KOROVKIN SETS FOR SEQUENCES OF SHAPE-PRESERVING OPERATORS

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Abstract: This paper examines Korovkin sets for sequences of operators which are shape-preserving relative to a cone of generalized convex functions, and gives some applications to shape-preserving linear approximation.

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1. Introduction

Let $L$ be a class of bounded operators on a Banach space $E$ equipped with the norm $\| \cdot \|_E$. Let $I$ be the identity operator, i.e. identity embedding operator of $E$ onto $E$.

A finite-dimensional subspace $S$ of $E$ is said to be $L$-Korovkin set for operator $I$ in $E$, if for any sequence of operators $\{L_n\}$ in $L$ the convergence of $L_n f$ to $I f$ for all $f \in S$ (in the norm $\| \cdot \|_E$) implies the convergence of $L_n f$ to $I f$ for all $f \in E$ (in the norm $\| \cdot \|_E$).

The case when $L$ is the set of linear positive operators is deeply examined in [1], [2], [3], [4], [5], [6], [7], [8], [9].

In various applications it is necessary to approximate a function preserving such its shape properties as monotonicity, convexity, concavity, etc. In the the-
ory of shape preserving approximation by means of polynomials and splines the last 25 years have seen extensive research. The most significant results were summarized in [10], [11]. Due to increased attention to shape-preserving approximation, properties of shape-preserving operators are of interest. Korovkin-type theorems for sequences of operators preserving shape properties of approximated functions relative to signs of fixed orders derivatives was presented in [16].

Using ideas and methods of [2], [3], [4], [5], [9], this paper characterizes Korovkin sets for the identity operator $I$ for the set of linear operators which are shape-preserving relative to a cone of generalized convex functions.

2. The Cone of Generalized Convex Functions

Let $C^k[0,1]$ be the space of all real-valued and $k$-times continuously differentiable functions defined on $[0,1]$. Let $B^k[0,1]$ denote the space of all functions whose derivatives of order $k$ are bounded on $[0,1]$.

A function $f$, defined on $[0,1]$, is said to be convex relative to the system $\{u_0, \ldots, u_p\}$ (we will write $f \in C(u_0, \ldots, u_p)$), if

$$
\begin{vmatrix}
  u_0(t_0) & u_0(t_1) & \cdots & u_0(t_{p+1}) \\
  \cdots & \cdots & \cdots & \cdots \\
  u_p(t_0) & u_p(t_1) & \cdots & u_p(t_{p+1}) \\
  f(t_0) & f(t_1) & \cdots & f(t_{p+1})
\end{vmatrix} \geq 0
$$

for all choices of $0 < t_0 < t_1 < \ldots < t_{p+1} < 1$.

In particular, if $u_0 \equiv 1$, then $C(u_0)$ is a cone of all increasing functions on $(0,1)$. If $u_0 \equiv 1$, $u_1(x) = x$, then $C(u_0, u_1)$ is a cone of all convex functions on $(0,1)$. The review of some results of the theory of generalized convex functions can be found in the book [12].

Let $k$ be an integer, $\sigma = (\sigma_0, \ldots, \sigma_k) \in \mathbb{R}^{k+1}$, $\sigma_p \in \{-1, 0, 1\}$ and $\sigma_0 \sigma_k \neq 0$.

We will suppose that functions $\{u_0, \ldots, u_{k-1}\}$ are linearly independent on $[0,1]$.

Denote $V_{p+1} := \{f \in C^{p+1}[0,1] : f \in C(u_0, \ldots, u_p)\}$, $p = 0, \ldots, k - 1$, $V_0 := \{f \in C[0,1] : f \geq 0\}$. Following ideas of [16] let us consider the cone

$$V_{0,k}(\sigma) = \bigcap_{p=0}^{k} \sigma_p V_p.$$  \hspace{1cm} (1)

A linear operator $L : C^k[0,1] \to B^k[0,1]$ is said to be conservative relative to the cone $V_{0,k}(\sigma)$, if

$$L(V_{0,k}(\sigma)) \subset V_{0,k}(\sigma).$$  \hspace{1cm} (2)
Note that a linear operator satisfying (2) is not necessarily positive. Let $L_{0,k}(\sigma)$ denote the set of all linear operators defined in $C^k[0,1]$, with range in $B^k[0,1]$, satisfying the shape-preserving property

$$L(V_{0,k}(\sigma)) \subset \sigma_0V_0.$$  \hfill (3)

