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DIFFERENTIAL EQUATION OF GENERALIZED K-WRIGHT FUNCTION

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Abstract: In this paper, authors introduced a homogeneous liner differential equation whose one of the solution in the form of Generalized K -Wright function. Special cases also deduced in terms of Generalized Wright function and Generalized K-Mittag-Leffler function.

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1. Introduction

Generalized K-Gamma Function $\Gamma_k(x)$ defined as (see Diaz and Pariguan [1])

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$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!\ k^n (nk)^{\frac{x}{k}\ -\ 1}}{(x)_{n,k}}\ ,\ k>0,\ x \in \mathbb{C}\backslash k\mathbb{Z}^-, \tag{1}$$

where $(x)_{n,k}$ is the k-Pochhammer symbol and is given by

$$(x)_{n,k} = x(x+k)(x+2k)\dots(x+(n-1)k), \qquad x\in\mathbb{C},\ k\in\mathbb{R},\ n\in\mathbb{N}^+ \tag{2}$$

For $Re(x) > 0$, $\Gamma_k(x)$ is K-Gamma function defined as the integral

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{\frac{-t^k}{k}}\ dt \tag{3}$$

It is easy to prove following results

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right) \tag{4}$$

and

$$(\gamma)_{nq,k} = (k)^{nq}\left(\frac{\gamma}{k}\right)_{nq} \tag{5}$$

The Generalized K-Wright function, introduced by [2], as

Definition 1. Generalized K-Wright Function is defined as ${}_p\Psi_q^k(z)$ for $k \in \mathbb{R}^+; z\in\mathbb{C}, \alpha_i, \beta_j \in \mathbb{R}(\alpha_i, \beta_j \neq 0; i = 1,2,...,p; j = 1,2,...,q)$ and $(a_i+\alpha_i n)$, $(b_j+\beta_j n) \in \mathbb{C}\backslash k\mathbb{Z}^-$,

$${}_p\Psi_q^k(z) = {}_p\Psi_q^k\left[\begin{array}{c} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array}\middle| z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma_k(a_i+\alpha_i n) z^n}{\prod_{j=1}^{q}\Gamma_k(b_j+\beta_j n) n!} \tag{6}$$

Theorem 1.1. (see Diaz and Pariguan [1]) *For $k \in \mathbb{R}^+$; $z\in\mathbb{C}$, $\alpha_i,\beta_j \in \mathbb{R}(\alpha_i,\beta_j \neq 0; i=1,2,...,p; j=1,2,...,q)$ and $(a_i+\alpha_i n), (b_j+\beta_j n) \in \mathbb{C}\backslash k\mathbb{Z}^-$,*

(a) *if $\Delta > -1$ then series (6) is absolutely convergent for all $z\in\mathbb{C}$ and Generalized K-Wright function ${}_p\Psi_q^k(z)$ is an entire function of z.*

(b) *if $\Delta = -1$ then series (6) is absolutely convergent for all $|z| < \delta$ and of $|z| = \delta$, $Re(\mu) > \frac{1}{2}$.*

We use following notations for describing convergence condition

$$\Delta = \sum_{j=1}^{q}\left(\frac{\beta_j}{k}\right) - \sum_{i=1}^{p}\left(\frac{\alpha_i}{k}\right); \delta = \prod_{i=1}^{p}\left|\frac{\alpha_i}{k}\right|^{\frac{-\alpha_i}{k}} \prod_{j=1}^{q}\left|\frac{\beta_j}{k}\right|^{\frac{\beta_j}{k}},$$

$$\mu = \sum_{j=1}^{q}\left(\frac{b_j}{k}\right) - \sum_{i=1}^{p}\left(\frac{a_i}{k}\right) + \frac{p-q}{2}.$$

The generalized Pochhammer symbol(Rainville [5]),

$$(\gamma)_{nq} = \frac{\Gamma(\gamma+nq)}{\Gamma(\gamma)} = q^{qn}\prod_{r=1}^{q}\left(\frac{\gamma+r-1}{q}\right)_n, \quad \text{if } q \in \mathbb{N}. \tag{7}$$

