*,\ j \in M$

$$\langle \mu^*, f(x) \rangle \leq \langle \mu^*, f(\bar{x}) \rangle \Rightarrow (\mu^* \circ f)'(\bar{x};\ \eta(E(x), \bar{x})) \leq 0; \tag{2.4}$$

$$-\sum_{j=1}^{m} \langle v_j^*,\ g_j(\bar{x}) \rangle \leq 0 \Rightarrow \sum_{j=1}^{m} (v_j^* \circ g_j)'(\bar{x};\ \eta(E(x), \bar{x})) \leq 0. \tag{2.5}$$

Definition 2.9. (f, g) is called pseudo semi E-type-I at $\bar{x} \in K$ with respect to η, if for each $x \in K$, there exist two maps E and η such that $E(\bar{x}) = \bar{x}$ and for all $\mu^* \in C^*,\ v_j^* \in D_j^*,\ j \in M$

$$(\mu^* \circ f)'(\bar{x};\ \eta(E(x), \bar{x})) \geq 0 \Rightarrow \langle \mu^*, f(x) \rangle \geq \langle \mu^*, f(\bar{x}) \rangle; \tag{2.6}$$

$$\sum_{j=1}^{m}(v_j^* \circ g_j)'(\bar{x};\ \eta(E(x),\bar{x})) \geq 0 \Rightarrow -\sum_{j=1}^{m}\langle v_j^*,\ g_j(\bar{x})\rangle \geq 0. \tag{2.7}$$

Definition 2.10. (f, g) is called quasipseudo semi E-type-I at $\bar{x} \in K$ with respect to η, if for each $x \in K$, there exist two maps E and η such that $E(\bar{x}) = \bar{x}$ and for all $\mu^* \in C^*,\ v_j^* \in D_j^*,\ j \in M$

$$\langle \mu^*, f(x)\rangle \leq \langle \mu^*, f(\bar{x})\rangle \Rightarrow (\mu^* \circ f)'(\bar{x};\ \eta(E(x),\bar{x})) \leq 0; \tag{2.8}$$

$$\sum_{j=1}^{m}(v_j^* \circ g_j)'(\bar{x};\ \eta(E(x),\bar{x})) \geq 0 \Rightarrow -\sum_{j=1}^{m}\langle v_j^*,\ g_j(\bar{x})\rangle \geq 0. \tag{2.9}$$

If in the above relation, we have

$$\sum_{j=1}^{m}(v_j^* \circ g_j)'(\bar{x};\ \eta(E(x),\bar{x})) \geq 0 \Rightarrow -\sum_{j=1}^{m}\langle v_j^*,\ g_j(\bar{x})\rangle > 0. \tag{2.10}$$

Then, we say that (f, g) is quasistrictlypseudo semi E-type-I at $\bar{x} \in K$.

Definition 2.11. (f, g) is called pseudoquasi semi E-type-I at $\bar{x} \in K$ with respect to η, if for each $x \in K$, there exist two maps E and η such that $E(\bar{x}) = \bar{x}$ and for all $\mu^* \in C^*,\ v_j^* \in D_j^*,\ j \in M$

$$(\mu^* \circ f)'(\bar{x};\ \eta(E(x),\bar{x})) \geq 0 \Rightarrow \langle \mu^*, f(x)\rangle \geq \langle \mu^*, f(\bar{x})\rangle; \tag{2.11}$$

$$-\sum_{j=1}^{m}\langle v_j^*,\ g_j(\bar{x})\rangle \leq 0 \Rightarrow \sum_{j=1}^{m}(v_j^* \circ g_j)'(\bar{x};\ \eta(E(x),\bar{x})) \leq 0. \tag{2.12}$$

Remark 2.3. If (f, g) is semi E-type-I at $\bar{x} \in K$ with respect to η, then (f, g) is both quasi semi E-type-I and pseudo semi E-type-I at $\bar{x} \in K$ with respect to η. If E is an identity map and $m = 1$, then the definitions (2.7)–(2.11) reduce to generalized type-I maps defined by Yu and Liu [18].

3. Optimality Criteria

In this section, we establish a necessary and a few sufficient optimality conditions for problem (P).

To obtain the necessary optimality condition, we need to prove the following generalized Gordan type alternative theorem.