The main goal of this paper is to characterize $L_{0,k}(\sigma)$–Korovkin sets for identity operator $I$ in $C^k[0,1]$. Since the set of operators satisfying (2) is a subset of the set of operators satisfying (3), all sufficient results which are valid for the set of operators $L_{0,k}(\sigma)$, will also be valid for the set of all operators which are conservative relative to the cone $V_{0,k}(\sigma)$. Due to linearity of the operators we will assume that $\sigma_0 = 1$.

Following ideas of [13], Bernstein-type operators that are conservative relative to some cone of generalized convex function were examined in [14].

3. The Characterization of Korovkin Sets

For a bounded linear operator $L$, let $L^*$ denote the adjoint of $L$. Given a point $x \in [0,1]$, let $\delta_x \in E^*$ denote Dirac functional, i.e. point evaluation of $f$ at $x$, $\delta_x(f) = f(x)$. Let $F$ be a subset of $E^*$ and let $L$ be the set of all bounded linear operators $L$ on $E$, such that $L^*\delta_x \in F$ for all $x \in [0,1]$. For example, if $F$ is the set of all positive functionals on $E$, then $L$ would be the set of all positive operators.

Let $\lambda \in F$ be a functional and let $S$ be a finite-dimensional subspace of $E$. The set $S$ is said to be $F$-Korovkin set for $\lambda$, if for any sequence $\{\lambda_n\} \subset F$ the convergence of $\lambda_n(f) \to \lambda(f)$ for all $f \in S$ implies the convergence $\lambda_n(f) \to \lambda(f)$ for all $f \in E$.

Following [5], $S$ is said to be $F$-determining set for $\lambda$, if for any functional $\mu \in F$ the equality $\mu(f) = \lambda(f)$ for all $f \in S$ implies $\mu = \lambda$.

Given $\mu \in E^*$ and $A \subset E$, let $\mu |_A \in A^*$ denotes the functional $\mu$ restricted to $A$.

This section presents necessary and sufficient conditions of $L_{0,k}(\sigma)$–Korovkin sets for identity operator $I$ in $E = C^k[0,1]$.

Let $V_{0,k}^*(\sigma)$ denote the set of all linear functionals in $E^*$ which are non-negative on $V_{0,k}(\sigma)$.

Given $y = (y_1, \ldots, y_r) \in [0,1]^r$, $r \in \mathbb{N}$, $0 \leq y_1 < \ldots < y_r \leq 1$, let us denote $\delta_y = (\delta_{y_1}, \ldots, \delta_{y_r})$.

Given $\alpha \in \mathbb{R}^r$ and $y = (y_1, \ldots, y_r) \in [0,1]^r$, denote $\alpha \delta_y = \sum_{i=1}^r \alpha_i \delta_{y_i} \in E^*$.

The expression $\alpha \delta_y \mid_{V_{0,k}(\sigma)} \geq 0$ means that for any $f \in V_{0,k}(\sigma)$ the inequality

$$\sum_{i=1}^r \alpha_i f(y_i) \geq 0.$$
\( \alpha \delta_y(f) \geq 0 \) holds. Let us denote

\[
V^*(y) := \{ \alpha \in \mathbb{R}^r : \alpha \delta_y |_{V_{0,k}(\sigma)} \geq 0 \};
\]

**Theorem 1.** The following are equivalent:

1. for any \( x \in [0,1] \) the finite-dimensional subspace \( S \) is \( V^*_{0,k}(\sigma) \)-determining set for \( \delta_x \).
2. \( \forall r \in \mathbb{N}, \forall y = (y_1, \ldots, y_r) \in [0,1]^r, \forall x \in [0,1], x \neq y_i, \# \alpha \in V^*(y) \), such that \( \delta_x |_S = \alpha \delta_y |_S \).

