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SEMIDEFINITE QUASICONCAVE PROGRAMMING

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Abstract: We introduce so-called semidefinite quasiconcave programming or equivalently semidefinite quasiconvex maximization problem. We derive new global optimality conditions by generalizing [1]. Based on the global optimality conditions, we construct an algorithm which generates a sequence of local maximizers that converges to a global solution. Subproblems of the proposed algorithm are semidefinite linear programming.

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1. Introduction

Semidefinite linear programming can be regarded as an extension of linear programming and solves the following problem

$$\min\langle C, X\rangle_F,$$
$$\langle A_j, X\rangle_F \leq b_j, j = 1, 2, .., s, \tag{1}$$
$$X \geq 0,$$

here $X \in \mathbb{R}^{n\times n}$ is a matrix of variables and $A_j \in \mathbb{R}^{n\times n}, j = 1, 2, ..., s$. $X \geq 0$ is notation for " X is positive semidefinite ". $\langle\cdot,\cdot\rangle_F$ denotes Frobenius norm and $\|X\|_F = \sqrt{\langle A, A\rangle_F}$.

Semidefinite programming finds many applications in engineering and optimization [5]. Most interior-point methods for linear programming have been generalized to semidefinite convex programming [3, 4, 5]. There are many works devoted to the semidefinite convex programming problem but less attention so for has been paid to semidefinite quasiconcave programming or equivalently semidefinite quasiconvex maximization problem.

Aim of this paper is to develop theory and algorithms for semidefinite quasiconcave programming. The paper is organized as follows. Section 2 is devoted to formulation of semidefinite concave programming and its global optimality conditions. In Section 3, we consider an approximation of the level set of the objective function and its properties.

2. Problem Definition and Optimality Conditions

Let X be matrices in $\mathbb{R}^{n\times n}$, and define a scalar matrix function as

$$f : \mathbb{R}^{n\times n} \to \mathbb{R}.$$

Definition 2.1. Let $f(X)$ be a differentiable function of the matrix X. Then

$$f'(X) = \left(\frac{\partial f(X)}{\partial x_{ij}}\right)_{n\times n}.$$

Introduce the Frobenius scalar product as

$$\langle X, Y\rangle_F = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}y_{ij}, \forall X, Y \in \mathbb{R}^{n\times n}.$$

If $f(\cdot)$ is differentiable, then it can be checked that

$$f(X+H) - f(X) = \langle f'(X), H\rangle_F + o(\|H\|_F).$$

Definition 2.2. A set $\mathbb{D} \subset \mathbb{R}^{n\times n}$ is convex if $\alpha X + (1-\alpha)Y \in \mathbb{D}$ for all $X, Y \in \mathbb{D}$ and $\alpha \in [0, 1]$.

Definition 2.3. The function $f : \mathbb{D} \to \mathbb{R}$ is said to be quasiconvex on $\mathbb{D}$ if $f(\alpha X + (1-\alpha)Y) \leq \max\{f(X), f(Y)\}$ for all $X, Y \in \mathbb{D}$ and $\alpha \in [0, 1]$.

The well known property of a convex function [2] can be easily generalized as follows:

Lemma 2.1. *A function $f : R^n \to R$ is quasiconvex if and only if the set*

$$L_c(f) = \{X \in R^n \mid f(X) \leq c\}$$

is convex for all $c \in R$.

Proof. Necessity. Suppose that $c \in R$ is an arbitrary number and $X, Y \in L_c(f)$. By the definition of quasiconvexity, we have

$$f(\alpha X + (1-\alpha)Y) \leq \max\{f(X), f(Y)\} \leq c \quad \text{for all } \alpha \in [0, 1],$$

which means that the set $L_c(f)$ is convex.

Sufficiency. Let $L_c(f)$ be a convex set for all $c \in R$. For arbitrary $X, Y \in R^n$, define $c^o = \max\{f(X), f(Y)\}$. Then $X \in L_{c^o}(f)$ and $Y \in L_{c^o}(f)$. Consequently, $\alpha X + (1 - \alpha Y) \in L_{c^o}(f)$, for any $\alpha \in [0, 1]$. This completes the proof.

