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FINSLERIAN HYPERSURFACES AND RANDERS CONFORMAL CHANGE OF A FINSLER METRIC

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Abstract: In the present paper we have studied the Finslerian hypersurfaces and Randers conformal change of a Finsler metric. The relations between the Finslerian hypersurface and the other which is Finslerian hypersurface given by Randers conformal change have been obtained. We have also proved that Randers conformal change makes three types of hypersurfaces invariant under certain condition. These three types of hypersurfaces are hyperplanes of first, second and third kind.

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1. Introduction

Let (M^n, L) be an n-dimensional Finsler space on a differentiable manifold M^n, equipped with the fundamental function $L(x, y)$. In 1984, Shibata [3] introduced

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the transformation of Finsler metric:

$$L^{'}(x,y) = f(L,\beta) \tag{1}$$

where $\beta = b_i(x)y^i$, $b_i(x)$ are components of a covariant vector in (M^n, L) and f is positively homogeneous function of degree one in L and β. This change of metric is called a β-change.

The conformal theory of Finsler spaces has been initiated by M.S. Knebelman [4] in 1929 and has been investigated in detail by many authors [5, 6, 7, 8] etc. The conformal change is defined as

$$L(x,y) \to e^{\sigma(x)} L(x,y),$$

where $\sigma(x)$ is a function of position only and known as conformal factor.

In the year 2012 [10] we studied Randers conformal change defining as

$$L(x,y) \to L^*(x,y) = e^{\sigma(x)} L(x,y) + \beta(x,y), \tag{2}$$

where $\sigma(x)$ is a function of x and $\beta(x,y) = b_i(x)y^i$ is a 1- form on M^n. This change generalizes various types of changes. When $\beta = 0$, it reduces to a conformal change. When $\sigma = 0$, it reduces to a Randers change. When $\beta = 0$ and σ is a non-zero constant then it reduces to homothetic change.

On the other hand, in 1985 M. Matsumoto investigated the theory of Finslerian hypersurface [2]. He has defined three types of hypersurfaces that were called a hyperplane of the first, second and third kinds.

In the year 2005, Prasad and Tripathi [12] studied the Finslerian Hypersurfaces and Kropina change of a Finsler metric and obtained different results in his paper. In the present paper, using the field of linear frame [1, 9, 11] we shall consider Finslerian hypersurfaces given by a Randers conformal change of a Finsler metric. Our purpose is to give some relations between the original Finslerian hypersurface and the other which is Finselrian hypersurface given by Randers conformal change. We also obtain that a Randers conformal change makes three types of hypersurfaces invariant under certain condition.

2. Finslerian Hypersurfaces

Let M^n be an n-dimensional smooth manifold and $F^n = (M^n, L)$ be an n-dimensional Finsler space equipped with the fundamental function $L(x,y)$ on M^n. The metric tensor $g_{ij}(x,y)$ and Cartan's C-tensor $C_{ijk}(x,y)$ are given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \qquad C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$$

respectively and we introduce the Cartan's connection $C\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ in F^n.

A hypersurface M^{n-1} of the underlying smooth manifold M^n may be parametrically represented by the equation $x^i = x^i(u^\alpha)$, where u^α are Gaussian coordinates on M^{n-1} and Greek indices vary from 1 to n-1. Here we shall assume that the matrix consisting of the projection factors $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ is of rank n-1. The following notations are also employed:

$$B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \qquad B^i_{0\beta} = v^\alpha B^i_{\alpha\beta}$$

If the supporting element y^i at a point (u^α) of M^{n-1} is assumed to be tangential to M^{n-1}, we may then write $y^i = B^i_\alpha(u)v^\alpha$ i.e. v^α is thought of as the supporting element of M^{n-1} at the point (u^α). Since the function $\bar{L}(u,v) = L\{x(u), y(u,v)\}$ gives rise to a Finsler metric of M^{n-1}, we get a $(n-1)$-dimensional Finsler space $F^{n-1} = \{M^{n-1}, \bar{L}(u,v)\}$.

At each point (u^α) of F^{n-1}, the unit normal vector $N^i(u,v)$ is defined by

$$g_{ij}B^i_\alpha N^j = 0, \qquad g_{ij}N^i N^j = 1 \tag{3}$$

If B^α_i, N_i is the inverse matrix of (B^i_α, N^i), we have

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \qquad B^i_\alpha N_i = 0, \qquad N^i N_i = 1 \qquad \text{and} \qquad B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j.$$

Making use of the inverse matrix $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get

$$B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta, \qquad N_i = g_{ij}N^j \tag{4}$$