**Proof.** Assume that the proposition 1 of Theorem holds. Let \( r \in \mathbb{N}, x \in [0,1], y \in [0,1]^r, \alpha \in V^*(y) \) be such that

\[
\delta_x |_S = \alpha \delta_y |_S.
\]

(4)

Then values of functionals \( \delta_x, \delta_x \in V^*_{0,k}(\sigma) \), and \( \lambda = \alpha \delta_y, \lambda \in V^*_{0,k}(\sigma) \), coincide on every function of the set \( S \) and differ on every function \( g \in E \), such that \( \delta_x(g) = 1, \delta_y_i(g) = 0, i = 1, \ldots, r \).

Assume that the proposition 2 of Theorem holds. Suppose that for a point \( x \in [0,1] \) there exists a functional \( \mu \in V^*_{0,k}(\sigma), \mu \neq \delta_x \), such that \( \mu |_S = \delta_x |_S \).

Consider the set

\[
M = \{ \alpha \delta_y |_S : y \in [0,1]^r, \alpha \in V^*(y), r \in \mathbb{N} \}.
\]

Let \( m \) denote the dimension of the set \( S \), \( m = \dim S \). The set \( M \) is a convex subset of \( \mathbb{R}^m \). It follows from Carateodory’s theorem that there exist \( \beta_i \geq 0, y^i \in [0,1]^{r_i}, \alpha_i \in V^*(y^i), i = 1, \ldots, m + 1, r_i \in \mathbb{N} \), such that

\[
\mu(f) = \sum_{i=1}^{m+1} \beta_i \alpha_i \delta_{y^i}(f)
\]

for every \( f \in S \). This contradicts the proposition 2 of Theorem. \( \square \)

**Lemma 2.** If \( \forall x \in [0,1] \) a finite-dimensional set \( S \) is \( V^*_{0,k}(\sigma) \)-determining set for \( \delta_x \), then the set \( S \) is \( V^*_{0,k}(\sigma) \)-Korovkin set for \( \delta_x \) for every \( x \in [0,1] \).

**Proof.** It is necessary to show that if a sequence \( \{ \lambda_n \} \subset V^*_{0,k}(\sigma) \) be such that

\[
\lambda_n(f) \to \delta_x(f)
\]

(5)

for every \( f \in S \), then \( \lambda_n(f) \to \delta_x(f) \) for every \( f \in E \).
First it should be shown that \(\{\|\lambda_n\|\}\) is bounded above.

Let us denote
\[
M = \{\alpha \delta_y : y \in [0, 1]^r, \alpha \in V^*(y), r \in \mathbb{N}\}.
\]

The set \(M \subset \mathbb{R}^m\) is convex, \(m = \dim S\), and does not contain the origin of the space \(\mathbb{R}^m\), \(0_m \notin M\). Indeed, if \(0_m \in M\) then there exist \(y \in [0, 1]^r\) and \(\alpha \in V^*(y)\), such that
\[
\alpha \delta_y |_S = 0.
\]
This contradicts the proposition 2 of Theorem 1. It follows from Separation Theorem that in the space \(\mathbb{R}^m\) there exists hyperplane separating \(0_m\) and \(M\). Therefore, there are \(f \in S\) and \(d > 0\), such that the inequality \(\alpha \delta_y > d\) holds for every \(y \in [0, 1]^r\) and \(\alpha \in V^*(y)\).

It follows from (5) that \(\lim_{n \to \infty} \lambda_n(f) = \delta_x(f)\), and consequently, \(\lambda_n(f) < M < \infty\). I.e., for any \(g\) such that \(|g| = 1\), \(\alpha \delta_y(g) = d\), we have \(\lambda_n(g) < M\), and therefore \(\|\lambda_n\| < \frac{M}{d}\).

Since \(\{\lambda_n\}\) satisfies (5), for any \(\varepsilon > 0\) there exist such \(y \in [0, 1]^r\), \(\alpha \in V^*(y)\), that
\[
|\lambda_n(f) - \alpha \delta_y(f)| < \varepsilon.
\]

Since \(E\) is a separable space, any bounded sequence \(\lambda_n \in E^*\) is a weakly compact sequence. Therefore, there exists a weakly convergent subsequence of \(\lambda_n \in E^*\). Moreover, the sequence \(\lambda_n\) will be converge weakly to functional \(\delta_x\), since each accumulation (limit) functional \(\lambda\) of the sequence must satisfy equalities
\[
\lambda |_S = \delta_x |_S,
\]
and therefore, coincide with \(\delta_x\). \(\square\)

**Theorem 3.** A finite-dimensional subspace \(S\) is \(L_{0,k}(\sigma)\)-Korovkin set for the identity operator \(I\) in \(E = C^k[0, 1]\), if and only if for each \(x \in [0, 1]\) the set \(S\) is \(V_{0,k}^*(\sigma)\)-determining set for \(\delta_x\).

