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LÉVY-KHINCHIN TYPE FORMULA FOR BASIC COMPLETELY MONOTONE FUNCTIONS

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Abstract: This paper is devoted to give the integral representations of basic completely monotone functions. The q-analogue of Laplace transform is introduced and some properties of the class of basic completely monotone functions are showed. Finally, many applied examples are given.

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1. Introduction

Recently, quantum calculus received a lot of attentions, and most of the published work has been concerned with some problems of q-difference equations [1, 2, 5, 7, 8, 11-16]. There are several branches of mathematics and engineering

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in which positive definite functions and some of its related functions play an important role [3, 4, 6, 12, 17-20], many areas of these branches and applications including completely monotone functions [8, 10, 12]. A function f is called completely monotone if for all n, $(-1)^n f^{(n)}(x) \geq 0$ on $(0, \infty)$. Bernstien's theorem asserts that f is completely monotone if and only if $f(x) = \int_{\mathbb{R}} e^{-xt} d\mu(t)$ where μ is a positive measure supported on a subset of $[0, \infty)$. The main aim of this paper is to give integral representations of basic completely monotone functions. A function $f :]0, \infty[\to \mathbb{R}$ will be called basic completely monotonic if it satisfies the following axioms:

(1) f has q-derivative of all orders.

(2) $(-1)^n D_q^{(n)} f(x) \geq 0$ for all $x > 0$, $n = 0, 1, 2, \cdots$.

Here the q-difference operator D_q is defined in [12] by

$$D_q f(x) = \frac{f(x) - f(qx)}{x - qx}, \quad q \neq 1 \tag{1.1}$$

We will denote the class of all basic completely monotone functions on $]0, \infty[$ by CM_q.

Example. The function E_q^{-x} belongs to the class CM_q, where

$$E_q^x = \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{x^j}{[j]!} \tag{1.2}$$

A central idea in this paper is to pass from series representation with positive coefficients as in Eqn. (1.2), to integral transform with non-negative densities. Let

$$F(x) = \int_c^d k(x,t) f(t) d_q t, \quad 0 \leq c < d \leq \infty. \tag{1.3}$$

Obviously, if $k(x,t)$ belongs to the class of basic completely monotone (CM_q) functions on the parameter x for all $t \in (0, \infty)$ and $f(t)$ is non-negative, then the q-derivative shows that $F(x)$ also belongs to CM_q. This paper is devoted to give integral representations for functions belongs to the class CM_q.

2. The Integral Representation

The formulas for the q-difference D_q of a product and a quotient of functions are

$$D_q(f(x)g(x)) = f(qx)D_qg(x) + g(x)D_qf(x) \tag{2.1}$$

$$D_q(\frac{f(x)}{g(x)}) = \frac{g(x)D_qf(x) - f(x)D_qg(x)}{g(qx)g(x)}, \quad g(qx)g(x) \neq 0 \tag{2.2}$$

also, the general Leibniz rule for action of powers of the q-derivative operator on a product of functions is

$$D_q^{(n)}(fg)(x) = \sum_{k=0}^{n} \binom{n}{k}_q D_q^{(k)} f(xq^{n-k}) D_q^{(n-k)} g(x) \tag{2.3}$$

Here we use the q-binomial coefficients defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \tag{2.4}$$

for $k = 0, 1, 2, ..., n$, where

$$[n]_q = \sum_{k=1}^{n} q^{k-1}, \quad [0]_q = 0,$$

and

$$[n]_q! = \prod_{k=1}^{n} [k]_q, \quad [0]_q! = 1.$$

are the q-analogue of the natural numbers and the factorial function. The q-binomial coefficient $\binom{n}{k}_q$ are polynomials in q with integer coefficients. If $q = 1$, then $[n]_q = n$. If $q \neq 1$, then $[n]_q = \frac{1-q^n}{1-q}$. For more properties of the difference operator D_q see [7,16]. Jackson [13-15] introduced the q-integral defined by

$$\int_0^x f(t)d_qt := \sum_{n=0}^{\infty} f(xq^n)(xq^n - xq^{n+1}) \tag{2.5}$$

and defined an integral on $(0, \infty)$by

$$\int_0^{\infty} f(t)d_qt := (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n) \tag{2.6}$$

Notice that

$$lim_{N\to\infty}\int_0^{q^{-N}} f(x)d_qx = \int_0^{\infty} f(x)dx$$

The idea here is that on $(1,\infty)$ the division points are at $q^{-1}, q^{-2}, q^{-3}, ...$ when $0 < q < 1$.