Equation (6) reduced in the following form by substituting $\alpha_i = km_i$, $\beta_j = kl_j$; $m_i, l_j \in \mathbb{N}$ and using (4), (5), (7) we obtain,

$${}_p\Psi_q^k(z) = A\sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\prod_{r=1}^{m_i}\left(\frac{\frac{a_i}{k}+r-1}{m_i}\right)_n}{\prod_{j=1}^{q}\prod_{s=1}^{l_j}\left(\frac{\frac{b_j}{k}+s-1}{l_j}\right)_n} \frac{(Bz)^n}{(n!)} \tag{8}$$

where

$$A = \frac{\prod_{i=1}^{p}\Gamma(\frac{a_i}{k})}{\prod_{j=1}^{q}\Gamma(\frac{b_j}{k})} K^{(\sum_{i=1}^{p}\frac{a_i}{k} - \sum_{j=1}^{q}\frac{b_j}{k} + q - p)} \tag{9}$$

and

$$B = \frac{\prod_{i=1}^{p}(m_i)^{m_i}}{\prod_{j=1}^{q}(l_j)^{l_j}} K^{(\sum_{i=1}^{p} m_i - \sum_{j=1}^{q} l_j)} \tag{10}$$

2. Main Result

In this section, authors established linear homogeneous differential equation known as Generalized K-Wright differential equation. One of its solution is Generalized K-Wright function (8). Finally, this differential equation deduced in terms of Generalized Wright function and Generalized K-Mitteg-Leffler function.

Theorem 2.1. *The Generalized K-Wright differential equation is defined as,*

$$\left[\theta\prod_{j=1}^{q}\prod_{s=1}^{l_j}\left(\theta + \frac{\frac{b_j}{k}+s-1}{l_j} - 1\right) - Bz\prod_{i=1}^{p}\prod_{r=1}^{m_i}\left(\theta + \frac{\frac{a_i}{k}+r-1}{m_i}\right)\right] W = 0, \tag{11}$$

for

$$p \leq q+1 \qquad (i=1,2,...,p \text{ and } j=1,2,...,q),$$

where no $\dfrac{\frac{b_j}{k}+s-1}{l_j}$ *is a negative integer or Zero and no two* $\dfrac{\frac{b_j}{k}+s-1}{l_j}$*'s differ by an integer or Zero, then the solution is*

$$W=\sum_{t=0}^{q} C_t W_t \tag{12}$$

where C_t *are arbitrary constant, and*

$$W_0 = {}_p\Psi_q^k(z) \tag{13}$$

and for $t=1,2,3,...,q$

$$W_t =$$

$$\sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\prod_{r=1}^{m_i}(1+\frac{\frac{a_i}{k}+r-1}{m_i}-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n}{\prod_{j=1}^{q}\prod_{s=1}^{l_j}(1+\frac{\frac{b_j}{k}+s-1}{l_t}-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n} \frac{(Bz)^{n+1-\prod_{s=1}^{l_t}(\frac{\frac{b_t}{k}+s-1}{l_t})}}{(2-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n}. \tag{14}$$

Here $\theta = z\frac{d}{dz}$ *and B is given by (10).*

Proof. From given condition, no $\dfrac{\frac{b_j}{k}+s-1}{l_j}$ is a positive integer then the linear combination (12) is the general solution of equation (11) around $z=0$. Note that if $p \leq q(i=1,2,...,p$ and $j=1,2,...,q)$ then the series $W_t; t = 0,1,2,...,q$, converge for all finite z and for $p \leq q+1, (i=1,2,...,p$ and $j = 1,2,...,q)$ the series W_t converges for $\mid z \mid < 1$.

To verify that W_0, satisfies (11).