**Proof.** First it should be proved necessary condition of Theorem. Assume that \(S\) is Korovkin set. Suppose that for a point \(x_0 \in [0, 1]\) there exists two different linear functionals \(\mu, \lambda \in V_{0,k}^*(\sigma)\), such that \(\mu |_S = \lambda |_S = \delta_{x_0}\). For simplicity we will suppose \(\lambda |_S = \delta_{x_0}\). Then, as it was shown in the proof of Theorem 1, there exist \(r \in \mathbb{N}\), \(y \in [0, 1]^r\) and \(\alpha \in V^*(y)\), such that
\[
\delta_{x_0} |_S = \alpha \delta_y |_S. \quad (6)
\]
Define a linear operator $L_0 : C^k[0,1] \to B^k[0,1]$ by

\[
L_0 f(x) = \begin{cases} 
  f(x), & x \neq x_0 \\
  \alpha \delta_y (f), & x = x_0.
\end{cases}
\]

Since $\alpha \in V^*(y)$, we have $L_0(V_{0,k}(\sigma)) \subset \sigma_0 V_0$. It follows from equalities (6) that $L_0$ is the identity operator on the subspace $S$, i.e. $L_0 f = I f = f$ for all $f \in S$.

Let

\[
L_n f = L_0 f, \quad n = 1, 2, \ldots.
\]

Then the sequence $\{L_n\}_{n \geq 0}$ be such that

1. $L_n(V_{0,k}(\sigma)) \subset \sigma_0 V_0$;

2. $\lim_{n \to \infty} \| (L_n - I) f \| = 0$ for all $f \in S$.

On the other hand, for every $f \in E$ such that $\delta_{x_0}(f) = 1$ and $\delta_y (f) = 0$, i.e. $S$ is not $L_{0,k}(\sigma)$-Korovkin set for $I$ in $E$.

To prove the sufficient condition of Theorem, we assume that $S$ is $V_{0,k}^*(\sigma)$-Korovkin set for $\delta_x$ for all $x \in [0,1]$. Let $\{L_n\}_{n \geq 1}$, $L_n : C^k[0,1] \to B^k[0,1]$, be a sequence of such linear operators, that

1. $L_n(V_{0,k}(\sigma)) \subset \sigma_0 V_0$;

2. for every $f \in S$

\[
\lim_{n \to \infty} \| (L_n - I) f \| = 0.
\]  \hspace{1cm} (7)

It is well-known that a sequence $\{f_n\} \subset E$ converges uniformly to $f \in E$ if and only if for each $x \in [0,1]$ $f_n(x_n) \to f(x)$ as $n \to \infty$ for every sequence $\{x_n\} \subset [0,1]$ such that $x_n \to x$ as $n \to \infty$.

Let $\{x_n\} \subset [0,1]$ be a sequence such that $x_n \to x$. For every $f \in S$ we have $L_n f(x_n) \to f(x)$, or, in the other words, $L_n^* \delta_{x_n}(f) \to \delta_x (f)$. It follows from Lemma 2 that $L_n^* \delta_{x_n}(f) \to \delta_x(f)$ for every $f \in E$, i.e. $L_n f \to f$ for every $f \in E$, and therefore, $S$ is Korovkin set for the identity operator $I$. \hfill \Box
4. Corollaries and Applications

Throughout this section it is assumed that the system of functions \( \{u_0, \ldots, u_{k-1}\} \), \( u_l \in C^{k-1}[0,1], \ l = 0, \ldots, k-1 \), is an extended complete Tchebychev (ECT) system on \([0,1]\). The cone \( V_{0,k}(\sigma) \) is defined in (1) and based on the system \( \{u_0, \ldots, u_{k-1}\} \).

First it should be proved two preliminary lemmas.