Lemma 3.1. *Let a map $f : X \to Y$ be semi E-preinvex on E-invex set $K \subset X$ with respect to η, if $C \subset Y$ is a convex cone with nonempty interior. Then, either*
(a) there exists $x \in K$, such that $-f(x) \in intC$, or
(b) there exists $p \in C^* \setminus \{0\}$, such that $(p \circ f)(K) \subset R_+$, where $R_+ = \{\alpha \in R : \alpha \geq 0\}$.

Proof. We assume that systems (a) and (b) have solutions $x \in K$ and $p \in C^* \setminus \{0\}$. Then, from Lemma 2.1, we have that $(p \circ f)(x) < 0,\ x \in K$, which is a contradiction to (b).

Now, we assume that system (a) has no solution. We will prove that system (b) has a solution.
We put $A = f(K) + intC$. Then, set A is open. In fact, let $u \in A$, there exists $x \in K$ and $s \in intC$, such that $u = f(x) + s$. Since $s \in intC$, there exists a ball N with center at zero, such that $s + N \subset C$. However, $u + N = f(x) + (s + N) \subset A$, and consequently, A is open.

Next, we will prove that A is a convex set. Let $u_1, u_2 \in A$, and $\tau \in (0, 1)$. Then, $u_1 = f(x_1) + s_1,\ u_2 = f(x_2) + s_2$, with $x_1,\ x_2 \in K$ and $s_1,\ s_2 \in intC$.

$$(1-\tau)u_1 + \tau u_2 = [(1-\tau)f(x_1) + \tau f(x_2)] + [(1-\tau)s_1 + \tau s_2]. \tag{3.1}$$

Since f is semi E-preinvex map, we have

$$f[E(x_1)+\tau\eta(E(x_2), E(x_1))] \leqq_C (1-\tau)f(x_1)+\tau f(x_2), \quad \forall\tau \in (0,1),\ x_1, x_2 \in K,$$

namely,

$$(1-\tau)f(x_1) + \tau f(x_2) \in f[E(x_1) + \tau\eta(E(x_2), E(x_1))] + C, \tag{3.2}$$

and

$$(1-\tau)s_1 + \tau s_2 \in intC. \tag{3.3}$$

By hypothesis, K is an E-invex set, that is,

$$E(x_1) + \tau\eta(E(x_2), E(x_1)) \in K, \qquad \forall\tau \in (0, 1),\ x_1, x_2 \in K. \tag{3.4}$$

From relations (3.1)-(3.4), we obtain $(1-\tau)u_1 + \tau u_2 \in A$, i.e., the set A is convex.
Since system (a) has no solution, then $0 \notin A$. From Hahn-Banach theorem, there exists $p \in Y^* \setminus \{0\}$, such that

$$p(A) \subset R_+. \tag{3.5}$$

We fix $s \in intC$. We would like to prove: $p(f(x)) \geq 0, \quad \forall x \in K$. Since $s \in intC$, we have

$$s + N \subset intC, \quad for\ some\ ball\ N. \tag{3.6}$$

For $\tau \in R_+$ sufficiently big, we have $\frac{1}{\tau}f(x) \in N$ and from (3.6) we have $s - \frac{1}{\tau}f(x) \in intC$ and considering that $intC$ is also a cone, we obtain $\tau s - f(x) \in intC$, that is $\tau s \in f(x) + intC \subset A$, and therefore, by (3.5) we have

$$p(s) \geq 0, \quad \forall s \in intC. \tag{3.7}$$

However, for each $\varepsilon > 0$ sufficiently small, such that $u = f(x) + \varepsilon s \in A$, and therefore,

$$(p \circ f)(x) = p(u) - \varepsilon P(s) \geq -\varepsilon P(s) \to 0, \quad as\ \ \varepsilon \to 0^+,$$

consequently,

$$(p \circ f)(x) \geq 0, \quad \forall x \in K. \tag{3.8}$$

For each $s_0 \in C,\ p(s_0) = \frac{1}{\tau}p(\tau s_0)$ and for $\tau > 0$ small, $\tau s_0 \in intC$, therefore, by (3.7), we have $p(s_0) \geq 0,\ \forall s_0 \in intC$, that is,

$$p \in C^* \setminus \{0\}. \tag{3.9}$$

Thus, (3.8) and (3.9) imply that p is a solution of system (b).
The proof is completed. □

