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A STUDY ON UNCONDITIONALLY STABLE EXPLICIT DIFFERENCE SCHEMES FOR THE VARIABLE COEFFICIENTS PARABOLIC DIFFERENTIAL EQUATION

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Abstract: In [2], we have presented some new algorithms for solving the linear variable coefficients parabolic differential equations. The proposed scheme is stable without any restriction to step-sizes of space and time. The scheme is required the condition of step size ratio $\frac{k}{h^2}, \to 0$ as $k, h \to 0$ in the convergence, where k and h are step sizes for space and time respectively. In this paper, we will present the explicit unconditional stable scheme which has no restriction on the step size ratio $\frac{k}{h^2}$ in the convergence. We will also present analysis for the present scheme.

AMS Subject Classification: 65L06, 65L07
Key Words: Runge-Kutta methods, method of lines, difference equation

1. Introduction

A number of difference schemes for solving partial difference equations have been proposed. The methods of lines are recognized as useful tools in the numerical solution of PDEs. E.C. Du Fort and S.P. Frankel [1] and some others have proposed difference schemes based on methods of lines. However, in using the explicit lines methods, stability of algorithms is a serious problems. We[2,3]

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have proposed some explicit difference schemes by using the idea of methods of lines and overcome this problems. The problem considered in [2] is

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}(a(x,t)\frac{\partial u(x,t)}{\partial x}) + b(x,t)\frac{\partial u(x,t)}{\partial x} + c(x,t)u(x,t), \tag{1.1}$$

$$a(x,t) > 0,$$

$$(x,t) \in \Omega = \{(x,t); 0 < x < 1, 0 < t \le t_F\},$$

with the initial Dirichlet boundary condition

$$u(x,t) = \begin{cases} f(t), & (0,t) \in \partial\Omega \cup \Omega, \\ 0, & (1,t) \in \partial\Omega \cup \Omega. \end{cases} \tag{1.2}$$

In the proposed scheme [2], it is required the condition of step size ratio

$$\frac{k}{h^2} \to 0 \quad \text{as} \quad h, k \to 0,$$

in the convergence, where h and k for space and time respectively. In this paper, we propose the difference approximation to (1.1) where the step size ratio is defined by

$$\frac{k}{h^2} = c \qquad (c \text{ is any positive constant}). \tag{1.3}$$

The outline of this paper is as follows. In Section 2, by using idea of methods of lines, we present the explicit difference approximation to (1.1). In Section 3, we study the truncation errors of our scheme. In Section 4, we study the convergence of the scheme with the condition (1.3) and we will show that our scheme converges to the true solution of (1.1). In Section 5, we study stability of the scheme, and we will show that our scheme is stable for any step size k and h with the condition (1.3).

2. Difference Scheme

In the same way as in [2], we will approximate (1.1) by replacing the derivative for space and time in the difference operator

$$\frac{\partial}{\partial x}(a(x,t)\frac{\partial u(x,t)}{\partial x}) \cong \frac{1}{h^2}\delta(a(x,t)\delta u(x,t)),$$

$$\frac{\partial u(x,t)}{\partial x} \cong \frac{1}{h}\triangle u(x,t),\ \frac{1}{h}\nabla u(x,t),$$

$$\frac{\partial u(x,t)}{\partial t} \cong \frac{1}{k}\triangle u(x,t),\ \frac{1}{k}\nabla u(x,t), \tag{2.1}$$

where δ is the central difference operator, $\triangle$ forward difference operator,∇ backward difference operator. We divide $x-$space to N_1 points, $t-$space to N_2 points where h and k are the mesh size for $x-$space,$t-$space respectively. We denote the approximation to (1.1) at the mesh point $(x,t) = (jh, nk)$

$$u_j^n \cong u(jh, nk).$$

We set the vector $\|U^n\|$ by

$$\|U^n\| = \max\{|u_j^n|; 0 \le j \le 1/h\}.$$

We define the difference approximation to (1.1) by the following scheme.