Theorem 2.1. *The sum, the product, and the pointwise limit of basic completely monotonic functions are also basic completely monotonic.*

Proof. Let $\{f_m\} \subseteq CM_q$, $f_m \to f$ as $m \to \infty$. Since the q-derivative and the limit commute, then we have

$$\begin{aligned} 0 &\leq \lim_{m\to\infty}(-1)^n D_q^{(n)} f_m(x) \\ &= (-1)^n D_q^{(n)} \lim_{m\to\infty} f_m(x) \\ &= (-1)^n D_q^{(n)} f(x), \end{aligned}$$

so, $f \in CM_q$. Let $f, g \in CM_q$, the linearity of D_q implies $f + g \in CM_q$. Also, since

$$D_q(f(x)g(x)) = f(qx)D_qg(x) + (D_qf(x))g(x),$$

and $f(x), g(x)$ are nonnegative for all $x > 0$, so at $n = 1$ we have

$$(-1)D_q(f(x)g(x)) \geq 0 \quad \text{for all} \quad x > 0.$$

Suppose

$$(-1)^{(n-1)} D_q^{(n-1)}(f(x)g(x)) \geq 0 \quad \text{for all} \quad x > 0,$$

this implies

$$(-1)^{(n)} D_q^{(n)}(f(x)g(x)) = (-1)^{(n-1)} D_q^{(n-1)}[(-1)D_q(f(x)g(x))] \geq 0 \quad \text{for all} \quad x > 0,$$

By using mathematical induction we get $fg \in CM_q$.

Theorem 2.2. *Every basic completely monotone function $f \in CM_q$ has an integral representation of the form*

$$f(s) = \int_0^{\infty} E_q^{-sx}\mu(x)d_qx := \mathcal{L}_q(\mu) \tag{2.7}$$

for some measure μ on the real line.

Proof. Since $E_q^{x+y} \neq E_q^x E_q^y$ in general [16], so we must use the definition of q-addition (compare [2])

$$(x \oplus_q y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}, \quad n = 0, 1, 2, \cdots, y \neq x$$

and so,

$$E_q^{x\oplus_q y} = E_q^x E_q^y.$$

Defining a semicharacter $\rho_a :]0,\infty[\to \mathbb{R}$ by $\rho_a(s) = E_q^{-as}$, it is clear that $0 \le \rho_a(s) \le 1$ i.e., ρ_a is decreasing then the integral

$$\int_0^\infty E_q^{-sx} w(x) d_q x$$

exists for some weight function $w(x)$ defined on $\mathbb{R}$. Applying corollary 4.5. Page 114 [3], we get

$$f(s) = \int_0^\infty E_q^{-sx} w(x) d_q x$$

which completes the proof of the Theorem.

Remark. The operator $L_q(.)$ which introduced in Eqn.(2.7) will be called q-Laplace transform and can be considered as main player for solving q-difference equations[8].

Theorem 2.3. *The set* S_q, *of all functions* $f :]0,\infty[\to \mathbb{R}$ *which can be expressed in the form*

$$f(y) = a + \int_0^\infty \frac{1}{y \oplus_q x}\mu(x) d_q x$$

for a unique constant $a \ge 0$, *is a convex cone contained in* CM_q.

Proof. Since,

$$\begin{aligned}\int_0^\infty \frac{1}{y \oplus_q x} w(x) d_q x &= \int_0^\infty \int_1^\infty E_q^{-(y\oplus_q x)\xi} d_q\xi w(x) d_q x \\ &= \int_1^\infty \int_0^\infty E_q^{-y\xi} E_q^{-x\xi} w(x) d_q x d_q \xi \\ &= \int_1^\infty E_q^{-y\xi} \mathcal{L}_q w(\xi) d_q \xi\end{aligned}$$

For $n \ge 1$ and $y > 0$ we find,

$$\frac{(-1)^n}{[n]_q!} D_q^{(n)} f(y) = \int_0^\infty \frac{1}{(y \oplus_q x)^{(n+1)}} w(x) d_q x$$

by applying Cauchy-Schwartz inequality we get that the sequence

$$\Phi_n(y) = \frac{(-1)^n}{[n]_q!} D_q^{(n)} f(y) \tag{2.8}$$

satisfying the relation

$$(-1)^{(k)} D_q^{(k)}[\log \Phi(y)] \geq 0, \quad \text{for} \quad k = 1, 2, \cdots$$

and noting that Φ satisfies (3.5) if and only if $-D_q(\log \Phi)$ belongs to CM_q, we get the desired.