Theorem 3.1. *(Necessary Optimality) Suppose that f and $g_j,\ j \in M$ are semi E-preinvex maps on E-invex set $K \subset X$ with respect to η and all η-E-semidifferentiable at $\bar{x} \in K$, where $E(\bar{x}) = \bar{x}$. If $\bar{x}$ is a weakly efficient solution of (P), then there exist $\mu^* \in C^*,\ v_j^* \in D_j^*$, not all zero, such that*

$$(\mu^* \circ f)'(\bar{x};\ \eta(E(x), \bar{x})) + \sum_{j=1}^{m} (v_j^* \circ g_j)'(\bar{x};\ \eta(E(x), \bar{x})) \geq 0, \quad \forall x \in F, \tag{3.10}$$

and

$$\sum_{j=1}^{m} \langle v_j^*,\ g_j(\bar{x}) \rangle = 0. \tag{3.11}$$

Proof. From Proposition 3 in [20], it follows that the feasible set $F = \{x \in K : -g_j(x) \in D_j, j \in M\}$ is E-invex set with respect to η. Let $\bar{x}$ be a weakly efficient solution of (P). In this case, the system

$$-[(f(x) - f(\bar{x})) \times g_j(x)] \in int(C \times D_j), \quad j \in M,$$

has no solution $x \in F$.
From Lemma 3.1, there exists $p = (\tau^*, v_j^*) \in (C^*, D_j^*) \setminus \{0, 0\}$, such that

$$\tau^* \circ [f(x) - f(\bar{x})] + v_j^* \circ g_j(x) \geq 0, \quad j \in M, \quad x \in F, \tag{3.12}$$

consequently,

$$v_j^* \circ g_j(\bar{x}) \geq 0, \quad j \in M.$$

Also, $\bar{x} \in F$ implies

$$v_j^* \circ g_j(\bar{x}) \leq 0, \quad j \in M.$$

Thus, from the above two relations, it follows that

$$v_j^* \circ g_j(\bar{x}) = 0, \quad j \in M. \tag{3.13}$$

Since F is E-invex set and $f,\ g_j,\ j \in M$ are η-E-semidifferentiable at $\bar{x}$, where $E(\bar{x}) = \bar{x}$, from (3.12) and (3.13), we obtain

$$\lim_{\lambda \to 0^+} \frac{\tau^* \circ f(\bar{x} + \lambda\eta(E(x), \bar{x})) - \tau^* \circ f(\bar{x}) + v_j^* \circ g_j(\bar{x} + \lambda\eta(E(x), \bar{x})) - v_j^* \circ g_j(\bar{x})}{\lambda}$$

$$= (\tau^* \circ f)'(\bar{x};\ \eta(E(x), \bar{x})) + (v_j^* \circ g_j)'(\bar{x};\ \eta(E(x), \bar{x})) \geq 0, \quad j \in M, \quad x \in F. \tag{3.14}$$

Hence, from (3.13) and (3.14), respectively, it follows

$$\sum_{j=1}^{m} \langle v_j^*, g_j(\bar{x}) \rangle = 0,$$

and

$$m(\tau^* \circ f)'(\bar{x};\ \eta(E(x), \bar{x})) + \sum_{j=1}^{m} (v_j^* \circ g_j)'(\bar{x};\ \eta(E(x), \bar{x})) \geq 0, \quad x \in F. \tag{3.15}$$

Setting in (3.15) $m\tau^* = \mu^*$, we obtain the desirable result. □

The following corollary follows directly from Theorem 3.1 and the proof is omitted here.

Corollary 3.1. *On the hypotheses of Theorem 3.1, if the Slater regularity condition is verified, then* $\mu^* \neq 0$.

Now, we establish some sufficient optimality conditions for (P) using semi E-type-I maps.