If $\|U^n\| < 1$. Then we set

$$u_j^{n+1} = u_j^n + \frac{k}{1+\hat{k}L_1}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n). \tag{2.2}$$

If $\|U^n\| \ge 1$. Then we set

$$u_j^{n+1} = u_j^n + \frac{k}{1+\hat{k}L_1\|U^n\|}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n), \tag{2.3}$$

where

$$\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n) = \{\eta(jh,nk)u_{j-1}^n + \rho(jh,nk)u_j^n + \tau(jh,nk)u_{j+1}^n\}, \tag{2.4}$$

with

$$\rho(jh,nk) = -\{\frac{1}{h^2}\{a_{j+\frac{1}{2}}(nk) + a_{j-\frac{1}{2}}(nk)\} - \frac{|b_j(nk)|}{h} + c_j(nk),$$

$$\eta(jh,nk) = \frac{1}{h^2}a_{j-\frac{1}{2}}(nk) + \frac{1}{2h}\{|b_j(nk)| - b_j(nk)\},$$

$$\tau(jh,nk) = \frac{1}{h^2}a_{j+\frac{1}{2}}(nk) + \frac{1}{2h}\{|b_j(nk)| + b_j(nk)\},$$

$$(n = 0,1,..,N_1+1,\ j = 0,1,..,N_2+1)\ (h = \frac{1}{N_1+1}, k = \frac{t_F}{N_2+1}),$$

and

$$a_j(t) = a(jh,t), b_j(t) = b(jh,t), c_j(t) = c(jh,t),$$

$$L_1 = \max_{0\leq i\leq \frac{1}{h}, 0\leq j\leq \frac{t_F}{k}} \{(\frac{1}{h^2})\{a_{i+\frac{1}{2}}(jk) + a_{i-\frac{1}{2}}(jk)\} + \frac{1}{h}|b(ih,jk)| - c(ih,jk)\}. \quad (2.5)$$

The step size $\hat{k}$ in (2.2), (2.3) is defined by

$$\hat{k} = k^{1+\rho}. \quad (\rho > 0) \quad (2.6)$$

3. Truncation Error

We define the truncation error $T(jh,nk)$ of (2.2), (2.3)

$$T(jh,nk) = u(jh,(n+1)k) - u(jh,nk) - \frac{k}{(1+\hat{k}L_1\|\tilde{U^n}\|)}\Phi(u((j-1)h,nk),u(jh,nk),u((j+1)h,nk)), \quad (3.1)$$

where

$$\|\tilde{U^n}\| = \max\{1, \|U^n\|\}.$$

By Taylor series expansions of the exact solution $u(jh,nk)$, we have

$$\begin{aligned} T(jh,nk) &= ku_t(jh,nk) + O(k^2) \\ &\quad - \frac{k}{(1+\hat{k}L_1\|\tilde{U^n}\|)} + \{a_x(jh,nk)u_x(jh,nk) + a(jh,nk)u_{xx}(jh,nk) \\ &\quad + b(jh,nk)\tilde{u}_x(jh,nk) + c(jh,nk) + \frac{h}{2}|b(jh,nk)|u_{xx}(jh,nk) + O(h^2)\} \\ &= ku_t(jh,nk) - \frac{k}{(1+\hat{k}L_1\|\tilde{U^n}\|)}\{u_x(jh,nk) + \frac{h}{2}|b(jh,nk)|u_{xx}(jh,nk) \\ &\quad + O(h^2)\}. \end{aligned} \quad (3.2)$$

From (3.2), we have

$$\begin{aligned} T(jh,nk) = k\{&\frac{\hat{k}L_1\|\tilde{U^n}\|}{(1+\hat{k}L_1\|\tilde{U^n}\|)}u_t(jh,nk) \\ &- \frac{1}{2}\frac{h}{(1+\hat{k}L_1\|\tilde{U^n}\|)}|b(jh,nk)|u_{xx}(jh,nk) + O(h^2\}. \end{aligned} \quad (3.3)$$

From (3.3), we have the result.

Theorem 1. *The truncation error of the difference approximation (2.2), (2.3) to (1.1) is given by*

$$T(jh, nk) = k^{(1+\tilde{\rho})} w(jh, nk), \tag{3.4}$$

where

$$w(jh, nk) = \frac{k^{(\rho-\tilde{\rho})}(kL_1)\|\tilde{U^n}\|}{(1+\hat{k}L_1\|\tilde{U^n}\|)} u_t(jh, nk) - \frac{1}{2\sqrt{c}} \frac{k^{(\frac{1}{2}-\tilde{\rho})}}{(1+\hat{k}L_1\|\tilde{U^n}\|)} |b(jh, nk)| u_{xx}(jh, nk) + O(h^2),$$

with

$$\tilde{\rho} = \min\{\rho, \frac{1}{2}\}. \tag{3.5}$$

4. Convergence

In this section, we study the convergence of the scheme (2.2), (2.3).

