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INTEGRAL REPRESENTATIONS FOR BASIC COMPLETELY ALTERNATING FUNCTIONS AND SOME OF ITS RELATED FUNCTIONS

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Abstract: Lévy Khinchin type formula for completely alternating functions are given. Some properties of the class of basic completely alternating functions are showed. The relations between the classes of completely monotone and completely alternating functions are investigated.

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1. Introduction

Apart from this old history of quantum calculus, the subject received a considerable interest of many mathematicians and from many aspects, theoretical and practical. It is hard to collect all such axes in a short notice, but to give

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the reader a reasonable idea we will mention some papers [10,20,21] one finds general study of the theory of q-difference equations. Since investigating q-difference equations using function theory tools explores more properties, this direction is also considered in many works, like [7,19,27]. The solutions of difference equations and q-analogues of existing classical ones, especially orthogonal polynomials could be found in [2,3,4]. Applicable problems involving q-difference equations and q-analogues of mathematical physical problems are studied in [11]. There are several branches of mathematics and engineering in which positive definite functions and some of its related functions play an important role[5,6,8,15,22-25], many areas of these branches and applications including completely monotone functions[11,13,15]. A function f is called completely alternating if for all n, $(-1)^n f^{(n)}(x) \leq 0$ on $(0,\infty)$. Bernstien's theorem asserts that f is completely alternating if and only if $f(x) = \int_{\mathbb{R}}(1-e^{-xt})d\mu(t)$ where μ is a positive measure supported on a subset of $[0,\infty)$. The main aim of this paper is to give integral representations of basic completely alternating functions. A function $f :]0,\infty[\to \mathbb{R}$ will be called basic completely alternating if it satisfies the following axioms:

1. (1) f has q-derivative of all orders.

2. (2) $(-1)^n D_q^{(n)} f(x) \leq 0$ for all $x > 0$, $n = 0, 1, 2, \cdots$.

Comparing the definitions of the class of basic completely monotone functions CM_q[11], and the class of basic completely alternating functions CA_q.

Remark. If the function ϕ belongs to the class CM_q, then $-\phi$ belongs to the class CA_q.

We will denote the class of all basic completely alternating functions on $]0,\infty[$ by CA_q. Many authors were gave a full investigation for the properties of the class of all completely monotone(alternating) functions see Widder[26], Ismail et al[15], Feller [8], Choquet[6] and Berg[5]. This paper is devoted to give integral representations for functions belongs to the class CA_q. Some properties of the class of basic completely alternating functions are showed. The relations between the classes of basic completely monotone and alternating functions are investigated. i.e, in this papers, I will go to answer the following questions:

1. (1) What is the relation between the class of positive definite functions and the class CM_q.

2. (2) What is the relation between the class of basic completely monotone functions and the class of completely alternating functions CA_q.

3. (3) What is the relation between the class of negative definite functions and the class CA_q.

4. (4) How are the integral representations of basic completely monotone functions will lead us to find the integral representations of basic completely alternating functions.

2. Basic Relations

In this Section 1 will concern my efforts to investigate the basic relations between the classes of basic completely monotone, alternating functions, positive and negative definite functions. For a semigroup S with involution $*$, a continuous bounded function $\phi : S \to \mathbb{R}$ is called positive definite, if

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda_j} \phi(s_i.s_j^*) \geq 0$$

for all $s_1, s_2, ..., s_n \in S, \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{C}$ and $n \in \mathbb{N}$. While, the function $\psi : S \to \mathbb{R}$ is called negative definite if

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda_j} \psi(s_i.s_j^*) \leq 0$$

for all $s_1, s_2, ..., s_n \in S, \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{C}$ and $n \in \mathbb{N}$, for $\sum_{i=1}^{n} \lambda_i$.

Theorem 2.1. *The sum, the product, and the pointwise limit of basic completely alternating functions are also basic completely alternating.*

Proof. Let $\{f_m\} \subseteq CA_q$, $f_m \to f$ as $m \to \infty$. Since the q-derivative and the limit commute, then we have

$$\begin{aligned} 0 &\geq \lim_{m\to\infty} (-1)^n D_q^{(n)} f_m(x) \\ &= (-1)^n D_q^{(n)} \lim_{m\to\infty} f_m(x) \\ &= (-1)^n D_q^{(n)} f(x), \end{aligned}$$

so, $f \in CA_q$. Let $f, g \in CA_q$, the linearity of D_q implies $f + g \in CA_q$. Also, since

$$D_q(f(x)g(x)) = f(qx)D_q g(x) + (D_q f(x))g(x),$$

and $f(x), g(x)$ are nonnegative for all $x > 0$, so at $n = 1$ we have

$$(-1)D_q(f(x)g(x)) \leq 0 \quad \text{for all} \quad x > 0.$$

Suppose

$$(-1)^{(n-1)}D_q^{(n-1)}(f(x)g(x)) \leq 0 \quad \text{for all} \quad x > 0,$$

this implies

$$(-1)^{(n)}D_q^{(n)}(f(x)g(x)) = (-1)^{(n-1)}D_q^{(n-1)}[(-1)D_q(f(x)g(x))] \leq 0 \qquad \text{for all} \quad x > 0,$$

By using mathematical induction we get $fg \in CA_q$.