Lemma 2.2. *Let $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ be a quasiconvex and differentiable function.Then the inequality $f(X) \leq f(Y)$ for $X, Y \in \mathbb{R}^{n\times n}$ implies that $\langle f'(Y), X - Y\rangle_F \leq$*

where $f'(X) = \left(\frac{\partial f(X)}{\partial x_{ij}}\right)_{n\times n}$ and $\langle\cdot,\cdot\rangle_F$ denotes the Frobenius scalar product of two matrix

Proof. Since f is quasiconvex,

$$f(\alpha X + (1-\alpha)Y) \leq \max\{f(X), f(Y)\} = f(Y)$$

for all $\alpha \in [0, 1]$ and $X, Y \in R^{n\times n}$ such that $f(X) \leq f(Y)$. By Taylor's formula, there is a neighborhood of the point Y on which:

$$f(Y + \alpha(X - Y)) - f(Y) =$$
$$\alpha\left(\langle f'(Y), X - Y\rangle_F + \frac{o(\alpha\|X - Y\|_F)}{\alpha}\right) \leq 0,\ \alpha > 0.$$

From the fact that $\dfrac{o(\alpha\|x - y\|_F)}{\alpha} \overset{\alpha\to o}{\longrightarrow} 0$, we obtain $\langle f'(Y), X - Y\rangle_F \leq 0$ which completes the proof.

Consider the problem of maximizing a differentiable quasiconvex matrix function subject to constraints:

$$\max f(X) \tag{2.1}$$

subject to:

$$\langle A_j, X\rangle_F \leq b_j, j = 1, 2, .., s, \tag{2.2}$$

$$X \geq 0, \tag{2.3}$$

where $A_j \in \mathbb{R}^{n\times n}, j = 1, 2, ..., s$ and $X \geq 0$ are positive semidefinite matrices, $b_j \in \mathbb{R}$.

We call problem (2.1)-(2.3) as the semidefinite quasiconvex maximization problem or equivalently, semidefinite quasiconcave programming.

Denote by $\mathbb{D}$ constraints of the problem:

$$\mathbb{D} = \{X \in \mathbb{R}^{n\times n} | \langle A_j, X\rangle_F \leq b_j, j = 1, 2, .., s; X \geq 0\}.$$

Then problem (2.1)-(2.3) reduces to

$$\max_{x\in\mathbb{D}} f(X) \tag{2.4}$$

It can be checked that the set $\mathbb{D}$ is convex. Problem (2.4) is nonconvex and belongs to a class of global optimization problems in Banach space.

3. Global Optimality Conditions

Introduce the level set $\mathrm{E}_{f(Z)}(f)$ of the function $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ at a point $Z \in \mathbb{R}^{n\times n}$:

$$\mathrm{E}_{f(Z)}(f) = \{Y \in \mathbb{R}^{n\times n} | f(Y) = f(Z)\}.$$

The global optimality condition for problem (2.4) can be formulated in the following theorem.

Theorem 3.1. *If $Z \in \mathbb{D}$ is a global solution to problem (2.4) then*

$$\langle f'(Y), X - Y\rangle_F \leq 0 \tag{2.5}$$

hold for all $Y \in \mathrm{E}_{f(Z)}(f)$ and $X \in \mathbb{D}$. If in addition, $f'(Y) \neq 0$ holds for all $Y \in E_{f(Z)}(f)$, then condition (2.5) is sufficient for $Z \in D$ being a solution to problem (2.4).

Proof. Necessity. Assume that Z is a solution of problem (2.4) and let $Y \in E_{f(Z)}(f)$ and $X \in D$. Then we have $f(X) \leq f(Y)$. Applying lemma 2.2, we obtain $\langle f'(Y), X - Y\rangle_F \leq 0$.

Sufficiency. Suppose, on the contrary, that Z is not a solution to problem (2.4), i.e, there exists a $U \in \mathbb{D}$ such that $f(U) > f(Z)$. The closed set