We set the approximation error by

$$e(jh, nk) = u(jh, nk) - u_j^n. \tag{4.1}$$

We use the abbreviation's

$$\begin{aligned} e_j^n &= e(jh, nh), \\ T(j, n) &= T(jh, nh), \\ u(j, n) &= u(jh, nk). \end{aligned} \tag{4.2}$$

From (2.2), (2.3), (3.1), and (4.1), we have

$$e_j^{n+1} = e_j^n + p\,\Phi(e_{j-1}^n, e_j^n, e_{j+1}^n) + T(j, n), \tag{4.3}$$

with

$$p = \frac{k}{1+\hat{k}L_1\|\tilde{U^n}\|}. \tag{4.4}$$

We set the initial conditions of (4.3)

$$e_j^0 = 0,$$
$$e_j^1 = T(j,1) \quad (0 < j < 1/h). \tag{4.5}$$

We use Taylor series expansions of $\Phi(e_{j-1}^n, e_j^n, e_{j+1}^n)$

$$\Phi(e_{j-1}^n, e_j^n, e_{j+1}^n) = a_x(j,n)e_x(j,n) + a(j,n)e_{xx}(j,n) + b(j,n)e_x(j,n) + c(j,n)e(j,n) + O(h). \tag{4.6}$$

Let us denote

$$q[m, u(x,m)] = a(x,mk)\frac{\partial^2}{\partial x^2}u(x,m) + a_x(x,mk)\frac{\partial}{\partial x}u(x,m) + b(x,mk)\frac{\partial}{\partial x}u(x,m) + c(x,mk)u(x,m) \quad (m = 1,2,3,..),$$

where

$$u(x,1) = T(x,k),$$
$$u(x,m) = \sum_{l=1}^{m} T(x,lk) \quad (m = 2,...,n) \tag{4.7}$$

Then we have

$$q[1, u(x,1)] = a(x,k)\frac{\partial^2}{\partial x^2}T(x,k) + a_x(x,k)\frac{\partial}{\partial x}T(x,k) + b(x,k)\frac{\partial}{\partial x}T(x,k) + c(x,k)T(x,k),$$

$$q[2, u(x,2)] = a(x,2k)(\frac{\partial^2}{\partial x^2}\sum_{l=1}^{2} T(x,lk))$$
$$+\frac{\partial}{\partial x}a(x,2k)(\frac{\partial}{\partial x}\sum_{l=1}^{2} T(x,lk)) + b(x,2k)(\frac{\partial}{\partial x}\sum_{l=1}^{2} T(x,lk)) + c(x,2k)(\sum_{l=1}^{2} T(x,lk)),$$

$$q[m, u(x,m)] = a(x,mk)(\frac{\partial^2}{\partial x^2}\sum_{l=1}^{m} T(x,lk))$$
$$+\frac{\partial}{\partial x}a(x,mk)(\frac{\partial}{\partial x}\sum_{l=1}^{m} T(x,lk)) + b(x,mk)(\frac{\partial}{\partial x}\sum_{l=1}^{m} T(x,lk)) + c(x,m)(\sum_{l=1}^{m} T(x,lk)),$$

$$q[2,q[1,u(x,1)]] = a(x,2k)(\frac{\partial^2}{\partial x^2}q[1,u(x,1)]) + \frac{\partial}{\partial x}a(x,2k)(\frac{\partial}{\partial x}q[1,u(x,1)])$$
$$+ b(x,2k)(\frac{\partial}{\partial x}q[1,u(x,1)]) + c(x,2k)q[1,u(x,1)],$$

$$q[m,q[m-1,u(x,m-1)]]] = a(x,mk)(\frac{\partial^2}{\partial x^2}q[m-1,u(x,m-1)])$$
$$+ (\frac{\partial}{\partial x}a(x,mk))(\frac{\partial}{\partial x}q[m-1,u(x,m-1)])$$
$$+ b(x,mk)(\frac{\partial}{\partial x}q[m-1,u(x,m-1)]) + c(x,mk)q[m-1,u(x,m-1)]. \quad (4.8)$$

From (4.3), (4.6), (4.8), we have

$$e_j^n = \Sigma_{l=1}^n T(j,l) + \{pq[1,u(x,1)] + p^2 q[2,q[1,u(x,1)]] + p^3 q[3,q[2,q[1,u(x,1)]]]$$
$$+ p^{n-1}q[n-1,q[n-2,q[n-3,[......[q[1,u(x,1)]...]]]]]\}$$
$$+ \{pq[2,u(x,2)] + p^2 q[3,q[2,u(x,2)]] + p^3 q[4,q[3,[q[2,u(x,2)]]]$$
$$+ ... + p^{n-2}q[n-1,q[n-2,[....[.q[2,u(x,2)]]..]]...]\} + ... + pq[n-1,u(x,n-1)]. \quad (4.9)$$