Let \( L_{k-1} f \left( \cdots; y_0, y_1, \ldots, y_{k-1} \right) \in \text{span}\{u_0, \ldots, u_{k-1}\} \) denotes the generalized polynomial, which interpolates \( f \in E \) at points \( 0 \leq y_0 < y_1 < \ldots < y_{k-1} < 1 \):

\[
L_{k-1} f (y_i; y_0, y_1, \ldots, y_{k-1}) = f(y_i), \ i = 0, \ldots, k-1. \tag{8}
\]

Let us set \( y_{-1} = -\infty, y_k = +\infty \).

**Lemma 4.** Let \( f \in V_{0,k}(\sigma) \).

1. If \( \sigma_0 \sigma_k > 0 \), then for all \( x \in \bigcup_{i=0}^{\lfloor(k-1)/2\rfloor} \left[ y_{k-1-(2i+1)}, y_{k-1-2i} \right] \)

\[
\sigma_0 L_{k-1} f (x; y_0, \ldots, y_{k-1}) \geq 0. \tag{9}
\]

2. If \( \sigma_0 \sigma_k < 0 \), then for all \( x \in \bigcup_{i=-1}^{\lfloor(k-2)/2\rfloor} \left[ y_{k-1-(2i+2)}, y_{k-1-(2i+1)} \right] \) the inequality (9) holds.

**Proof.** Suppose that \( x \in (y_{l-1}, y_l), \ l = 0, \ldots, k \). It follows from \( f \in V_{0,k}(\sigma) \), that

\[
\sigma_k \Delta_{k-1} f (x; y_0, \ldots, y_{k-1}) \geq 0,
\]

where

\[
\Delta_{k-1} f (x; y_0, \ldots, y_{k-1}) = (-1)^l \begin{vmatrix}
  u_0(x) & u_0(y_0) & \cdots & u_0(y_{k-1}) \\
  \cdots & \cdots & \cdots & \cdots \\
  u_{k-1}(x) & u_{k-1}(y_0) & \cdots & u_{k-1}(y_{k-1}) \\
  f(x) & f(y_0) & \cdots & f(y_{k-1})
\end{vmatrix}
\]

It follows from

\[
\Delta_{k-1} f (x; y_0, \ldots, y_{k-1}) = (-1)^{k-1+l} \left( L_{k-1} f (x; y_0, \ldots, y_{k-1}) - f(x) \right) \det(u_i(y_j))_{i=0,\ldots,k-1}^{j=0,\ldots,k-1}, \tag{10}
\]

that \( \sigma_k (-1)^{k-1+l} L_{k-1} f (x; y_0, \ldots, y_{k-1}) \geq \sigma_k (-1)^{k-1+l} f(x) \). Since \( \sigma_0 f \geq 0 \), the inequality (9) holds for appropriate \( x \).

We need the following property of interpolatory polynomials.
Lemma 5. Let $L_{k-1}f(\cdot; y_0, y_1, \ldots, y_{k-1}) \in \text{span}\{u_0, \ldots, u_{k-1}\}$ be the polynomial, which interpolates $f$ at points $0 \leq y_0 < y_1 < \ldots < y_{k-1} < 1$. Then

$$L_{k-1}u_i(\cdot; y_0, \ldots, y_{k-1}) = u_i, \quad i = 0, \ldots, k - 1.$$

The following three propositions are main results of this section.

Corollary 6. Assume that $S$ is $L_{0,k}(\sigma)$–Korovkin set for the identity operator $I$ in $C^k[0, 1]$. If $\{u_0, \ldots, u_{k-1}\} \subset S$, then $\dim S \geq k + 1$.

Proof. It should be shown that if $\dim S = k$, then there exists a sequence of linear operators $\{L_n\} \in L_{0,k}(\sigma)$, such that

1. $L_n f = f$ for all $f \in S$;
2. there exists such $g \in E$ and $\theta \in (0, 1)$, that $\lim_{n \to \infty} L_n g(\theta) \neq g(\theta)$.

Take arbitrary points $0 < y_1 < y_2 < \ldots < y_k < 1$. Let us denote by

$$L_{k-1}f(\cdot; y_0, y_1, \ldots, y_{k-1}) \in \text{span}\{u_0, \ldots, u_{k-1}\}$$

the generalized polynomials which interpolates $f \in E$ at points $0 < y_0 < y_1 < \ldots < y_{k-1} < 1$.