3. Totally Basic Completely Monotone TCM_q

Definition 3.1. A linear homomorphic function f on $\mathbb{R}^d$ will be called totally basic completely monotone if it satisfies: (1)

$$\partial_{x_i}^{(m,q)} f(x) = \partial_{x_i}^{(q)} \partial_{x_i}^{(m-1,q)} f(x) \quad \text{exist for all} \quad m \in \mathbb{N}_0.$$

(2)

$$(-1)^m \partial_{x_i}^{(m,q)} f(x) \geq 0 \quad \text{for all} \quad m \in \mathbb{N}_0,$$

where the q-difference operators $\partial_{x_i}^{(q)}, \quad i = 1, 2, ..., d$ are given by

$$\partial_{x_i}^{(q)} f(x) = \partial_{x_i}^{(q)} f(x_1, \cdots, x_d) \equiv \frac{f(x_1, \cdots, x_{i-1}, qx_i, x_{i+1}, \cdots, x_d) - f(x)}{(q-1)x}$$

and

$$\partial_{x_i}^{(1,q)} = \partial_{x_i}^{(q)} \quad \textit{and} \quad \partial_{x_i}^{(0,q)} = 1.$$

Remark. It is clear that if $d = 1$, then

$$D_q f(x) = \partial_x^{(q)} f(x)$$

Lemma 3.2. *Let f, g be two linear homogenous functions on $\mathbb{R}^d$ then we have*

$$\partial_{x_i}^{(m,q)}(f(x)g(x)) = f(\widetilde{x_i})\partial_{x_i}^{(m,q)} g(x) + g(x)\partial_{x_i}^{(m,q)} f(x)$$

where $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ and $m = (m_1, \cdots, m_d) \in \mathbb{N}_0^d$ and

$$\widetilde{x_i} = (x_1, \cdots, x_{i-1}, q^{n_i} x_i^{n_i}, x_{i+1}, \cdots, x_d)$$

We leave the proof for the readers.

Theorem 3.3. *The sum, the product, and the point wise limit of the elements of the set TM_q of all totally basic completely monotone functions are* *also totally basic completely monotone functions and every function $f \in TM_q$ has an integral representation of the form*

$$f(s) = \underbrace{\int_0^\infty ... \int_0^\infty}_{m-times} E_q^{-s.x} \mu(x) d_q x \tag{3.1}$$

for some Borel measure μ on $\mathbb{R}^m$ where $s = (s_1, \cdots, s_m) \in \mathbb{R}^m$, $x = (x_1, \cdots, x_m)$ and $s.x = s_1 x_1 \oplus_q \cdots \oplus_q s_m x_m$.

Proof. It is clear that the operators $\partial_{x_i}^{(q)}$ for all $i \in \mathbb{N}_0$ is linear, so $f + g \in TM_q$ for all $f, g \in TM_q$ Since the operator $\partial_{x_i}^{(q)}$ and the limit commute then we directly have that TM_q is a closed set over the Euclidean space $\mathbb{R}^m$. Let $f, g \in TM_q$, by virtue of the last Lemma and from the non negativity of $f(x), g(x)$; $x \in \mathbb{R}_+^m$ we get:

$$(-1)\partial_{x_i}^{(q)}(f(x)g(x)) \geq 0 \quad \forall\, x \in \mathbb{R}_+^m$$

Letting

$$(-1)^{n-1}\partial_{x_i}^{(n-1,q)}(f(x)g(x)) \geq 0 \quad \forall\, x \in \mathbb{R}_+^m$$

then

$$(-1)^n \partial_{x_i}^{(n,q)}(f(x)g(x)) = (-1)^{n-1}\partial_{x_i}^{(n-1,q)}[(-1)\partial_{x_i}^{(q)} f(x)g(x)] \geq 0$$

applying mathematical induction we get $fg \in CM_q$. Combining Theorem 3.2 with proposition 4.7 page 115 [3] we get

$$f(s) = \underbrace{\int_0^\infty ... \int_0^\infty}_{m-times} E_q^{-s_1.x_1} E_q^{-s_2.x_2} ... E_q^{-s_m.x_m} \mu_1(x_1) d_q x_1 \mu_2(x_2) d_q x_2 ... \mu_m(x_m) d_q x_m$$

for some measures $\mu_1, \mu_2, ..., \mu_m$ on $\mathbb{R}$. Putting $\mu(x) = \mu_1(x_1)\mu_2(x_2)...\mu_m(x_m)$ implies (2.2) which completes the proof of the theorem.

