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KOROVKIN SETS FOR SEQUENCES OF SHAPE-PRESERVING OPERATORS

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Abstract: This paper examines Korovkin sets for sequences of operators which are shape-preserving relative to a cone of generalized convex functions, and gives some applications to shape-preserving linear approximation.

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1. Introduction

Let $\mathbf{L}$ be a class of bounded operators on a Banach space E equipped with the norm $\|\cdot\|_E$. Let I be the identity operator, i.e. identity embedding operator of E onto E.

A finite-dimensional subspace S of E is said to be $\mathbf{L}$-*Korovkin set* for operator I in E, if for any sequence of operators $\{L_n\}$ in $\mathbf{L}$ the convergence of $L_n f$ to If for all $f \in S$ (in the norm $\|\cdot\|_E$) implies the convergence of $L_n f$ to If for all $f \in E$ (in the norm $\|\cdot\|_E$).

The case when $\mathbf{L}$ is the set of linear positive operators is deeply examined in [1], [2], [3], [4], [5], [6], [7], [8], [9].

In various applications it is necessary to approximate a function preserving such its shape properties as monotonicity, convexity, concavity, etc. In the the-

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ory of shape preserving approximation by means of polynomials and splines the last 25 years have seen extensive research. The most significant results were summarized in [10], [11]. Due to increased attention to shape-preserving approximation, properties of shape-preserving operators are of interest. Korovkin-type theorems for sequences of operators preserving shape properties of approximated functions relative to signs of fixed orders derivatives was presented in [16].

Using ideas and methods of [2], [3], [4], [5], [9], this paper characterizes Korovkin sets for the identity operator I for the set of linear operators which are shape-preserving relative to a cone of generalized convex functions.

2. The Cone of Generalized Convex Functions

Let $C^k[0,1]$ be the space of all real-valued and k-times continuously differentiable functions defined on [0,1]. Let $B^k[0,1]$ denote the space of all functions whose derivatives of order k are bounded on [0,1].

A function f, defined on [0,1], is said to be *convex relative to the system* $\{u_0, \ldots, u_p\}$ (we will write $f \in C(u_0, \ldots, u_p)$), if

$$\begin{vmatrix} u_0(t_0) & u_0(t_1) & \ldots & u_0(t_{p+1}) \\ \ldots & \ldots & \ldots & \ldots \\ u_p(t_0) & u_p(t_1) & \ldots & u_p(t_{p+1}) \\ f(t_0) & f(t_1) & \ldots & f(t_{p+1}) \end{vmatrix} \geq 0$$

for all choices of $0 < t_0 < t_1 < \ldots < t_{p+1} < 1$.

In particular, if $u_0 \equiv 1$, then $C(u_0)$ is a cone of all increasing functions on $(0,1)$. If $u_0 \equiv 1$, $u_1(x) = x$, then $C(u_0, u_1)$ is a cone of all convex functions on $(0,1)$. The review of some results of the theory of generalized convex functions can be found in the book [12].

Let k be an integer, $\sigma = (\sigma_0, \ldots, \sigma_k) \in R^{k+1}$, $\sigma_p \in \{-1, 0, 1\}$ and $\sigma_0\sigma_k \neq 0$.

We will suppose that functions $\{u_0, \ldots, u_{k-1}\}$ are linearly independent on $[0,1]$.

Denote $V_{p+1} := \{f \in C^{p+1}[0,1] \ : \ f \in C(u_0, \ldots, u_p)\}$, $p = 0, \ldots, k-1$, $V_0 := \{f \in C[0,1] \ : \ f \geq 0\}$. Following ideas of [16] let us consider the cone

$$V_{0,k}(\sigma) = \cap_{p=0}^{k} \sigma_p V_p. \tag{1}$$

A linear operator $L : C^k[0,1] \to B^k[0,1]$ is said to be conservative relative to the cone $V_{0,k}(\sigma)$, if

$$L(V_{0,k}(\sigma)) \subset V_{0,k}(\sigma). \tag{2}$$

Note that a linear operator satisfying (2) is not necessary positive. Let $\mathbf{L}_{0,k}(\sigma)$ denote the set of all linear operators defined in $C^k[0,1]$, with range in $B^k[0,1]$, satisfying the shape-preserving property