Theorem 3.2. *Assume that there exist* $\bar{x} \in F$ *and* $\mu^* \in C^* \setminus \{0_{Y^*}\}$ *[or,* $\mu^* \in intC^*$*],* $v_j^* \in D_j^*,\ j \in M$ *such that (3.10) and (3.11) hold. Furthermore, if any one of the following conditions holds:*
(a) (f, g) *is semi E-type-I at* $\bar{x} \in F$ *with respect to the same* η*;*
(b) (f, g) *is pseudoquasi semi E-type-I at* $\bar{x} \in F$ *with respect to the same* η*;*
(c) (f, g) *is quasistrictlypseudo semi E-type-I at* $\bar{x} \in F$ *with respect to the same* η*.*
Then $\bar{x}$ *is a weakly efficient solution [or, an efficient solution] of* (P)*.*

Proof. By contradiction, we assume that $\bar{x}$ is not a weakly efficient solution [or, an efficient solution] of (P). Then there is a feasible solution $\breve{x}$ of problem (P) such that

$$f(\breve{x}) <_C f(\bar{x}) \quad [or, f(\breve{x}) \leq_C f(\bar{x})].$$

From $\mu^* \in C^* \setminus \{0_{Y^*}\}$ [or, $\mu^* \in intC^*$] and Lemma 2.1, we have

$$\langle \mu^*, f(\breve{x}) - f(\bar{x}) \rangle < 0. \tag{3.16}$$

By condition (a), we get

$$(\mu^* \circ f)'(\bar{x};\ \eta(E(\breve{x}), \bar{x})) < 0. \tag{3.17}$$

According to relation (3.11) and condition (a), we also obtain

$$\sum_{j=1}^{m} (v_j^* \circ g_j)'(\bar{x};\ \eta(E(\breve{x}), \bar{x})) \leq 0. \tag{3.18}$$

Adding (3.17) and (3.18), we have

$$(\mu^* \circ f)'(\bar{x};\ \eta(E(\breve{x}), \bar{x})) + \sum_{j=1}^{m} (v_j^* \circ g_j)'(\bar{x};\ \eta(E(\breve{x}), \bar{x})) < 0,$$

which is in contradiction with (3.10).

By condition (b) and relation (3.11), we get

$$\sum_{j=1}^{m} (v_j^* \circ g_j)'(\bar{x};\ \eta(E(\breve{x}), \bar{x})) \leq 0.$$

Considering (3.10), we also get

$$(\mu^* \circ f)'(\bar{x};\ \eta(E(\breve{x}), \bar{x})) \geq 0.$$

By condition (b) again, we have

$$\langle \mu^*, f(\breve{x}) - f(\bar{x}) \rangle \geq 0,$$

which is a contradiction to (3.16).

By condition (c) and relation (3.16), we obtain

$$(\mu^* \circ f)'(\bar{x};\ \eta(E(\breve{x}), \bar{x})) \leq 0.$$

Combining the above inequality with (3.10), we get

$$\sum_{j=1}^{m} (v_j^* \circ g_j)'(\bar{x};\ \eta(E(\breve{x}), \bar{x})) \geq 0.$$

From condition (c) again, it leads to

$$-\sum_{j=1}^{m} \langle v_j^*,\ g_j(\bar{x}) \rangle > 0,$$

which contradicts (3.11).

Therefore, the theorem is proved. □

4. Duality

Consider the following dual for problem (P):

$$(D) \quad \begin{cases} max \quad f(y) \\ s.t. \quad (\mu^* \circ f)'(y;\ \eta(E(x), y)) + \sum_{j=1}^{m} (v_j^* \circ g_j)'(y;\ \eta(E(x), y)) \geq 0, \\ \forall x \in F, \\ \sum_{j=1}^{m} \langle v_j^*, g_j(y) \rangle \geq 0, \\ y \in K, \quad \mu^* \in C^*, \quad v_j^* \in D_j^*, \quad j \in M. \end{cases} \tag{4.1}$$

Denote the feasible set of problem (D) by G, i.e., $G = \{(y, \mu^*, v_j^*) : (\mu^* \circ f)'(y;\ \eta(E(x), y)) + \sum_{j=1}^{m} (v_j^* \circ g_j)'(y;\ \eta(E(x), y)) \geq 0,\ \sum_{j=1}^{m} \langle v_j^*, g_j(y) \rangle \geq 0,\ \forall x \in F,\ y \in K,\ \mu^* \in C^*,\ v_j^* \in D_j^*,\ j \in M\}$.

In this section, we establish weak, strong and converse duality results.