Consider,

$$\Delta \equiv \theta \prod_{j=1}^{q}\prod_{s=1}^{l_j}(\theta+\frac{\frac{b_j}{k}+s-1}{l_j}-1)W_0,$$

using (8), it follows that

$$\Delta \equiv A\sum_{n=0}^{\infty} \frac{\theta\prod_{j=1}^{q}\prod_{s=1}^{l_j}(\theta+\frac{\frac{b_j}{k}+s-1}{l_j}-1)\prod_{i=1}^{p}\prod_{r=1}^{m_i}(\frac{\frac{a_i}{k}+r-1}{m_i})_n}{\prod_{j=1}^{q}\prod_{s=1}^{l_j}(\frac{\frac{b_j}{k}+s-1}{l_j})_n}\frac{(Bz)^n}{(n!)},$$

where A and B are given by (9) and (10), respectively.

Since $\theta(Bz)^n = n(Bz)^n$, we have

$$\Delta \equiv A\sum_{n=0}^{\infty}\frac{n\prod_{j=1}^{q}\prod_{s=1}^{l_j}(n+\frac{\frac{b_j}{k}+s-1}{l_j}-1)\prod_{i=1}^{p}\prod_{r=1}^{m_i}(\frac{\frac{a_i}{k}+r-1}{m_i})_n}{\prod_{j=1}^{q}\prod_{s=1}^{l_j}(\frac{\frac{b_j}{k}+s-1}{l_j})_n}\frac{(Bz)^n}{(n!)},$$

this gives,

$$\Delta \equiv A\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{p}\prod_{r=1}^{m_i}(\frac{\frac{a_i}{k}+r-1}{m_i})_n}{\prod_{j=1}^{q}\prod_{s=1}^{l_j}(\frac{\frac{b_j}{k}+s-1}{l_j})_{n-1}}\frac{(Bz)^n}{(n-1)!},$$

above equation can be written as,

$$\Delta \equiv A\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\prod_{r=1}^{m_i}(\frac{\frac{a_i}{k}+r-1}{m_i})_{n+1}}{\prod_{j=1}^{q}\prod_{s=1}^{l_j}(\frac{\frac{b_j}{k}+s-1}{l_j})_n}\frac{(Bz)^{n+1}}{(n)!},$$

above equation reduces to,

$$\Delta \equiv Bz\,A\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\prod_{r=1}^{m_i}(\frac{\frac{a_i}{k}+r-1}{m_i}+n)(\frac{\frac{a_i}{k}+r-1}{m_i})_n}{\prod_{j=1}^{q}\prod_{s=1}^{l_j}(\frac{\frac{b_j}{k}+s-1}{l_j})_n}\frac{(Bz)^n}{(n)!},$$

this leads,

$$\Delta \equiv Bz\prod_{i=1}^{p}\prod_{r=1}^{m_i}(\theta+\frac{\frac{a_i}{k}+r-1}{m_i})W_0.$$

This shows that W_0 is a solution of the differential equation (11).

To verify that $W_t, t=1,2,...,q$, satisfies equation (11).

Consider,

$$\Omega \equiv \theta\prod_{j=1}^{q}\prod_{s=1}^{l_j}(\theta+\frac{\frac{b_j}{k}+s-1}{l_j}-1)W_t,$$

from (14), we get

$$\Omega \equiv \theta\prod_{j=1}^{q}\prod_{s=1}^{l_j}(\theta+\frac{\frac{b_j}{k}+s-1}{l_j}-1)\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\prod_{r=1}^{m_i}(1+\frac{\frac{a_i}{k}+r-1}{m_i}-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n}{\prod_{j=1}^{q}\prod_{s=1}^{l_j}(1+\frac{\frac{b_j}{k}+s-1}{l_t}-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n}$$

$$\times\frac{(Bz)^{n+1-\prod_{s=1}^{l_t}(\frac{\frac{b_t}{k}+s-1}{l_t})}}{(2-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n},$$