$$L(V_{0,k}(\sigma)) \subset \sigma_0 V_0. \tag{3}$$

The main goal of this paper is to characterize $\mathbf{L}_{0,k}(\sigma)$–Korovkin sets for identity operator I in $C^k[0,1]$. Since the set of operators satisfying (2) is subset of the set of operators satisfying (3), all sufficient results which are valid for the set of operators $\mathbf{L}_{0,k}(\sigma)$, will also be valid for the set of all operators which are conservative relative to the cone $V_{0,k}(\sigma)$. Due to linearity of the operators we will assume that $\sigma_0 = 1$.

Following ideas of [13], Bernstein-type operators that are conservative relative to some cone of generalized convex function were examined in [14].

3. The Characterization of Korovkin Sets

For a bounded linear operator L, let L^* denote the adjoint of L. Given a point $x \in [0,1]$, let $\delta_x \in E^*$ denote Dirac functional, i.e. point evaluation of f at x, $\delta_x(f) = f(x)$, $f \in E$.

Let F be a subset of E^* and let $\mathbf{L}$ be the set of all bounded linear operators L on E, such that $L^*\delta_x \in F$ for all $x \in [0,1]$. For example, if F is the set of all positive functionals on E, then $\mathbf{L}$ would be the set of all positive operators.

Let $\lambda \in F$ be a functional and let S be a finite-dimensional subspace of E.

The set S is said to be F-Korovkin set for λ, if for any sequence $\{\lambda_n\} \subset F$ the convergence of $\lambda_n(f) \to \lambda(f)$ for all $f \in S$ implies the convergence $\lambda_n(f) \to \lambda(f)$ for all $f \in E$.

Following [5], S is said to be F-determining set for λ, if for any functional $\mu \in F$ the equality $\mu(f) = \lambda(f)$ for all $f \in S$ implies $\mu = \lambda$.

Given $\mu \in E^*$ and $A \subset E$, let $\mu\mid_A \in A^*$ denotes the functional μ restricted to A.

This section presents necessary and sufficient conditions of $\mathbf{L}_{0,k}(\sigma)$–Korovkin sets for identity operator I in $E = C^k[0,1]$.

Let $V^*_{0,k}(\sigma)$ denote the set of all linear functionals in E^* which are non-negative on $V_{0,k}(\sigma)$.

Given $\mathbf{y} = (y_1, \ldots, y_r) \in [0,1]^r$, $r \in \mathbb{N}$, $0 \le y_1 < \ldots < y_r \le 1$, let us denote $\delta_{\mathbf{y}} = (\delta_{y_1}, \ldots, \delta_{y_r})$.

Given $\alpha \in \mathbb{R}^r$ and $\mathbf{y} = (y_1, \ldots, y_r) \in [0,1]^r$, denote $\alpha\delta_{\mathbf{y}} = \sum_{i=1}^r \alpha_i \delta_{y_i} \in E^*$. The expression $\alpha\delta_{\mathbf{y}}\mid_{V_{0,k}(\sigma)} \ge 0$ means that for any $f \in V_{0,k}(\sigma)$ the inequality

$\alpha\delta_{\mathbf{y}}(f) \geq 0$ holds. Let us denote

$$V^*(\mathbf{y}) := \{\alpha \in \mathbb{R}^r : \ \alpha\delta_{\mathbf{y}} \mid_{V_{0,k}(\sigma)} \geq 0\},$$

Theorem 1. *The following are equivalent:*

1. *for any $x \in [0,1]$ the finite-dimensional subspace S is $V^*_{0,k}(\sigma)$-determining set for δ_x.*
2. *$\forall r \in \mathbb{N}$, $\forall \mathbf{y} = (y_1, \dots, y_r) \in [0,1]^r$, $\forall x \in [0,1]$, $x \neq y_i$, $\nexists \alpha \in V^*(\mathbf{y})$, such that $\delta_x \mid_S = \alpha\delta_{\mathbf{y}} \mid_S$.*