$L_{f(Z)}(f) = \{X \in \mathbb{R}^{n\times n} | f(X) \leq f(Z)\}$ is convex by Lemma (2.1). Let Y be the projection of U on $L_{f(Z)}(f)$ such that

$$\|Y - U\|_F = \min_{X \in L_{f(Z)}(f)} \|X - U\|_F.$$

obviously,

$$\|Y - U\|_F > 0 \tag{2.7}$$

holds since $U \notin L_{f(Z)}(f)$. The point Y can be considered as a solution of the convex minimization problem:

$$\min_{X \in L_{f(Z)}(f)} \{g(X) = \frac{1}{2}\|X - U\|_F^2\} \tag{2.8}$$

Applying the Lagrange method to problem (2.8), we obtain the following optimality conditions at the point Y:

$$\begin{cases} \lambda_0 \geq 0, \quad \lambda \geq 0, \quad \lambda_0 + \lambda > 0, \\ \lambda_0 g'(Y) + \lambda f'(Y) = 0, \\ \lambda(f(Y) - f(Z)) = 0 \end{cases} \tag{2.9}$$

or equivalently,

$$\begin{cases} \lambda_0 \geq 0, \quad \lambda \geq 0, \quad \lambda_0 + \lambda > 0, \\ \lambda_0(Y - U) + \lambda f'(Y) = 0, \\ \lambda(f(Y) - f(Z)) = 0. \end{cases} \tag{2.10}$$

If $\lambda = 0$, then (2.10) implies that $\lambda > 0, f(Y) = f(Z)$, and $f'(Y) = 0$ which contradicts the assumption in the theorem. If $\lambda = 0$, then we have $\lambda_0 > 0$, and $g'(Y) = Y - U = 0$ which also contradicts (2.7). So, without loss of generality, we can set $\lambda_0 = 1$ and $\lambda > 0$ in (2.10). Hence, we have

$$Y - U + \lambda f'(Y) = 0, \quad \lambda > 0.$$

From this, we can conclude that

$$\lambda f'(Y) = U - Y$$

and

$$\lambda\langle f'(Y), U - Y\rangle_F = \|U - Y\|_F^2 > 0$$

which contradicts (2.5). Last contradiction implies that the assumption that Z is not a global solution to problem (2.4) must be false. This completes the proof.

Remark 3.1. For a fixed $Y \in \mathrm{E}_{f(Z)}(f)$ checking condition (2.5) reduces to

$$\max_{x\in\mathbb{D}}\langle f'(Y), X\rangle_F \le \langle f'(Y), Y\rangle_F$$

or equivalently to semidefinite linear programming:

$$\max\langle f'(Y), X\rangle_F$$

subject to:

$$\langle A_j, X\rangle_F \le b_j, j = 1, 2, ..., s,$$

$$X \ge 0.$$

Remark 3.2. In order to conclude that a point $\widetilde{Z} \in \mathbb{D}$ is not a global solution to problem (2.4), we need to find a pair $(U, \widetilde{Y})$ such that

$$\langle f'(\widetilde{Y}), U - \widetilde{Y}\rangle_F \ge 0, \widetilde{Y} \in \mathrm{E}_{f(Z)}(f), U \in \mathbb{D}.$$

The following example illustrates the use of this property.

Example 3.1. Consider the problem

$$\max_{x\in\mathbb{D}} \|CX\|_F^2,$$

$$\mathbb{D} = \{X \in \mathbb{R}^{n\times n} | \underline{X} \le X \le \overline{X}, X \ge 0\},$$

where

$$C = \begin{pmatrix} 2 & -3 \\ 4 & 5 \end{pmatrix}, \qquad \underline{X} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}, \qquad \overline{X} = \begin{pmatrix} 7 & 5 \\ 8 & 9 \end{pmatrix}.$$

We can evaluate the gradient of f as:

$$f'(X) = 2C^TCX.$$

We check whether a point X^0

$$X^0 = \begin{pmatrix} 5 & 4 \\ 6 & 7 \end{pmatrix}$$

is a global solution or not. It can be computed that

$$f(X^0) = 5334.$$

Take a point $U \in \mathbb{D}$ such that

$$U \geq 0 \qquad \text{and} \qquad U = \begin{pmatrix} 4 & 4 \\ 7 & 8 \end{pmatrix}$$

Find a $\widetilde{Y}$ so that $\widetilde{Y} \in \mathrm{E}_{f(X^0)}(f)$ and

$$\widetilde{Y} = \begin{pmatrix} 0.1 & 0.01 \\ 0.2 & 0.3 \end{pmatrix}$$

If we evaluate $\langle f'(\widetilde{Y}), U - \widetilde{Y}\rangle_F$, then $\langle f'(\widetilde{Y}), U - \widetilde{Y}\rangle_F = 5.2660 > 0$ which means that X^0 is not a global solution.In fact, the global solution X^* is:

$$X^* = \begin{pmatrix} 7 & 5 \\ 8 & 9 \end{pmatrix}$$

4. Approximation of the level set and Algorithm

As we have seen in Section 2 that in order to check condition (2.5), we need to solve the following semidefinite linear programming for each given $Y \in \mathrm{E}_{f(Z)}(f)$.

$$\max_{x \in \mathbb{D}} \langle f'(Y), X\rangle_F. \tag{3.1}$$

For this purpose, we need to approximate the level set of the function f with a finite number of points so that one could solve a finite number of problems (3.1).