Theorem 4.1. *(Weak duality) Let $x \in F$, $(y, \mu^*, v_j^*) \in G$, and $\mu^* \in C^* \setminus \{0_{Y^*}\}$ [or, $\mu^* \in intC^*$]. Furthermore, if any one of the following conditions is satisfied:*

(a) (f, g) *is semi E-type-I at* $y \in F$ *with respect to the same* η*;*
(b) (f, g) *is pseudoquasi semi E-type-I at* $y \in F$ *with respect to the same* η*;*
(c) (f, g) *is quasistrictlypseudo semi E-type-I at* $y \in F$ *with respect to the same* η*.*
Then, $f(x) \not<_C f(y)$ *[or,* $f(x) \not\leq_C f(y)$*].*

Proof. Assume to the contrary that there exist $\breve{x} \in F$, $(y, \mu^*, v_j^*) \in G$ such that

$$f(\breve{x}) <_C f(y) \quad [or,\ f(\breve{x}) \leq_C f(y)].$$

By $\mu^* \in C^* \setminus \{0_{Y^*}\}$ [or, $\mu^* \in intC^*$] and Lemma 2.1, we have

$$\langle \mu^*,\ f(\breve{x}) - f(y) \rangle < 0. \tag{4.2}$$

From $(y, \mu^*, v_j^*) \in G$, it follows that

$$-\sum_{j=1}^{m} \langle v_j^*, g_j(y) \rangle \leq 0. \tag{4.3}$$

According to the first inequality in (4.1) and $\breve{x} \in F$, we get

$$(\mu^* \circ f)'(y;\ \eta(E(\breve{x}), y)) + \sum_{j=1}^{m} (v_j^* \circ g_j)'(y;\ \eta(E(\breve{x}), y)) \geq 0. \tag{4.4}$$

Utilizing relations (4.2), (4.3) and condition (a), we obtain

$$(\mu^* \circ f)'(y;\ \eta(E(\breve{x}), y)) < 0,$$

$$\sum_{j=1}^{m} (v_j^* \circ g_j)'(y;\ \eta(E(\breve{x}), y)) \leq 0.$$

Summing the above two inequalities, we have

$$(\mu^* \circ f)'(y;\ \eta(E(\breve{x}), y)) + \sum_{j=1}^{m} (v_j^* \circ g_j)'(y;\ \eta(E(\breve{x}), y)) < 0,$$

which is a contradiction to relation (4.4).

If condition (b) holds, then $-\sum_{j=1}^{m} \langle v_j^*, g_j(y) \rangle \leq 0$ implies that

$$\sum_{j=1}^{m} (v_j^* \circ g_j)'(y;\ \eta(E(\breve{x}), y)) \leq 0.$$

Taking (4.4) into account, we obtain

$$(\mu^* \circ f)'(y;\ \eta(E(\breve{x}), y)) \geq 0.$$

By condition (b) again, the above relation means that

$$\langle \mu^*, f(\breve{x}) - f(y) \rangle \geq 0,$$

which contradicts (4.2).

If condition (C) holds, then (4.2) leads to

$$(\mu^* \circ f)'(y;\ \eta(E(\breve{x}), y)) \leq 0.$$

On account of (4.4), we have

$$\sum_{j=1}^{m} (v_j^* \circ g_j)'(y;\ \eta(E(\breve{x}), y)) \geq 0.$$

Using condition (C) again, we get

$$-\sum_{j=1}^{m} \langle v_j^*, g_j(y) \rangle > 0,$$

which is in contradiction with (4.3).
Therefore, the theorem is proved. □

Theorem 4.2. *(Strong duality) Let $\bar{x}$ be a weakly efficient solution of (P). Assume that the maps f and g_j, $j \in M$ are semi E-preinvex with respect to the same η on E-invex set $K \subset X$, are η-E-semidifferentiable at $\bar{x} \subset K$, where $E(\bar{x}) = \bar{x}$, and problem (P) satisfies the Slater regularity condition. Moreover, if any one of the following conditions holds:*
(a) (f, g) is semi E-type-I at $x \in F$ with respect to η;
(b) (f, g) is pseudoquasi semi E-type-I at $x \in F$ with respect to η;
(c) (f, g) is quasistrictlypseudo semi E-type-I at $x \in F$ with respect to η.
Then, there exist $(\bar{\mu}^*, \bar{v}_j^*) \in C^* \times D_j^*$ with $\bar{\mu}^* \neq 0_{Y^*}$ such that $\sum_{j=1}^{m} \langle \bar{v}_j^*, g_j(\bar{x}) \rangle = 0$, $(\bar{x}, \bar{\mu}^*, \bar{v}_j^*)$ is a weakly efficient solution for (D), and the objective values of the two problems are equal.