Proof. Assume that the proposition 1 of Theorem holds. Let $r \in \mathbb{N}$, $x \in [0,1]$, $\mathbf{y} \in [0,1]^r$, $\alpha \in V^*(\mathbf{y})$ be such that

$$\delta_x \mid_S = \alpha\delta_{\mathbf{y}} \mid_S . \tag{4}$$

Then values of functionals δ_x, $\delta_x \in V^*_{0,k}(\sigma)$, and $\lambda = \alpha\delta_{\mathbf{y}}$, $\lambda \in V^*_{0,k}(\sigma)$, coincide on every function of the set S and differ on every function $g \in E$, such that $\delta_x(g) = 1$, $\delta_{y_i}(g) = 0$, $i = 1, \dots, r$.

Assume that the proposition 2 of Theorem holds. Suppose that for a point $x \in [0,1]$ there exists a functional $\mu \in V^*_{0,k}(\sigma)$, $\mu \neq \delta_x$, such that $\mu \mid_S = \delta_x \mid_S$.

Consider the set

$$M = \{\alpha\delta_{\mathbf{y}} \mid_S : \ \mathbf{y} \in [0,1]^r, \ \alpha \in V^*(\mathbf{y}), \ r \in \mathbb{N}\}.$$

Let m denote the dimension of the set S, $m = \dim S$. The set M is a convex subset of $\mathbb{R}^m$. It follows from Carateodory's theorem that there exist $\beta_i \geq 0$, $\mathbf{y}^i \in [0,1]^{r_i}$, $\alpha_i \in V^*(\mathbf{y}^i)$, $i = 1, \dots, m+1$, $r_i \in N$, such that

$$\mu(f) = \sum_{i=1}^{m+1} \beta_i \alpha_i \delta_{\mathbf{y}^i}(f)$$

for every $f \in S$. This contradicts the proposition 2 of Theorem. □

Lemma 2. *If $\forall\, x \in [0,1]$ a finite-dimensional set S is $V^*_{0,k}(\sigma)$-determining set for δ_x, then the set S is $V^*_{0,k}(\sigma)$-Korovkin set for δ_x for every $x \in [0,1]$.*

Proof. It is necessary to show that if a sequence $\{\lambda_n\} \subset V^*_{0,k}(\sigma)$ be such that

$$\lambda_n(f) \to \delta_x(f) \tag{5}$$

for every $f \in S$, then $\lambda_n(f) \to \delta_x(f)$ for every $f \in E$.

First it should be shown that $\{\|\lambda_n\|\}$ is bounded above.

Let us denote

$$M = \{\alpha\delta_{\mathbf{y}} : \ \mathbf{y} \in [0,1]^r, \ \alpha \in V^*(\mathbf{y}), \ r \in \mathbb{N}\}.$$

The set $M \subset \mathbb{R}^m$ is convex, $m = \dim S$, and does not contain the origin of the space R^m, $0_m \notin M$. Indeed, if $0_m \in M$ then there exist $\mathbf{y} \in [0,1]^r$ and $\alpha \in V^*(\mathbf{y})$, such that

$$\alpha\delta_{\mathbf{y}} \mid_S = 0.$$

This contradicts the proposition 2 of Theorem 1. It follows from Separation Theorem that in the space $\mathbb{R}^m$ there exists hyperplane separating 0_m and M. Therefore, there are $f \in S$ and $d > 0$, such that the inequality $\alpha\delta_{\mathbf{y}} > d$ holds for every $\mathbf{y} \in [0,1]^r$ and $\alpha \in V^*(\mathbf{y})$.

It follows from (5) that $\lim_{n\to\infty} \lambda_n(f) = \delta_x(f)$, and consequently, $\lambda_n(f) < M < \infty$. I.e., for any g such that $|g| = 1$, $\alpha\delta_{\mathbf{y}}(g) = d$, we have $\lambda_n(g) < M$, and therefore $\|\lambda_n\| < \frac{M}{d}$.