We assume

$$\|\frac{\partial^{l_1}}{\partial x^{l_1}}f_1(x,t)\| \le K_1 \quad (l_1 = 0,1,2),$$
$$\|\frac{\partial^{l_2}}{\partial x^{l_2}}\left(f_2(x,t)(\frac{\partial^{l_1}}{\partial x^{l_1}}f_1(x,t))\right)\| \le K_1^2 \quad (l_1,l_2 = 0,1,2),$$

$$\|\frac{\partial^{l_m}}{\partial x^{l_m}}\left(f_m(x,t)\left(\frac{\partial^{l_{m-1}}}{\partial x^{l_{m-1}}}f_{m-1}(x,t)\left(\frac{\partial^{l_{m-2}}}{\partial x^{l_{m-2}}}f_{m-2}(x,t)...\left(\frac{\partial^{l_1}}{\partial x^{l_1}}f_1(x,t)\right)\right)\right)...\right)\|$$
$$\le K_1^m. \quad (4.10)$$

$((x,t) \in \Omega = \{(x,t); 0 < x < 1, 0 < tF\})$ (K_1 : constant) $(l_1, l_2, , .l_m = 0,1,2)$, where

$f_i(x,t)(i = 1,2,..,m)$ are one of the functions $\{a(x,t), b(x,t), c(x,t)\}$.

We assume

$$|\frac{\partial^l}{\partial x^l}w(x,t)| \le \tilde{w}(x,t),$$

$$\|\tilde{w}(x,t)\| \leq K_1. \tag{4.11}$$

$$(l = 0, 1, 2, ...) \quad (x,t) \in \Omega = \{(x,t); 0 < x < 1, 0 < t \leq t_F\}.$$

We use the following notation

$$\tilde{w}(j,l) = \tilde{w}(jh, lk).$$

Then (4.9), (4.10), and (4.11), we have

$$\begin{aligned} |e_j^n| &\leq \Sigma_{l=1}^{n} k^{(1+\tilde{\rho})}\tilde{w}(j,l) + 4K_1p\Sigma_{l=0}^{(n-2)} r^l \tilde{w}(j,l) + 4K_1p\Sigma_{l=0}^{(n-3)} r^l \tilde{w}(j,l) \\ &\qquad + ... + 4K_1p\Sigma_{l=1}^{(n-1)}\tilde{w}(j,l) \\ &= \Sigma_{l=1}^{n} k^{(1+\tilde{\rho})}\tilde{w}(j,l) + 4K_1p\{(1-r^{n-1}) + (1-r^{n-2})\Sigma_{l=1}^{2}\tilde{w}(j,l) \\ &\qquad + ... + (1-r)\Sigma_{l=1}^{(n-1)}\tilde{w}(j,l), \end{aligned} \tag{4.12}$$

with

$$r = 9pK_1.$$

We set the maximum norm of the function $\tilde{w}(j,l)$

$$\|W^n\| = \max_{1\leq j\leq 1/h}\{\tilde{w}(j,n)\}. \tag{4.13}$$

We have, after some computation,

$$\begin{aligned} |e_j^n| &\leq \Sigma_{l=1}^{n} k^{(1+\tilde{\rho})}\|W^n\| \\ &\quad + \{2K_1pn(n-1) - \frac{4K_1p}{(r-1)^2}\{\frac{r^{(n+1)} - r^2}{(r-1)} - (n-1)r\}\}\|W^n\|. \end{aligned} \tag{4.14}$$

Then,we have

$$|e_j^n| \leq nk^{(1+\tilde{\rho})}\|W^n\| + k^{(1+\tilde{\rho})}\{\frac{2K_1pn^2}{(1-r)} + \frac{4K_1p}{(1-r)^3}r^2 + \frac{4K_1p}{(1-r)^2}nr\}\|W^n\|. \tag{4.15}$$

From (1.2), (4.4), we have

$$kn \leq t_F,$$

$$pn \leq \frac{t_F}{(1 + kL_1\|\tilde{U^n}\|)},$$

$$k^{1+\tilde{\rho}}pn^2\|W^n\| \leq (t_F)^2 k^{\tilde{\rho}}\{\frac{\|W^n\|}{(1 + \hat{k}L_1\|\tilde{U^n}\|)}\}. \tag{4.16}$$

From (4.11), (4.15), and (4.16), we obtain the inequality

$$|e_j^n| \leq \{t_F + \frac{1}{(1-r)}(\frac{2rt_F^2}{(1+\hat{k}L_1\|U^n\|)} + \frac{4pr^3k}{(1-r)^2} + \frac{4r^2t_Fk}{(1-r)(1+\hat{k}L_1\|U^n\|)})\}k^{\tilde{\rho}}K_1. \quad (4.17)$$