According to my results obtained in the above Theorem, we can easily obtain the following Corollary:

Corollary 2.2. *The class of basic completely monotone (alternating) functions is a closed convex cone in* $\mathbb{R}^{(0,\infty)}$.

Proposition 2.3. *The function* $f : (0, \infty) \to \mathbb{R}$ *belongs to the class* CM_q *if and only if it is bounded positive definite on* $(0, \infty)$.

Proof. Firstly, we will prove that the function

$$f(s) = \int_0^\infty E_q^{-sx}\mu(x)d_qx; \qquad s \in (0, \infty) \tag{2.1}$$

is positive definite. In fact, for $s_1, s_2, ..., s_n \in (0, \infty)$ and $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$ we find

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda_j} f(s_i + s_j^*) = \sum_{i,j=1}^{n} \lambda_i \bar{\lambda_j} \int_0^\infty \rho_x(s_i + s_j^*)\mu(x)d_qx$$
$$= \int_0^\infty |\sum_{i=1}^{n} \lambda_i \rho_x(s_i)|^2 \mu(x)d_qx \geq 0.$$

So, every function of the form (2.1) is positive definite. Recalling, the integral representations of basic completely monotone functions [11], implies the "if part" of the Proposition. Conversely, suppose that f is bounded positive definite on $(0, \infty)$. According to [5, Theorem 2.3 page 93], the function f can be written in the form (2.1), this complete the proof of the Proposition.

3. The Integral Representation and Concluding Remarks

Jackson [16-18] introduced the q-integral defined by

$$\int_0^x f(t)d_qt := \sum_{n=0}^{\infty} f(xq^n)(xq^n - xq^{n+1}) \tag{3.1}$$

and defined an integral on $(0, \infty)$by

$$\int_0^\infty f(t)d_qt := (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n) \tag{3.2}$$

Notice that

$$\lim_{N\to\infty} \int_0^{q^{-N}} f(x)d_qx = \int_0^\infty f(x)dx$$

The idea here is that on $(1, \infty)$ the division points are at $q^{-1}, q^{-2}, q^{-3}, ...$ when $0 < q < 1$.

Theorem 3.1. Every basic completely alternating function $f \in CA_q$ has an integral representation of the form

$$f(s) = \int_0^\infty (1 - E_q^{-sx})\mu(x)d_qx \tag{3.3}$$

for some measure μ on the real line.

Proof. Since $E_q^{x+y} \neq E_q^x E_q^y$ in general [16], so we must use the definition of q-addition (compare [2])

$$(x \oplus_q y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}, \quad n = 0, 1, 2, \cdots, \ y \neq x$$

and so,

$$E_q^{x\oplus_q y} = E_q^x E_q^y.$$

Defining a semicharacter $\rho_a : (0, \infty) \to \mathbb{R}$ by $\rho_a(s) = E_q^{-as}$, it is clear that $0 \leq \rho_a(s) \leq 1$ i.e., $1 - \rho_a$ is decreasing then the integral

$$\int_0^\infty (1 - E_q^{-sx})w(x)d_qx$$

exists for some weight function $w(x)$ defined on $\mathbb{R}$. Collecting the above results, we get

$$f(s) = \int_0^\infty (1 - E_q^{-sx})w(x)d_qx$$

This completes the proof of the Theorem.

Theorem 3.2. *The function* $\psi : (0, \infty) \to \mathbb{R}$ *belongs to the class* CA_q*, if and only if* $\exp_q(-t\psi)$ *belongs to the class* CM_q *for all* $t > 0$.

Proof. If $\exp_q(-t\psi) \in CM_q$ for all $t > 0$ then $1 - \exp_q(-t\psi) \in CA_q$ is, of course, basic completely alternating and so is therefore the point wise limit

$$\psi = \lim_{t \to 0} \frac{1 - \exp_q(-t\psi)}{t} \in CA_q.$$

Since, the function ϕ belongs to the class CM_q implies the function $-\phi$ belongs to the class CA_q. So, according to Propsition 2.3 we have:

Proposition 3.3. The function $\psi : (0, \infty) \to \mathbb{R}$ *belongs to the class* CA_q *if and only if it is bounded negative definite on* $(0, \infty)$.

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