Since $\{\lambda_n\}$ satisfies (5), for any $\varepsilon > 0$ there exist such $\mathbf{y} \in [0,1]^r$, $\alpha \in V^*(\mathbf{y})$, that

$$|\lambda_n(f) - \alpha\delta_{\mathbf{y}}(f)| < \varepsilon.$$

Since E is a separable space, any bounded sequence $\lambda_n \in E^*$ is a weakly compact sequence. Therefore, there exists a weakly convergent subsequence of $\lambda_n \in E^*$. Moreover, the sequence λ_n will be converge weakly to functional δ_x, since each accumulation (limit) functional λ of the sequence must satisfy equalities

$$\lambda \mid_S = \delta_x \mid_S,$$

and therefore, coincide with δ_x. □

Theorem 3. *A finite-dimensional subspace S is $\mathbf{L}_{0,k}(\sigma)$–Korovkin set for the identity operator I in $E = C^k[0,1]$, if and only if for each $x \in [0,1]$ the set S is $V^*_{0,k}(\sigma)$-determining set for δ_x.*

Proof. First it should be proved necessary condition of Theorem. Assume that S is Korovkin set. Suppose that for a point $x_0 \in [0,1]$ there exists two different linear functionals $\mu, \lambda \in V^*_{0,k}(\sigma)$, such that $\mu \mid_S = \lambda \mid_S = \delta_{x_0}$. For simplicity we will suppose $\lambda \mid_S = \delta_{x_0}$. Then, as it was shown in the proof of Theorem 1, there exist $r \in \mathbb{N}$, $\mathbf{y} \in [0,1]^r$ and $\alpha \in V^*(\mathbf{y})$, such that

$$\delta_{x_0} \mid_S = \alpha\delta_{\mathbf{y}} \mid_S . \qquad (6)$$

Define a linear operator $L_0 : C^k[0,1] \to B^k[0,1]$ by

$$L_0 f(x) = \begin{cases} f(x), & x \neq x_0 \\ \alpha\delta_{\mathbf{y}}(f), & x = x_0. \end{cases}$$

Since $\alpha \in V^*(\mathbf{y})$, we have $L_0(V_{0,k}(\sigma)) \subset \sigma_0 V_0$. It follows from equalities (6) that L_0 is the identity operator on the subspace S, i.e. $L_0 f = If = f$ for all $f \in S$.

Let

$$L_n f = L_0 f, \; n = 1, 2, \ldots.$$

Then the sequence $\{L_n\}_{n \geq 0}$ be such that

1. $L_n(V_{0,k}(\sigma)) \subset \sigma_0 V_0$;

2. $\lim_{n\to\infty} \|(L_n - I)f\| = 0$ for all $f \in S$.

On the other hand, for every $f \in E$ such that $\delta_{x_0}(f) = 1$ and $\delta_{\mathbf{y}}(f) = 0_r \in \mathbb{R}^r$, we have $L_0 f \neq f$, i.e. S is not $\mathbf{L}_{0,k}(\sigma)$–Korovkin set for I in E.

To prove the sufficient condition of Theorem, we assume that S is $V^*_{0,k}(\sigma)$-Korovkin set for δ_x for all $x \in [0,1]$. Let $\{L_n\}_{n\geq 1}$, $L_n : C^k[0,1] \to B^k[0,1]$, be a sequence of such linear operators, that

1. $L_n(V_{0,k}(\sigma)) \subset \sigma_0 V_0$;

2. for every $f \in S$
$$\lim_{n\to\infty} \|(L_n - I)f\| = 0. \tag{7}$$

It is well-known that a sequence $\{f_n\} \subset E$ converges uniformly to $f \in E$ if and only if for each $x \in [0,1]$ $f_n(x_n) \to f(x)$ as $n \to \infty$ for every sequence $\{x_n\} \subset [0,1]$ such that $x_n \to x$ as $n \to \infty$.

Let $\{x_n\} \subset [0,1]$ be a sequence such that $x_n \to x$. For every $f \in S$ we have $L_n f(x_n) \to f(x)$, or, in the other words, $L_n^* \delta_{x_n}(f) \to \delta_x(f)$. It follows from Lemma 2 that $L_n^* \delta_{x_n}(f) \to \delta_x(f)$ for every $f \in E$, i.e. $L_n f \to f$ for every $f \in E$, and therefore, S is Korovkin set for the identity operator I. □

4. Corollaries and Applications

Throughout this section it is assumed that the system of functions $\{u_0, \dots, u_{k-1}\}$, $u_l \in C^{k-1}[0,1]$, $l = 0, \dots, k-1$, is an extended complete Tchebychev (ECT) system on $[0,1]$. The cone $V_{0,k}(\sigma)$ is defined in (1) and based on the system $\{u_0, \dots, u_{k-1}\}$.