We set the maximum error of $\{e_j^n\}$ by

$$\|E^n\| = \max_{1\leq j\leq 1/h}\{|e_j^n|\}.$$

Then, from (4.17) we have

$$\|E^n\| \leq \{t_F + \frac{1}{(1-r)}(\frac{2rt_F^2}{(1+\hat{k}L_1\|U^n\|)} + \frac{4pr^3k}{(1-r)^2} + \frac{4r^2t_Fk}{(1-r)(1+\hat{k}L_1\|U^n\|)})\}k^{\tilde{\rho}}K_1. \quad (4.18)$$

From (4.18), we have

$$\lim_{k->0}\|E^n\| = 0. \quad (4.19)$$

Theorem [2] We consider the convergence paths

$$\{(h_i,k_i);\ \frac{k_i}{{h_i}^2} = c\ (c: \text{constant})\ i = 1,2,..,h_i,k_i \to 0\ (i\to\infty)\}. \quad (4.20)$$

Suppose that for h_i and k_i, there exists positive numbers $j(h_i)$ and $n(k_i)$ such that

$$j(h_i)h_i \ \to\ x \in [0,1](i\to\infty) \quad n(k_i)k_i \ \to\ t \in [0,t_F].$$

If the conditions (4.10), (4.11) and $|u(x,t)|, |u_x(x,t)|, |u_{xx}(x,t)|$ are bounded in $[0,1]*[0,t_F]$.

Then the scheme (2.2), (2.3) converge to the solution $u(x,t)$ of the differential equation (1.1) uniformly.

5. Stability

In this section, we study the stability of the numerical process (2.2), (2.3) and define as follows.

Definition. The numerical processes $\{Y^n \in R_n\}$ is stable if there exists a positive constant K_2 such that

$$\|Y^n\| \leq K_2,$$

where $\|.\|$ denotes some norm and the constant K_2 is depends only on initial value.

We prove that the scheme (2.2), (2.3) are stable in mean of the von Neumann.

We use the following Lemma in proof of stability.

Lemma 1.

$$\|\Phi(e_{j-1}^n, e_j^n, e_{j+1}^n)\| \leq L_2, \quad (L_2 : constant) \tag{5.1}$$

under the conditions (4.10), (4.11).

Proof. From (2.4), we have

$$\Phi(e_{j-1}^n, e_j^n, e_{j+1}^n) = \frac{1}{h^2} a(jh, nk)(e_{j-1}^n - 2e_j^n + e_{j+1}^n) + \frac{1}{2h}\, a_x(jh, nk)(e_{j+1}^n - e_{j-1}^n)$$
$$+\frac{1}{8} a_{xx}(jh, nk)(e_{j-1}^n - 2e_j^n + e_{j+1}^n) + \frac{1}{2h}\{|b(jh, nk)|(e_{j-1}^n - 2e_j^n + e_{j+1}^n)$$
$$+b(jh, nk)(e_{j+1}^n - e_{j-1}^n)\} + c(jh, nk)T(jh, nk)$$
$$= a(jh, nk)e_{xx}(jh, nk) + a_x(jh, nk)e_x(jh, nk) + \frac{h^2}{8} a_{xx}(jh, nk)e_{xx}(jh, nk)$$
$$+\frac{h}{2}|b(jh, nk)|e_{xx}(jh, nk) + b(jh, nk)e_x(jh, nk) + c(jh, nk)e(jh, nk). \tag{5.2}$$

If we set the term of expansion e_j^n by $\{s_{m_1}^{m_2}\}$. Then from (4.9), we have

$$s_{m_1}^{m_2} = q[m_2 - 1, q[m_2 - 2, [..., q[m_1, u(x, m_1)]]]..]. \quad (m_1, m_2 = 1, 2, .., n)$$

The factor of $s_{m_1}^{m_2}$ is given by

$$s(x, t) = f_{m_2}(x, t) \frac{\partial^{l_{s_1}}}{\partial x^{l_{s_1}}} \Bigg(f_{m_2 - 1}(x, t) \Bigg(\frac{\partial^{l_{s_2}}}{\partial x^{l_{s_2}}} f_{m_2 - 2}(x, t)$$
$$... \Bigg(\frac{\partial^{l_{s_p}}}{\partial x^{l_{s_p}}} f_{m_1}(x, t) \Bigg) \Bigg) \Bigg) ... \Bigg) \frac{\partial^{l_{s_q}}}{\partial x^{l}{}_{s_q}} (\Sigma_{l=1}^{m_1} \tilde{T}(x, t)). \tag{5.3}$$