4. Examples and Discussion

As pointed in [9], for some special cases the hypergeometric functions satisfy the axioms of completely monotone functions. Similarly, we can easily finding that the four Jacobi functions:

$$\theta_1(z, q) = 2\sum_{n=0}^{\infty} (-1)^n q^{(n+0.5)^2} sin(2n+1)z, \tag{4.1}$$

$$\theta_2(z,q) = 2\sum_{n=0}^{\infty} q^{(n+0.5)^2} cos(2n+1)z, \tag{4.2}$$

$$\theta_3(z,q) = 1 + 2\sum_{n=0}^{\infty} q^{n^2} cos2nz, \tag{4.3}$$

$$\theta_4(z,q) = 1 + 2\sum_{n=0}^{\infty} (-1)^n q^{n^2} cos2nz, \tag{4.4}$$

belongs to the class CM_q. Moreover, the following functions have many applications in physics, applied mathematics and electricity[5], and all of them satisfy the basic completely monotone axioms:

(1) The q-exponential function e_q^{-x} where

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}. \tag{4.5}$$

(2) The q-logarithmic function $log_q[a + b/x]$ where $a > 1, b > 0$ and

$$log_q(x) = \sum_{n=1}^{\infty} \frac{x^n}{1-q^n}, \quad |x| < 1. \tag{4.6}$$

(3) The q-confluent hypergeometric function ${}_1\phi_1(a;b;q;-ax)$, where

$${}_1\phi_1(a;b;q;z0) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n (q;q)_n} z^n. \tag{4.7}$$

Many authors were gave a full investigation for the properties of the class of all completely monotone functions see Widder[21], Ismail et al[12], Feller [6], Choquet[4] and Berg[3]. This paper is devoted to give the integral representations of basic completely monotone functions. This representations will guide us, in the following papers, to answer the following questions:

(1) What is the relation between the class of positive definite functions and the class CM_q.

(2) What is the relation between the class of the class completely monotone functions and the class of completely alternating functions CA_q.

(3) What is the relation between the class of negative definite functions and the class CA_q.

(4) How are the integral representations of basic completely monotone functions implies to find the integral representations of basic completely alternating functions.

References

[1] W.A. Al-Salam, q-Bernoulli numbers and polynomials, *Math. Nachr.*, **17** (1959), 239-260.

[2] G.E. Andrews, R. Askey, R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.

[3] C. Berg, J.P.R. Christensen, P. Ressel, *Harmonic Analysis on Semigroups*, New York, Springer-Verlag, 1984.

[4] G. Chouquet, Theory of capacities, *Ann. Inst. Fourier*, Grenable, **5** (1954), 131-295.

[5] H. Exton, *q-Hypergeometric Functions and Applications*, Ellis Horwood Limited,Chichester, 1983.

[6] W. Feller, *An Introduction to Probability Theory and Its Application*, Academic Press, New York, 1966.

[7] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.

[8] Hossam A. Ghany, Basic completely monotone functions as coefficients and solutions of linear q-difference equations with some application, *Physics Essays*, **25** (2012).

[9] Hossam A. Ghany, Computations of some integrals involving logarithmic function with some thermodynamic application, *Physics Essays*, **24** (2011), 456.

[10] Hossam A. Ghany, Completely monotonic and completely alternating functions on hypergroup, In: *5-th Asian mathematical Conference*, Kulalumpor-Malysia, 22-26 Jun 2009. Pure Mathematics. Penang: Universiti Sains Malaysia, School of Mathematical Sciences. 1(2009) 121-128.

[11] Hossam A. Ghany, q-Derivative of Basic Hypergeometric Series with respect to parameters, *International Journal of Mathematical Analysis*, 2009.

[12] M.E.H. Ismail, L. Lorch, Muldoon, Completely monotonic functions associated with the gamma functions and its q-analogue, *J. Math Anal. Appl.*, **116** (1986), 1-9.

[13] F.H. Jackson, On q-functions and a certain difference operator, *Trans. Roy. Soc. Edin.*, **46** (1908), 253-281,

[14] F.H. Jackson, On q-definite integrals, *Quart. J. Pure Appl. Math.*, **41** (1910), 193-203.

[15] F.H. Jackson, On basic double hypergeometric functions, *Quart. J. Math.*, **13** (1942), 69-82

[16] V. Kac, P. Cheung, *Quantum Calculus*, New York, Springer, 2002.

[17] A.S. Okb El Bab, Hossam A. Ghany, Harmonic analysis on hypergroups, *International Journal of Pure and Applied Mathematics* (2010).

[18] A.S. Okb El Bab, Hossam A. Ghany, A. Hady, Remarks on some quasi classes of the moment problems, *International Journal of Pure and Applied Mathematics* (2012).

[19] A.S. Okb El Bab, Hossam A. Ghany, S. Ramadan, On strongly negative definite functions for the product of commutative hypergroups, *International Journal of Pure and Applied Mathematics* (2011).

[20] A.S. Okb El Bab, A.M. Zabel, Hossam A. Ghany, Harmonic analysis in hypercomplex system, *International Journal of Pure and Applied Mathematics* (2012).

[21] D.V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1941.