First it should be proved two preliminary lemmas.

Let $L_{k-1}f(\,\cdot\,; y_0, y_1, \dots, y_{k-1}) \in \operatorname{span}\{u_0, \dots, u_{k-1}\}$ denotes the generalized polynomial, which interpolates $f \in E$ at points $0 \le y_0 < y_1 < \dots < y_{k-1} < 1$:

$$L_{k-1}f(y_i; y_0, y_1, \dots, y_{k-1}) = f(y_i), \;\; i = 0, \dots, k-1. \tag{8}$$

Let us set $y_{-1} = -\infty$, $y_k = +\infty$.

Lemma 4. *Let $f \in V_{0,k}(\sigma)$.*

1. *If $\sigma_0\sigma_k > 0$, then for all $x \in \bigcup_{i=0}^{[(k-1)/2]} [y_{k-1-(2i+1)}, y_{k-1-2i}]$*

$$\sigma_0 L_{k-1}f(x; y_0, \dots, y_{k-1}) \geqslant 0. \tag{9}$$

2. *If $\sigma_0\sigma_k < 0$, then for all $x \in \bigcup_{i=-1}^{[(k-2)/2]} [y_{k-1-(2i+2)}, y_{k-1-(2i+1)}]$ the inequality (9) holds.*

Proof. Suppose that $x \in (y_{l-1}, y_l)$, $l = 0, \dots, k$. It follows from $f \in V_{0,k}(\sigma)$, that

$$\sigma_k \Delta_{k-1} f(x; y_0, \dots, y_{k-1}) \geqslant 0,$$

where

$$\Delta_{k-1} f(x; y_0, \dots, y_{k-1}) = (-1)^l \begin{vmatrix} u_0(x) & u_0(y_0) & \dots & u_0(y_{k-1}) \\ \dots & \dots & \dots & \dots \\ u_{k-1}(x) & u_{k-1}(y_0) & \dots & u_{k-1}(y_{k-1}) \\ f(x) & f(y_0) & \dots & f(y_{k-1}) \end{vmatrix}.$$

It follows from

$$\begin{aligned} &\Delta_{k-1} f(x; y_0, \dots, y_{k-1}) = \\ &\quad = (-1)^{k-1+l}(L_{k-1}f(x; y_0, \dots, y_{k-1}) - f(x)) \det(u_i(y_j))_{i=0,\dots,k-1}^{j=0,\dots,k-1}, \end{aligned} \tag{10}$$

that $\sigma_k(-1)^{k-1+l} L_{k-1}f(x; y_0, \dots, y_{k-1}) \geqslant \sigma_k(-1)^{k-1+l} f(x)$. Since $\sigma_0 f \geqslant 0$, the inequality (9) holds for appropriate x. □

We need the following property of interpolatory polynomials.

Lemma 5. *Let $L_{k-1}f(\,\cdot\,; y_0, y_1, \ldots, y_{k-1}) \in \operatorname{span}\{u_0, \ldots, u_{k-1}\}$ be the polynomial, which interpolates f at points $0 \le y_0 < y_1 < \ldots < y_{k-1} < 1$. Then*

$$L_{k-1}u_i(\,\cdot\,; y_0, \ldots, y_{k-1}) = u_i, \quad i = 0, \ldots, k-1.$$

The following three propositions are main results of this section.

Corollary 6. *Assume that S is $\mathbf{L}_{0,k}(\sigma)$–Korovkin set for the identity operator I in $C^k[0,1]$. If $\{u_0, \ldots, u_{k-1}\} \subset S$, then* $\dim S \ge k+1$.

Proof. It should be shown that if $\dim S = k$, then there exists a sequence of linear operators $\{L_n\} \in \mathbf{L}_{0,k}(\sigma)$, such that

1. $L_n f = f$ for all $f \in S$;
2. there exists such $g \in E$ and $\theta \in (0,1)$, that $\lim_{n\to\infty} L_n g(\theta) \ne g(\theta)$.