The first derivation of $s(x, t)$ is

$$\frac{\partial}{\partial x}s(x,t) = \frac{\partial}{\partial x}\Bigg(f_{m_2}(x,t)\frac{\partial^{l_{s_1}}}{\partial x^{l_{s_1}}}\Bigg(f_{m_2-1}(x,t)\Bigg(\frac{\partial^{l_{s_2}}}{\partial x^{l_{s_2}}}f_{m_2-2}(x,t) \\ ...\Bigg(\frac{\partial^{l_{s_p}}}{\partial x^{l_{s_p}}}f_{m_1}(x,t)\Bigg)\Bigg)\Bigg)...\Bigg)\frac{\partial^{l_{s_q}}}{\partial x^{l_{s_q}}}(\Sigma_{l=1}^{m_1}T(j,l)) \\ + f_{m_2}(x,t)\frac{\partial^{l_{s_1}}}{\partial x^{l_{s_1}}}\Bigg(f_{m_2-1}(x,t)\Bigg(\frac{\partial^{l_{s_2}}}{\partial x^{l_{s_2}}}f_{m_2-2}(x,t) \\ ...\Bigg(\frac{\partial^{l_{s_p}}}{\partial x^{l_{s_p}}}f_{m_1}(x,t)\Bigg)\Bigg)\Bigg)...\Bigg)\frac{\partial^{l_{s_q}+1}}{\partial x^{l_{s_q}+1}}(\Sigma_{l=1}^{m_2}T(j,l)).$$

The second deraivation of $s(x,t)$ is

$$\frac{\partial^2}{\partial x^2}s(x,t) = \frac{\partial^2}{\partial x^2}\Bigg(f_{m_2}(x,t)\frac{\partial^{l_{s_1}}}{\partial x^{l_{s_1}}}\Bigg(f_{m_2-1}(x,t)\Bigg(\frac{\partial^{l_{s_2}}}{\partial x^{l_{s_2}}}f_{m_2-2}(x,t) \\ ...\Bigg(\frac{\partial^{l_{s_p}}}{\partial x^{l_{s_p}}}f_{m_1}(x,t)\Bigg)\Bigg)\Bigg)...\Bigg)\frac{\partial^{l_{s_q}}}{\partial x^{l_{s_q}}}(\Sigma_{l=1}^{m_1}T(j,l)) \\ + 2\frac{\partial}{\partial x}\Bigg(f_{m_2}(x,t)\frac{\partial^{l_{s_1}}}{\partial x^{l_{s_1}}}\Bigg(f_{m_2-1}(x,t)\Bigg(\frac{\partial^{l_{s_2}}}{\partial x^{l_{s_2}}}f_{m_2-2}(x,t) \\ ...\Bigg(\frac{\partial^{l_{s_p}}}{\partial x^{l_{s_p}}}f_{m_1}(x,t)\Bigg)\Bigg)\Bigg)...\Bigg)\frac{\partial^{l_{s_q}+1}}{\partial x^{l_{s_q}+1}}(\Sigma_{l=1}^{m_1}T(j,l)) \\ + \Bigg(f_{m_2}(x,t)\frac{\partial^{l_{s_1}}}{\partial x^{l_{s_1}}}\Bigg(f_{m_2-1}(x,t)\Bigg(\frac{\partial^{l_{s_2}}}{\partial x^{l_{s_2}}}f_{m_2-2}(x,t) \\ ...\Bigg(\frac{\partial^{l_{s_p}}}{\partial x^{l_{s_p}}}f_{m_1}(x,t)\Bigg)\Bigg)\Bigg)...\Bigg)\frac{\partial^{l_{s_q}+2}}{\partial x^{l_{s_q}+2}}(\Sigma_{l=1}^{m_1}T(j,l)). \quad (5.4)$$

From (5.3), (5.4), we have

$$\|\frac{\partial}{\partial x}s_{m_1}^{m_2}\| \leq 2\times 4\times (7)^{(m_2-m_1)}K^{(m_2-m_1+1)}(\Sigma_{l=1}^{m_1}\|\frac{\partial^l}{\partial x^l}\tilde{T}(x,l)\|),$$

$$\|\frac{\partial^2}{\partial x^2}s_{m_1}^{m_2}\| \leq 4\times 4\times (7)^{(m_2-m_1)}K^{(m_2-m_1+1)}(\Sigma_{l=1}^{m_1}\|\frac{\partial^l}{\partial x^l}\tilde{T}(x,l)\|) \quad (5.5)$$