Definition 4.1. The set A_Z^m defined for a given $m \in \mathbb{N}$ and $Z \in \mathbb{R}^{n\times n}$ by

$$A_Z^m = \{Y^1, Y^2, ..., Y^m | Y^i \in \mathrm{E}_{f(Z)}(f), i = 1, 2, ..., m\} \qquad (3.2).$$

is called an approximation set to the level set $\mathrm{E}_{f(Z)}(f)$ at the point Z.

Assume that A_Z^m is given and $\mathbb{D}$ is compact in $\mathbb{R}^{n\times n}$. Let $U^i, i = 1, 2, ..., m$ be the solutions to the following problems:

$$\langle f'(Y^i), U^i\rangle_F = \max_{X\in\mathbb{D}} \langle f'(Y^i), X\rangle_F. \qquad (3.3)$$

Define θ_m as follows:

$$\theta_m = \max_{i=1,2..,m} \langle f'(Y^i), X\rangle_F.$$

Lemma 4.1. *If there is a point $Y^i \in A_Z^m$ for $Z \in \mathbb{D}$ such that $\langle f'(Y^i), U^i - Y^i\rangle_F > 0$, where U^i satisfies (3.3),then*

$$f(U^i) > f(Z).$$

Proof. By the definition of U^i, we have

$$\langle f'(Y^i), U^i - Y^i\rangle_F = \max_{X\in\mathbb{D}} \langle f'(Y^i), X - Y^i\rangle_F$$

Since f is quasiconvex, then by Lemma 2.2, we have $f(U^i) \le f(Y^i)$ which implies that $\langle f'(Y^i), U - Y^i\rangle_F \le 0$ for all $U, Y \in \mathbb{R}^{n\times n}$. The proof is complete.

Now we can formulate an algorithm for finding an approximate solutions for problem (2.4).

Algorithm (CDCP).

Input: A convex differentiable function f and a compact set $\mathbb{D}$ in $\mathbb{R}^{n\times n}$

Output: An approximate solution X to (2.4).

Step 1. Choose a point $X^0 \in \mathbb{D}$. Set $k := 0$.

Step 2. Find a local maximizer $Z^k \in \mathbb{D}$ of problem (2.4)using one of existing methods of semidefinite nonconvex programming.

Step 3. Construct an approximation set $A^m_{Z^k}$ at the point Z^k.

Step 4. For each $Y^i \in A^m_{Z^k}$ solve semidefinite linear programming

$$\max_{X \in \mathbb{D}} \langle f'(Y^i), X \rangle_F.$$

Let $U^i, i = 1, 2, ..., m$ be solutions, i.e.,

$$\langle f'(Y^i), U^i \rangle_F = \max_{X \in \mathbb{D}} \langle f'(Y^i), X \rangle_F.$$

Step 5. Find a number $j \in 1, 2, ..., m$ such that

$$\theta_m = \langle f'(Y^j), U^j - Y^j \rangle_F = \max_{j=1,2..,m} \langle f'(Y^i), U^i - Y^i \rangle_F$$

Step 6. If $\theta^k_m \leq 0$ then terminate and Z^k is an approximate solution.

Step 7. Set $X^{k+1} := U^i, k := k + 1$ and go to step 2.

We note that the algorithm generates a sequence of local maximizers $\{Z^k\}$ of problem (2.4) such that

$$f(Z^{k+1}) \geq f(Z^k), k = 0, 1...$$

This gives us an opportunity to approach the global solution in (2.4) using standard approach of semidefinite programming.

5. Conclusion

We introduced so-called semidefinite quasiconcave programming or equivalently semidefinite quasiconvex maximization problem. Unlike semidefinite quasiconvex programming, the problem is nonconvex and NP hard. We derived global optimality conditions for the problem. Based on the global optimality conditions, we propose an algorithm for solving the problem. Subproblems of the algorithm are semidefinite linear programming.

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