Take arbitrary points $0 < y_1 < y_2 < \ldots < y_k < 1$. Let us denote by

$$L_{k-1}f(\,\cdot\,; y_0, y_1, \ldots, y_{k-1}) \in \operatorname{span}\{u_0, \ldots, u_{k-1}\}$$

the generalized polynomials which interpolates $f \in E$ at points $0 < y_0 < y_1 < \ldots < y_{k-1} < 1$.

It follows from Lemma 4 that there exists a point $\theta \in (0,1)$, $\theta \ne y_i$, $i = 1, \ldots, k$, such that

$$L_{k-1}f(\theta; y_0, y_1, \ldots, y_{k-1}) \ge 0$$

for all $f \in V_{0,k}(\sigma)$

Given $n \in N$, let us define

$$L_n f(x) = \begin{cases} f(x), & x \ne \theta; \\ L_{k-1}f(\theta; y_0, y_1, \ldots, y_{k-1}). \end{cases}$$

It is obvious that operators $L_n \in \mathbf{L}_{0,k}(\sigma)$, i.e. $L_n(V_{0,k}(\sigma)) \subset V_0$. It follows from Lemma 5 that $L_n f = f$ on [0,1] for all $f \in S$. On the other hand, let a function $g \in E$ be such that

$$\det(u_i(y_j))_{i=0,\ldots,k}^{j=0,\ldots,k} \ne 0,$$

where $y_0 := \theta$, $u_k := g$. It follows from (10) that

$$L_{k-1}g(\theta; y_0, y_1, \ldots, y_{k-1}) - g(\theta) \ne 0.$$

Then we have $\lim_{n\to\infty} L_n g(\theta) \ne g(\theta)$. □

Denote $e_j(x) = x^j$, $j = 0, 1, \ldots$.

Corollary 7. *Let $k \geq 2$, and $u_k \in C^k[0,1]$ be such that system $u_0, u_1, \ldots, u_k$ is ECT–system on [0,1]. If $u_0 = e_0$, $u_1 = e_1$ and $\sigma_0\sigma_2 \neq -1$, then $S = \mathrm{span}\{u_0, \ldots, u_k\}$ is $\boldsymbol{L}_{0,k}(\sigma)$–Korovkin set for the identity operators I in $C^k[0,1]$.*

Proof. Suppose there exist such $\mathbf{y} = (y_1, \ldots, y_r)$, $x \neq y_i$, $x \in [0,1]$, and $\alpha \in V^*(\mathbf{y})$, that

$$\delta_x \mid_S = \alpha\delta_{\mathbf{y}} \mid_S . \tag{11}$$

On the other hand, there exists [15] such a generalized polynomial $f \in S$, that $\delta_x(f) = 0$ and $\alpha\delta_{\mathbf{y}}(f) > 0$. This contradicts to (11). □

Remark 8. *The following example shows that in the case $\sigma_0\sigma_2 = -1$, the statement of Corollary 7 is not true. Given $x \in [0,1]$, let us define the functional μ_x by*

$$\begin{aligned}\mu_x(f) &= (1/2)\delta_x(e_2)\ [0,s,2s]f + (1/3)\delta_x(e_3)\ ([1-2s,1-s,1]f - [0,s,2s]f) + \\ &+ \delta_x(e_1)\ (\delta_1(f) - \delta_0(f) - (1/3)[1-2s,1-s,1]f - (1/6)[0,s,2s]f) + \delta_0(f),\end{aligned}$$

where $s = 1/4$ and $[y_1, y_2, y_3]f$ denotes the second order divided difference of f at points y_1, y_2, y_3. It easy to check that

1. *$\mu_x e_0 = e_0(x)$, $\mu_x e_1 = e_1(x)$, $\mu_x e_2 = e_2(x)$, i.e. representation (11) holds for $S = \mathrm{span}\{e_0, e_1, e_2\}$ and for any $x \neq i/4$, $i = 0, \ldots, 4$;*