Then, in the same way as (4.17), we have

$$\|\frac{\partial}{\partial x}e(j,n)\| \leq 2L_n,$$

$$\|\frac{\partial^2}{\partial x^2}e(j,n)\| \leq 4L_n \quad (5.6)$$

with

$$L_n = \{t_F + \frac{1}{(1-r)}(\frac{2rt_F^2}{(1+\hat{k}L_1\|U^n\|)} + \frac{4pr^3k}{(1-r)^2} + \frac{4r^2t_Fk}{(1-r)(1+\hat{k}L_1\|U^n\|)})\}k^{\tilde{\rho}}\|W^n\|.$$

From (5.2), (5.5), (5.6), we have

$$\|\Phi(e_{j-1}^n, e_j^n, e_{j+1}^n)\| \le \|a(jh,nk)e_{xx}(jh,nk)\| + \|a_x(jh,nk)e_x(jh,nk)\|$$
$$+\frac{h}{2}\|b(jh,nk)e_{xx}(jh,nk)\| + \|b(jh,nk)e_x(jh,nk)\| + \|c(jh,nk)e(jh,nk)\| + O(h^2)$$
$$\le L_2. \quad (L_2 = K_1(9+\frac{h}{2})L_n) \quad (5.7)$$

If we assume

$$\|u(x,t)\| \le K_3, \|u_x(x,t)\| \le K_3, \|u_{xx}(x,t)\| \le K_3, \quad (K_3 : constant), \quad (5.8)$$
$$(x,t) \in \Omega = \{(x,t); 0 < x < 1, 0 < t \le t_F\}.$$

Then, in the same way to (5.7), we have

$$\|\Phi(u(j-1,n), u(j,n), u(j+1,n))\|$$
$$\le \|a_x(j,n)u_x(j,n)\| + \|a(j,n)u_{xx}(j,n)\|$$
$$+ \|b(j,n)u_x(j,n)\| + \|c(j,n)u(j,n)\| + \frac{h}{2}(\|b(j,n)u_{xx}(j,n))\|$$
$$\le (4+\frac{h}{2})K_1K_3. \quad (5.9)$$

From (5.7), (5.9), we obtain:

$$\|\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)\| \le \|\Phi(u(j-1,n), u(j,n), u(j+1,n))\| + \|\Phi(e_{j-1}^n, e_j^n, e_{j+1}^n)\|$$
$$\le L_3. \quad (L_3 = (4+\frac{h}{2})K_1K_3 + K_1(9+\frac{h}{2})L_n) \quad (5.10)$$

From (5.10), we have the following result.

Lemma 2.

$$\|\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)\| \le L_3, \quad (L_3 : constant) \quad (5.11)$$

with

$$L_3 = (4+\frac{h}{2})K_1K_3 + K_1(9+\frac{h}{2})L_n,$$

with the condition (4.10), (4.11), (5.8).

From (2.2), (2.3), we have

$$\begin{aligned} u_j^{n+1} &= u_j^n + \frac{k}{1+\hat{k}L_1\|\tilde{U^n}\|}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n) \\ &= u_j^n + \frac{k}{1+kL_1}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n) + \frac{k}{1+\hat{k}L_1\|\tilde{U^n}\|}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n) \\ &\quad - \frac{k}{1+kL_1}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n). \end{aligned} \tag{5.12}$$

From (5.12), we have

$$\begin{aligned} |u_j^{n+1}| \leq & \; |u_j^n + \frac{k}{1+kL_1}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)| + |\frac{k}{1+\hat{k}L_1\|\tilde{U^n}\|}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n) \\ & - \frac{k}{1+kL_1}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)|. \end{aligned} \tag{5.13}$$

We have the inequality

$$\begin{aligned} |u_j^n + \frac{k}{1+kL_1}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)| \leq & \; |1+\frac{k\rho(nh,jk)}{1+kL_1}| \, |u_j^n| + \frac{k}{1+kL_1}\eta(jh,nk)|u_{j-1}^n| \\ & + \frac{k}{1+kL_1}\tau(jh,nk)|u_{j+1}^n|. \end{aligned} \tag{5.14}$$

From (2.5), we have

$$1 + \frac{k\rho(jh,nk)}{1+kL_1} \geq 0,$$

it follows

$$\begin{aligned} & |u_j^n + \frac{k}{1+kL_1}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)| \\ \leq & \{1 + \frac{k}{1+kL_1}\{\rho(nh,jk) + \eta(jh,nk) + \tau(jh,nk)\}\}\|U^n\| \\ & \leq \|U^n\|. \end{aligned} \tag{5.15}$$