It follows from Lemma 4 that there exists a point $\theta \in (0, 1)$, $\theta \neq y_i$, $i = 1, \ldots, k$, such that

$$L_{k-1}f(\theta; y_0, y_1, \ldots, y_{k-1}) \geq 0$$

for all $f \in V_{0,k}(\sigma)$.

Given $n \in N$, let us define

$$L_n f(x) = \begin{cases} f(x), & x \neq \theta; \\ L_{k-1}f(\theta; y_0, y_1, \ldots, y_{k-1}). & \end{cases}$$

It is obvious that operators $L_n \in L_{0,k}(\sigma)$, i.e. $L_n(V_{0,k}(\sigma)) \subset V_0$. It follows from Lemma 5 that $L_n f = f$ on $[0, 1]$ for all $f \in S$. On the other hand, let a function $g \in E$ be such that

$$\det(u_i(y_j))_{i=0,\ldots,k}^{j=0,\ldots,k} \neq 0,$$

where $y_0 := \theta$, $u_k := g$. It follows from (10) that

$$L_{k-1}g(\theta; y_0, y_1, \ldots, y_{k-1}) - g(\theta) \neq 0.$$

Then we have $\lim_{n \to \infty} L_n g(\theta) \neq g(\theta)$.
Denote $e_j(x) = x^j, j = 0, 1, \ldots$.

**Corollary 7.** Let $k \geq 2$, and $u_k \in C^k[0,1]$ be such that system $u_0, u_1, \ldots, u_k$ is ECT–system on $[0,1]$. If $u_0 = e_0$, $u_1 = e_1$ and $\sigma_0 \sigma_2 \neq -1$, then $S = \text{span}\{u_0, \ldots, u_k\}$ is $L_{0,k}(\sigma)$–Korovkin set for the identity operators $I$ in $C^k[0,1]$.

**Proof.** Suppose there exist such $y = (y_1, \ldots, y_r), x \neq y_i, x \in [0,1]$, and $\alpha \in V^*(y)$, that

$$
\delta_x |_{\text{span} S} = \alpha \delta_y |_{\text{span} S}.
$$

On the other hand, there exists [15] such a generalized polynomial $f \in S$, that $\delta_x(f) = 0$ and $\alpha \delta_y(f) > 0$. This contradicts to (11). \qed

**Remark 8.** The following example shows that in the case $\sigma_0 \sigma_2 = -1$, the statement of Corollary 7 is not true. Given $x \in [0,1]$, let us define the functional $\mu_x$ by

$$
\mu_x(f) = (1/2)\delta_x(e_2) [0,s,2s]f + (1/3)\delta_x(e_3) ([1 - 2s,1-s,1]f - [0,s,2s]f) + \delta_x(e_1) (\delta_1(f) - \delta_0(f) - (1/3)[1 - 2s,1-s,1]f - (1/6)[0,s,2s]f) + \delta_0(f),
$$

where $s = 1/4$ and $[y_1, y_2, y_3]f$ denotes the second order divided difference of $f$ at points $y_1, y_2, y_3$. It easy to check that

1. $\mu_x e_0 = e_0(x), \mu_x e_1 = e_1(x), \mu_x e_2 = e_2(x)$, i.e. representation (11) holds for $S = \text{span}\{e_0, e_1, e_2\}$ and for any $x \neq i/4, i = 0, \ldots, 4$;

2. $\mu_x f \geq 0$ for every $f \in V_{0.2}(\sigma)$, where $\sigma = (1,0,-1)$.

**Corollary 9.** If $S = \text{span}\{u_0, \ldots, u_k\}$ is $L_{0,k}(\sigma)$–Korovkin set for the identity operator $I$ in $C^k[0,1]$, then $\{u_0, \ldots, u_k\}$ is ECT–system on $[0,1]$.