2. *$\mu_x f \geq 0$ for every $f \in V_{0,2}(\sigma)$, where $\sigma = (1, 0, -1)$.*

Corollary 9. *If $S = \mathrm{span}\{u_0, \ldots, u_k\}$ is $\boldsymbol{L}_{0,k}(\sigma)$–Korovkin set for the identity operator I in $C^k[0,1]$, then $\{u_0, \ldots, u_k\}$ is ECT–system on [0,1].*

Proof. Assume the contrary. Then there exist

$$\mathbf{y}^1 = (y_0^1, \ldots, y_k^1),\ \mathbf{y}^2 = (y_0^2, \ldots, y_k^2) \in [0,1]^{k+1},$$

satisfying

$$\det(u_i(y_j^1)) \cdot \det(u_i(y_j^2)) < 0.$$

Since $F(y_0, \ldots, y_k) = \det(u_i(y_j))$ is continuous on $\mathbb{R}^{k+1}$, there exists such $\mathbf{y}^0 = (y_0^0, \ldots, y_k^0) \in [0,1]^{k+1}$, that $\det(u_i(y_j^0)) = 0$.

It follows from Lemma 4 that there is a point $y_s^0 \in \{y_0^0, \ldots, y_k^0\}$, such that

$$L_{k-1}f(y_s^0; y_0^0, \ldots, y_{s-1}^0, y_{s+1}^0, \ldots, y_{k-1}^0) \geq 0,$$

for all $f \in V_{0,k}(\sigma)$.

Consider the functional

$$\alpha\delta_{\mathbf{y}}(f) := L_{k-1}f(y_s^0; y_0^0, \dots, y_{s-1}^0, y_{s+1}^0, \dots, y_{k-1}^0),$$

where $\mathbf{y} = (y_0^0, \dots, y_{s-1}^0, y_{s+1}^0, \dots, y_{k-1}^0)$. It is clear that $\alpha\delta_{\mathbf{y}} \in V^*(\mathbf{y})$. It follows from Lemma 5 and the equality (10), that $\alpha\delta_{\mathbf{y}}(f) = f(y_s)$ for all $f \in \operatorname{span}\{u_0, \dots, u_k\}$. □

Let D^i denote the i-th differential operator, $D^i f(x) = d^i f(x)/dx^i$. Let $k \geq 2$ be an integer, $\sigma = (\sigma_0, \dots, \sigma_k) \in R^{k+1}$, $\sigma_p \in \{-1, 0, 1\}$ and $\sigma_0\sigma_k \neq 0$. Let $\|f\| = \sup_{x\in[0,1]} |f(x)|$. In the paper of F. J. Muñoz-Delgado, V. Ramírez-González and D. Cárdenas-Morales [16] the following cone

$$C_{0,k}(\sigma) = \{f \in C^k[0,1] \ : \ \sigma_i D^i f \geq 0, \ 0 \leq i \leq k\}.$$

was considered. In particular, they proved [16] the next Korovkin-type result for a sequences of linear operators preserving shape.

Theorem 10. *Let $C_{0,k}(\sigma)$ be a cone, such that $\sigma_0\sigma_2 \neq -1$. Let $L_n : C^k[0,1] \to C^k[0,1]$, $n \geq 1$, be a sequence of linear operators. If*

1. $L_n(C_{0,k}(\sigma)) \subset \sigma_0 V_0$,
2. $\lim_{n\to\infty} \|L_n e_j - e_j\| = 0$, $j = 0, \dots, k$,

Then $\lim_{n\to\infty} \|L_n f - f\| = 0$ *for all* $f \in C^k[0,1]$.

The result of Theorem 10 follows from Corollary 7 with $u_j = e_j$, $j = 0, \dots, k$. It arises from the fact that the system $e_0, \dots, e_k$ is extended complete Tchebycheff system on $[0,1]$.

The next proposition follows from Corollary 7.