We study stability of the scheme (2.2). From condition (2.2), we have

$$\frac{k}{1+\hat{k}L_1\|U^n\|}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n) - \frac{k}{1+kL_1\|\tilde{U^n}\|}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)$$

$$= k\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)\{\frac{1}{(1+kL_1\|\tilde{U^n}\|)} - \frac{1}{(1+kL_1)}\}. \tag{5.16}$$

Then, from (5.16), we have the inequality,

$$\| \frac{k}{1+\hat{k}L_1\|U^n\|}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n) - \frac{k}{1+kL_1\|\tilde{U^n}\|}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)\|$$

$$\leq k\|\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)\|| \frac{1}{(1+\hat{k}L_1\|\tilde{U^n}\|)} - \frac{1}{(1+kL_1)}|,$$

and

$$|\frac{1}{(1+\hat{k}L_1)} - \frac{1}{(1+kL_1\|\tilde{U^n}\|)}| \leq 2. \tag{5.17}$$

Then, from (5.13), (5.14), (5.16), (5.17), we have

$$|u_j^{n+1}| \leq |u_j^n| + 2kL_3, \tag{5.18}$$

From (5.18), we have

$$|u^n| \leq |u_i^0| + 2nkL_3. \tag{5.19}$$

From (5.19), we have

$$\|U^n\| \leq \|U^0\| + 2nkL_3 \leq \|U^0\| + 2t_F L_3. \tag{5.20}$$

We study stability of the scheme (2.3). From condition (2.3), we have

$$\frac{k}{(1+\hat{k}L_1\|\tilde{U^n}\|)} = \frac{k}{(1+\hat{k}L_1\|U^n\|)}.$$

From (5.12), we have the inequality

$$|u_j^{n+1}| \leq |u_j^n + \frac{k}{1+kL_1\|U^n\|}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)|$$

$$+ |\frac{k}{1+\hat{k}L_1\|U^n\|}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n) - \frac{k}{1+kL_1\|U^n\|}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)|. \tag{5.21}$$

From the condition $\|U^n\| \geq 1$, we have

$$1 + \frac{k\rho(jh, nk)}{1+kL_1\|U^n\|} \geq 0.$$

In the same way to (5.15), we have

$$|u_j^n + \frac{k}{1+kL_1}\Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)| \leq \|U^n\|. \tag{5.22}$$

From Lemma [2], we have

$$\left| \frac{k}{1+\hat{k}L_1\|U^n\|}\Phi(u^n_{j-1},u^n_j,u^n_{j+1}) - \frac{k}{1+kL_1\|U^n\|}\Phi(u^n_{j-1},u^n_j,u^n_{j+1})\right|$$

$$\leq |k\Phi(u^n_{j-1},u^n_j,u^n_{j+1})\frac{|(k-\hat{k})|L_1\|U^n\|}{(1+kL_1\|U^n\|)(1+\hat{k}L_1\|U^n)\||}$$

$$\leq \{kL_1L_3 + O(k\sqrt{k})\}\|U^n\|. \quad (5.23)$$

From (5.21), (5.22), (5.23), we have

$$|u^j_{n+1}| \leq (1 + kL_1L_3 + O(k\sqrt{k}))\|U^n\|,$$

and we have the following result

$$\|U^{n+1}\| \leq \{1 + kL_1L_3 + O(k\sqrt{k})\}^n \|U^0\|. \quad (5.24)$$

From (5.24), we have

$$\|U^n\| \leq e^{w_1 t_F}\|U^0\|,$$

with

$$w_1 = L_1L_3.$$

Summarizing the results, we have

Theorem [3] For any given step size h, k, we set the constant $\hat{k}, L_1$ to be

$$L_1 = \max_{0\leq i\leq \frac{1}{h}, 0\leq j\leq \frac{t_F}{k}} \{(\frac{1}{h^2})\{a_{i+\frac{1}{2}}(jk) + a_{i-\frac{1}{2}}(jk)\} + \frac{1}{h}|b(ih,jk)| - c(ih,jk)\}$$

$$\hat{k} = k^{1+\alpha}, \qquad (\alpha > 0)$$

and if $|u(x,t), |u_x(x,t)|$, and $|u_{xx}(x,t)|$ are bounded in the region $(x,t) \in \Omega = \{(x,t); 0 < x < 1, 0 < t \leq t_F\})$ and the conditions (4.10), (4.11) are satisfied. Then the difference approximation (2.2), (2.3) are stable.

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