Since $\theta(Bz)^n = n(Bz)^n$, we have

$$\Omega \equiv \sum_{n=0}^{\infty}\frac{(n+1-\prod_{s=1}^{l_t}(\frac{\frac{b_t}{k}+s-1}{l_t}))\prod_{j=1}^{q}\prod_{s=1}^{l_j}(n-\prod_{s=1}^{l_t}(\frac{\frac{b_t}{k}+s-1}{l_t})+\frac{\frac{b_j}{k}+s-1}{l_j})}{\prod_{j=1}^{q}\prod_{s=1}^{l_j}(1+\frac{\frac{b_j}{k}+s-1}{l_t}-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n}$$

$$\times\frac{\prod_{i=1}^{p}\prod_{r=1}^{m_i}(1+\frac{\frac{a_i}{k}+r-1}{m_i}-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n(Bz)^{n+1-\prod_{s=1}^{l_t}(\frac{\frac{b_t}{k}+s-1}{l_t})}}{(2-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n}. \tag{15}$$

On simplification, we have,

$$\frac{(n+1-\prod_{s=1}^{l_t}(\frac{\frac{b_t}{k}+s-1}{l_t}))}{(2-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n}=\frac{1}{(2-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_{n-1}}. \tag{16}$$

and

$$\frac{\prod_{j=1}^{q}\prod_{s=1}^{l_j}(n-\prod_{s=1}^{l_t}(\frac{\frac{b_t}{k}+s-1}{l_t})+\frac{\frac{b_j}{k}+s-1}{l_j})}{\prod_{j=1}^{q}\prod_{s=1}^{l_j}(1+\frac{\frac{b_j}{k}+s-1}{l_t}-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n}$$

$$=\frac{1}{\prod_{j=1}^{q}\prod_{s=1}^{l_j}(1+\frac{\frac{b_j}{k}+s-1}{l_t}-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_{n-1}} \tag{17}$$

substituting (16) and (17) in (15), we obtain

$$\Omega \equiv \sum_{n=1}^{\infty}\frac{\prod_{i=1}^{p}\prod_{r=1}^{m_i}(1+\frac{\frac{a_i}{k}+r-1}{m_i}-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n}{\prod_{j=1}^{q}\prod_{s=1}^{l_j}(1+\frac{\frac{b_j}{k}+s-1}{l_t}-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_{n-1}}\frac{(Bz)^{n+1-\prod_{s=1}^{l_t}(\frac{\frac{b_t}{k}+s-1}{l_t})}}{(2-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_{n-1}},$$

now we replace n by $n+1$, we get

$$\Omega \equiv Bz\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\prod_{r=1}^{m_i}(1+\frac{\frac{a_i}{k}+r-1}{m_i}-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_{n+1}}{\prod_{j=1}^{q}\prod_{s=1}^{l_j}(1+\frac{\frac{b_j}{k}+s-1}{l_t}-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n}\frac{(Bz)^{n+1-\prod_{s=1}^{l_t}(\frac{\frac{b_t}{k}+s-1}{l_t})}}{(2-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n}.$$

From the equation above, it follows that

$$\Omega \equiv$$

$$Bz\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\prod_{r=1}^{m_i}(n+1+\frac{\frac{a_i}{k}+r-1}{m_i}-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})(1+\frac{\frac{a_i}{k}+r-1}{m_i}-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n}{\prod_{j=1}^{q}\prod_{s=1}^{l_j}(1+\frac{\frac{b_j}{k}+s-1}{l_t}-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n}$$

$$\times\frac{(Bz)^{n+1-\prod_{s=1}^{l_t}(\frac{\frac{b_t}{k}+s-1}{l_t})}}{(2-\prod_{s=1}^{l_t}\frac{\frac{b_t}{k}+s-1}{l_t})_n},$$

Hence,

$$\Omega \equiv Bz\prod_{i=1}^{p}\prod_{r=1}^{m_i}(\theta+\frac{\frac{a_i}{k}+r-1}{m_i})]W_t.$$

This shows that $W_t, t=1,2,...,q$ is the solutions of the differential equation (11).