**Proof.** Assume the contrary. Then there exist

$$
y^1 = (y_0^1, \ldots, y_k^1), y^2 = (y_0^2, \ldots, y_k^2) \in [0,1]^{k+1},
$$
satisfying

$$
\det(u_i(y_j^1)) \cdot \det(u_i(y_j^2)) < 0.
$$

Since $F(y_0, \ldots, y_k) = \det(u_i(y_j))$ is continuous on $\mathbb{R}^{k+1}$, there exists such $y^0 = (y_0^0, \ldots, y_k^0) \in [0,1]^{k+1}$, that $\det(u_i(y_j^0)) = 0$.

It follows from Lemma 4 that there is a point $y^0_s \in \{y_0^0, \ldots, y_k^0\}$, such that

$$
L_{k-1} f(y^0_s; y^0_0, \ldots, y^0_{s-1}, y^0_{s+1}, \ldots, y^0_{k-1}) \geq 0,
$$
for all \( f \in V_{0,k}(\sigma) \).

Consider the functional
\[
\alpha_{\delta_y}(f) := L_{k-1}f(y_0^0, y_0^1, \ldots, y_{s+1}^0, \ldots, y_{k-1}^0),
\]
where \( y = (y_0^0, \ldots, y_{s-1}^0, y_s^0, \ldots, y_{k-1}^0) \). It is clear that \( \alpha_{\delta_y} \in V^*(y) \). It follows from Lemma 5 and the equality (10), that \( \alpha_{\delta_y}(f) = f(y_s) \) for all \( f \in \text{span}\{u_0, \ldots, u_k\} \).

Let \( D_i \) denote the \( i \)-th differential operator, \( D_i f(x) = d^i f(x)/dx^i \). Let \( k \geq 2 \) be an integer, \( \sigma = (\sigma_0, \ldots, \sigma_k) \in \mathbb{R}^{k+1} \), \( \sigma_p \in \{-1, 0, 1\} \) and \( \sigma_0 \sigma_k \neq 0 \). Let \( \|f\| = \sup_{x \in [0,1]} |f(x)| \). In the paper of F. J. Muñoz-Delgado, V. Ramírez-González and D. Cárdenas-Morales [16] the following cone
\[
C_{0,k}(\sigma) = \{ f \in C^k[0,1] : \sigma_i D^i f \geq 0, 0 \leq i \leq k \}.
\]
was considered. In particular, they proved [16] the next Korovkin-type result for a sequences of linear operators preserving shape.

**Theorem 10.** Let \( C_{0,k}(\sigma) \) be a cone, such that \( \sigma_0 \sigma_2 \neq -1 \). Let \( L_n : C^k[0,1] \rightarrow C^k[0,1] \), \( n \geq 1 \), be a sequence of linear operators. If

1. \( L_n(C_{0,k}(\sigma)) \subset \sigma_0 V_0 \),
2. \( \lim_{n \rightarrow \infty} \|L_n e_j - e_j\| = 0, j = 0, \ldots, k \),

Then \( \lim_{n \rightarrow \infty} \|L_n f - f\| = 0 \) for all \( f \in C^k[0,1] \).

The result of Theorem 10 follows from Corollary 7 with \( u_j = e_j, j = 0, \ldots, k \). It arises from the fact that the system \( e_0, \ldots, e_k \) is extended complete Tchebycheff system on \([0,1]\).

The next proposition follows from Corollary 7.

**Corollary 11.** Let \( C_{0,k}(\sigma) \) be a cone, such that \( \sigma_0 \sigma_2 \neq -1 \). Let \( g \in C^k[0,1] \) be such that the system \( e_0, \ldots, e_{k-1}, g \) is an extended complete Tchebycheff system on \([0,1]\). Let \( L_n : C^k[0,1] \rightarrow C^k[0,1] \), \( n \geq 1 \), be a sequence of linear operators. If

1. \( L_n(C_{0,k}(\sigma)) \subset \sigma_0 V_0 \);
2. \( \lim_{n \rightarrow \infty} \|L_n e_j - e_j\| = 0, j = 0, \ldots, k-1 \),
3. \( \lim_{n \rightarrow \infty} \|L_n g - g\| = 0 \),
then \( \lim_{n \to \infty} \|L_n f - f\| = 0 \) for all \( f \in C^k[0, 1] \).

**Remark 12.** The results of this paper can also be derived using the properties of Minkowski duality and ideas of paper [17].

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**References**