For the induced Cartan's connection $IC\Gamma = (F^\alpha_{\beta\gamma}, N^\beta_\alpha, C^\alpha_{\beta\gamma})$ on F^{n-1}, the second fundamental h-tensor $H_{\alpha\beta}$ and the normal curvature vector H_α are respectively given by [2]

$$H_{\alpha\beta} = N_i(B^i_{\alpha\beta} + F^i_{jk}B^j_\alpha B^k_\beta) + M_\alpha H_\beta, \qquad H_\alpha = N_i(B^i_{0\beta} + N^i_j B^j_\beta) \tag{5}$$

where

$$M_\alpha = C_{ijk}B^i_\alpha N^j N^k \qquad .$$

Contracting $H_{\alpha\beta}$ by v^α, we immediately get $H_{0\beta} = H_{\alpha\beta}v^\alpha = H_\beta$. Furthermore the second fundamental v-tensor $M_{\alpha\beta}$ is given by [2]

$$M_{\alpha\beta} = C_{ijk}B^i_\alpha B^j_\beta N^k \tag{6}$$

3. Finsler Space with Randers Conformal Change

Let (M^n, L) be a Finsler space F^n, where M^n is an n-dimensional differentiable manifold equipped with a fundamental function L. A change in fundamental metric L, defined by equation (2), is called Randers conformal change, where $\sigma(x)$ is conformal factor and function of position only and $\beta(x,y) = b_i(x)y^i$ is a 1- form on M^n. A space equipped with fundamental metric $L^*(x,y)$ is called Randers conformally changed space F^{*n}.

Differentiating equation (2) with respect to y^i, the normalized supporting element $l_i^* = \dot{\partial}_i L^*$ is given by

$$l_i^*(x,y) = e^{\sigma(x)} l_i(x,y) + b_i(x), \tag{7}$$

where $l_i = \dot{\partial}_i L$ is the normalized supporting element in the Finsler space F^n.

Differentiating (7) with respect to y^j, the angular metric tensor $h_{ij}^* = L^*\dot{\partial}_i\dot{\partial}_j L^*$ is given by

$$h_{ij}^* = e^{\sigma(x)} \frac{L^*}{L} h_{ij} \tag{8}$$

where $h_{ij} = L\dot{\partial}_i\dot{\partial}_j L$ is the angular metric tensor in the Finsler space F^n.

Again the fundamental tensor $g_{ij}^* = \dot{\partial}_i\dot{\partial}_j \frac{L^{*2}}{2} = h_{ij}^* + l_i^* l_j^*$ is given by

$$g_{ij}^* = \tau g_{ij} + b_i b_j + e^{\sigma(x)} L^{-1}(b_i y_j + b_j y_i) - \beta e^{\sigma(x)} L^{-3} y_i y_j \tag{9}$$

where we put $y_i = g_{ij}(x,y)y^j$, $\tau = e^{\sigma(x)} \frac{L^*}{L}$ and g_{ij} is the fundamental tensor of the Finsler space F^n. It is easy to see that the $\det(g_{ij}^*)$ does not vanish, and the reciprocal tensor with components g^{*ij} is given by

$$g^{*ij} = \tau^{-1} g^{ij} + \phi y^i y^j - L^{-1}\tau^{-2}(y^i b^j + y^j b^i) \tag{10}$$

where $\phi = e^{-2\sigma(x)}(Le^{\sigma(x)}b^2 + \beta)L^{*-3}$, $b^2 = b_i b^i$, $b^i = g^{ij} b_j$ and g^{ij} is the reciprocal tensor of g_{ij}.

Here it will be more convenient to use the tensors

$$h_{ij} = g_{ij} - L^{-2} y_i y_j, \qquad a_i = \beta L^{-2} y_i - b_i \tag{11}$$

both of which have the following interesting property:

$$h_{ij} y^j = 0, \qquad a_i y^i = 0 \tag{12}$$

Now differentiating equation (9) with respect to y^k and using relation (11), the Cartan covariant tensor C^* with the components $C_{ijk}^* = \dot{\partial}_k(\frac{g_{ij}^*}{2})$ is given as:

$$C_{ijk}^* = \tau[C_{ijk} - \frac{1}{2L^*}(h_{ij} a_k + h_{jk} a_i + h_{ki} a_j)] \tag{13}$$

where C_{ijk} is (h)hv-torsion tensor of Cartan's connection $C\Gamma$ of Finsler space F^n.

In order to obtain the tensor with the components C^{*j}_{ik}, paying attention to (12), we obtain from (10) and (13),

$$C^{*j}_{ik} = C^j_{ik} - \frac{1}{2L^*}(h^j_i a_k + h^j_k a_i + h_{ik}a^j) - (\tau L)^{-1} C_{ikr} y^j b^r - \frac{\tau^{-1}}{2LL^*}(2a_i a_k + a^2 h_{ik}) y^j \tag{14}$$

where $a_i a^i = a^2$.