Particular Cases: For suitable values of the parameters, we can obtain certain differential equations for different generalized functions for $\alpha_i = km_i$, $\beta_j = kl_j$; $m_i, l_j \in \mathbb{N}; i=1,2,...,p; j=1,2,...,q$ and $k \in \mathbb{R}$.

[A] Taking $k=1$ in (11), we have arrived at

$$\left[\theta\prod_{j=1}^{q}\prod_{s=1}^{l_j}\left(\theta+\frac{b_j+s-1}{l_j}-1\right)-Bz\prod_{i=1}^{p}\prod_{r=1}^{m_i}\left(\theta+\frac{a_i+r-1}{m_i}\right)\right]W=0. \quad (18)$$

where $B=\dfrac{\prod_{i=1}^{p}(m_i)^{m_i}}{\prod_{j=1}^{q}(l_j)^{l_j}}$.

Equation (18), is the differential equation for Generalized Wright function $W_0 = {}_p\Psi_p(z)$, definded by Wright [10].

[B] Taking $p=1, q=2; \alpha_1 = km_1, \beta_1 = kl_1, \beta_2 = 0$ in (11), we obtain,

$$\left[\theta\prod_{j=1}^{2}\prod_{s=1}^{l_j}\left(\theta+\frac{b_j+s-1}{l_j}-1\right)-Bz\prod_{i=1}^{1}\prod_{r=1}^{m_i}\left(\theta+\frac{a_i+r-1}{m_i}\right)\right]W=0, \quad (19)$$

where

$$B=\frac{\prod_{i=1}^{1}(m_i)^{m_i}}{\prod_{j=1}^{2}(l_j)^{l_j}}K^{(\sum_{i=1}^{1}m_i-\sum_{j=1}^{2}l_j)}$$

Equation (19), is the differential equation for Generalized K-Mittag- Leffler function $GE^{a_1,m_1}_{k,km_1,kl_1}(z)$, definded by Gehlot [3,4].

The function defined in equation (10) is an extension of generalized Mittag-Leffler function defined by Shukla and Prajapati ([6], [7], [8]), Shukla et al [9].

References

[1] R. Diaz, E. Pariguan, On hypergeometric functions and Pochhammer k-symbol, *Divulgaciones Mathematicas*, **15** No. 2 (2007), 179-192.

[2] Gehlot Kuldeep Singh, Jyotindra C. Prajapati, Fractional calculus of generalized K-wright function, *Journal of Fractional Calculus and Applications*, **4**, No. 2 (2013), 283-289.

[3] Kuldeep Singh Gehlot, Differential equation of generalized K-Mittag-Leffler function, *Int. J. Contemp. Math. Sciences* (2013), Accepted.

[4] Kuldeep Singh Gehlot, The generalized K-Mittag-Leffler function, *Int. J. Contemp. Math. Sciences*, **7**, No. 45 (2012), 2213-2219.

[5] Earl D. Rainville, *Special Functions*, The Macmillan Company, New York (1963).

[6] A.K. Shukla, J.C. Prajapati, A general class of polynomials associated with generalized Mittag-Lefflerfunction, *Integral Transforms and Special Functions*, **19**, No. 1 (2008), 23-34.

[7] A.K. Shukla, J.C. Prajapati, Some remarks on generalized Mittag-Leffler function, *Proyecciones Journal of Mathematics*, **28**, No. 1 (2009), 27-34.

[8] A.K. Shukla, J.C. Prajapati, On a generalized Mittag-Leffler type function and generated integral operator, *Mathematical Sciences Research Journal*, **12** (2008), 283-290.

[9] A.K. Shukla, I.A. Salehbhai, J.C. Prajapati, On Laguerre transform in two variables, *Integral Transforms and Special Functions*, **20**, No. 6 (2009), 459-470.

[10] E.M. Wright, The asymptotic expansion of the grneralized hypergeometric function, *J. London Math. Soc.*, **10** (1935), 287-293.