Corollary 11. *Let $C_{0,k}(\sigma)$ be a cone, such that $\sigma_0\sigma_2 \neq -1$. Let $g \in C^k[0,1]$ be such that the system $e_0, \dots, e_{k-1}, g$ is an extended complete Tchebycheff system on $[0,1]$. Let $L_n : C^k[0,1] \to C^k[0,1]$, $n \geq 1$, be a sequence of linear operators. If*

1. $L_n(C_{0,k}(\sigma)) \subset \sigma_0 V_0$;
2. $\lim_{n\to\infty} \|L_n e_j - e_j\| = 0$, $j = 0, \dots, k-1$,
3. $\lim_{n\to\infty} \|L_n g - g\| = 0$,

then $\lim_{n\to\infty} \|L_n f - f\| = 0$ *for all* $f \in C^k[0,1]$.

Remark 12. *The results of this paper can also be derived using the properties of Minkowski duality and ideas of paper [17].*

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References

[1] P.P. Korovkin, On convergence of linear positive operators in the space of continuous functions, *Dokl. Akad. Nauk. SSR*, **90**, No. 6 (1953), 961-964.

[2] P.P. Korovkin, Conditions for the uniqueness of the moment problem and the convergence of sequences of linear operators, *Notes of Kalinin State Pedagogical Institute*, **26** (1958), 95-102.

[3] V.I. Volkov, On conditions for convergence of sequences of linear positive operators in the space of continuous functions assigned on closed surfaces, *Uspeki Mat. Nauk*, **15**, No. 1 (1960), 181-184.

[4] Y.A. Šaškin, Korovkin systems in spaces of continuous functions, *Izv. Akad. Nauk SSSR, Ser, Mat.*, **26**, No. 4 (1962), 495-512.

[5] Y.A. Šaškin, Finitely defined linear operators in spaces of continuous functions, *Uspeki Mat. Nauk*, **20**, No. 6 (1965), 175-180.

[6] G.G. Lorentz, Korovkin sets, in: *Regional Conference at the Univ. of Calif.* Riverside, June 15-19, 1972.

[7] J. Cavaretta, A.S., Korovkn theorem for finitely defined operators, in: G.G. Lorentz (Ed.), *Approximation Theory*, Academic Press (1973), 299-305.

[8] C.A. Micchelli, Chebyshev subspaces and convergence of positive linear operators, *Proc. Amer. Math. Soc*, **40** (1973), 448-452, **doi:** 10.2307/2039390.

[9] Le Baron, O. Ferguson, Michael D. Rusk, Korovkin sets for an operator on a space of continuous functions, *Pacific J. Math.*, **65**, No 2 (1976), 337-345.

[10] S. G. Gal, *Shape-Preserving Approximation by Real and Complex Polynomials*, Birkhäuser Boston, USA (2008), **doi:** 10.1007/978-0-8176-4703-2.

[11] K. A. Kopotun, D. Leviatan, A. Prymak, and I. A. Shevchuk, Uniform and Pointwise Shape Preserving Approximation by Algebraic Polynomials, *Surveys in Approximation Theory*, **6** (2011), 24-74.

[12] S. Karlin, W. Stadden, *Tchebycheff systems: With applications in analysis and statistics, Vol. Pure and Applied Mathematics XV*, Interscience Publishers John Wiley & Sons, New York (1966).

[13] H. Gonska, P. Piţul, I. Raşa, General King-type operators, Results Math. 53 (3-4) (2009), 279-286, **doi:** 10.1007/s00025-008-0338-9.

[14] D. Cárdenas-Morales, P. Garrancho, I. Raşa, Bernstein-type operators which preserve polynomials, Comput. Math. Appl. 62 (1) (2011), 158-163, **doi:** 10.1016/j.camwa.2011.04.063.

[15] S. P. Sidorov, Korovkin-type theorem for sequences of operators preserving shape, *Positivity*, **15**, No. 1 (2011), 1-8, **doi:** 10.1007/s11117-009-0038-z.

[16] F.J. Muñoz-Delgado, V. Ramírez-González, D. Cárdenas-Morales, Qualitative Korovkin-type results on conservative approximation, *J. Approx. Theory*, **94** (1998), 144-159, **doi:** 10.1006/jath.1998.3182.

[17] S.S. Kutateladze, A.M. Rubinov, Minkowsi duality and its applications, *Russ. Math. Surv.*, **27**, No. 3 (1972), 127-176.