Proof. Since $\bar{x}$ satisfies all the conditions of Theorem 3.1, there exist $\bar{\mu}^*$, $\bar{v}_j^*$, $j \in M$ such that $\sum_{j=1}^{m} \langle \bar{v}_j^*, g_j(\bar{x}) \rangle = 0$ and $(\bar{x}, \bar{\mu}^*, \bar{v}_j^*) \in G$. Also, by the weak duality, it follows that $(\bar{x}, \bar{\mu}^*, \bar{v}_j^*)$ is a weakly efficient solution for (D). It is obvious that the objective function values of (P) and (D) are equal at their respective weakly efficient solutions. □

Theorem 4.3. *(Converse duality) Let* $(\bar{y}, \bar{\mu}^*, \bar{v}_j^*)$, $j \in M$ *be a weakly efficient solution [or, an efficient solution] for problem* (D). *Assume that* $\bar{\mu}^* \in C^* \setminus \{0_{Y^*}\}$ *[or,* $\bar{\mu}^* \in intC^*$*] and all conditions in Theorem 4.1 hold at* $\bar{y}$. *Then* $\bar{y}$ *is a weakly efficient solution [or, an efficient solution] for* (P).

Proof. We proceed by contradicting. Assume that $\bar{y}$ is not a weakly efficient solution [or, an efficient solution] for (P), that is, there exists $\breve{y} \in F$ such that

$$f(\breve{y}) <_C f(\bar{y}) \quad [or,\ f(\breve{y}) \leq_C f(\bar{y})].$$

From $\bar{\mu}^* \in C^* \setminus \{0_{Y^*}\}$ [or, $\bar{\mu}^* \in intC^*$] and Lemma 2.1, we have

$$\langle \bar{\mu}^*, f(\breve{y}) - f(\bar{y}) \rangle < 0. \tag{4.5}$$

By $(\bar{y}, \bar{\mu}^*, \bar{v}_j^*) \in G$, $j \in M$, we also have

$$(\bar{\mu}^* \circ f)'(\bar{y};\ \eta(E(\breve{y}), \bar{y})) + \sum_{j=1}^{m} (\bar{v}_j^* \circ g_j)'(\bar{y};\ \eta(E(\breve{y}), \bar{y})) \geq 0, \tag{4.6}$$

and

$$\sum_{j=1}^{m} \langle \bar{v}_j^*, g_j(\bar{y}) \rangle \geq 0. \tag{4.7}$$

On account of condition (a) in Theorem 4.1 and (4.5), we get

$$(\bar{\mu}^* \circ f)'(\bar{y};\ \eta(E(\breve{y}), \bar{y})) < 0. \tag{4.8}$$

Since condition (a) of Theorem 4.1 holds and $\breve{y} \in F$, $(\bar{y}, \bar{\mu}^*, \bar{v}_j^*) \in G$, $j \in M$, it yields that

$$\sum_{j=1}^{m} (\bar{v}_j^* \circ g_j(\bar{y}))'(\bar{y};\ \eta(E(\breve{y}), \bar{y})) \leq 0. \tag{4.9}$$

From relations (4.8) and (4.9), it follows that

$$(\bar{\mu}^* \circ f)'(\bar{y};\ \eta(E(\breve{y}), \bar{y})) + \sum_{j=1}^{m} (\bar{v}_j^* \circ g_j(\bar{y}))'(\bar{y};\ \eta(E(\breve{y}), \bar{y})) < 0$$

which is a contradiction to (4.6).

If condition (b) or (c) of Theorem 4.1 holds, by the similar argument to that of Theorem 4.1, we obtain

$$\langle \bar{\mu}^*, f(\breve{y}) - f(\bar{y}) \rangle \geq 0, \tag{4.10}$$

or

$$-\sum_{j=1}^{m}\langle\bar{v}_j^*, g_j(\bar{y})\rangle > 0. \tag{4.11}$$

The inequalities (4.10) and (4.11) contradict (4.5) and (4.7), respectively. So the theorem is proved. □

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