4. Hypersurfaces Given by a Randers Conformal Change

Consider a Finslerian hypersurface $F^{n-1} = \{M^{n-1}, \bar{L}(u,v)\}$ of the F^n and another Finslerian hypersurface $F^{*n-1} = \{M^{n-1}, \bar{L}^*(u,v)\}$ of the F^{*n} given by the Randers conformal change. Let N^i be the unit vector at each point of F^{n-1} and (B^α_i, N_i) be the inverse matrix of (B^i_α, N^i). The function B^i_α may be considered as components of $(n-1)$ linearly independent tangent vectors of F^{n-1} and they are invariant under Randers conformal change. Thus we shall show that a unit normal vector $N^{*i}(u,v)$ of F^{*n-1} is uniquely determined by

$$g^*_{ij} B^i_\alpha N^{*j} = 0, \qquad g^*_{ij} N^{*i} N^{*j} = 1 \tag{15}$$

Contracting (9) by $N^i N^j$ and paying attention to (3) and the fact that $l_i N^i = 0$, we have

$$g^*_{ij} N^i N^j = \tau + (b_i N^i)^2 \tag{16}$$

Therefore we obtain

$$g^*_{ij}\{\pm\frac{N^i}{\sqrt{\tau+(b_iN^i)^2}}\}\{\pm\frac{N^j}{\sqrt{\tau+(b_iN^i)^2}}\} = 1$$

Hence we can put

$$N^{*i} = \frac{N^i}{\sqrt{\tau + (b_i N^i)^2}} \tag{17}$$

where we have chosen the positive sign in order to fix an orientation.

Using equation (9), (17) and from first condition of (15) we have

$$(b_i B^i_\alpha + e^{\sigma(x)} l_i B^i_\alpha)\frac{b_j N^j}{\sqrt{\tau + (b_i N^i)^2}} = 0 \tag{18}$$

If $b_i B^i_\alpha + e^{\sigma(x)} l_i B^i_\alpha = 0$, then contracting it by v^α and using $y^i = B^i_\alpha v^\alpha$ we get $\beta + e^{\sigma(x)} L = L^* = 0$ which is contradiction to the assumption that $L^* > 0$. Hence $b_i N^i = 0$. Therefore equation (17) can be written as

$$N^{*i} = \frac{1}{\sqrt{\tau}} N^i \tag{19}$$

Summarizing the above we have

Proposition 4.1. *If* $\{(B^i_\alpha, N^i),\ \alpha = 1, 2, ...(n-1)\}$ *be the filed of linear frame of the Finsler space* F^n, *there exist a field of linear frame* $\{(B^i_\alpha, N^{*i} = \frac{1}{\sqrt{\tau}} N^i),\ \alpha = 1, 2, ...(n-1)\}$ *of the Finsler space* F^{*n} *such that (15) is satisfied along* F^{*n-1} *and then* b_i *is tangential to both the hypersurfaces* F^{n-1} *and* F^{*n-1}.

The quantities $B^{*\alpha}_i$ are uniquely defined along F^{*n-1} by

$$B^{*\alpha}_i = g^{*\alpha\beta} g^*_{ij} B^j_\beta$$

where $g^{*\alpha\beta}$ is the inverse matrix of $g^*_{\alpha\beta}$. Let $(B^{*\alpha}_i, N^*_i)$ be the inverse matrix of (B^i_α, N^{*i}), then we have

$$B^i_\alpha B^{*\beta}_i = \delta^\beta_\alpha, \qquad B^i_\alpha N^*_i = 0, \qquad N^{*i} N^*_i = 1$$

Furthermore $B^i_\alpha B^{*\alpha}_j + N^{*i} N^*_j = \delta^i_j$. We also get $N^*_i = g^*_{ij} N^{*j}$ which in view of (7), (9) and (19) gives

$$N^*_i = \sqrt{\tau} N_i \tag{20}$$

We denote the Cartan's connection of F^n and F^{*n} by $(F^i_{jk}, N^i_j, C^i_{jk})$ and $(F^{*i}_{jk}, N^{*i}_j, C^{*i}_{jk})$ respectively and put $D^i_{jk} = F^{*i}_{jk} - F^i_{jk}$ which will be called difference tensor. We choose the vector field b_i in F^n such that

$$D^i_{jk} = A_{jk} b^i - B_{jk} l^i \tag{21}$$

where A_{jk} and B_{jk} are components of symmetric covariant tensors of second order. Since $b_i N^i = 0$ and $N_i l^i = 0$, from (21) we get $N_i D^i_{jk} = 0$ and $N_i D^i_{0k} = 0$. Therefore from (5) and (20) we get

$$H^*_\alpha = \sqrt{\tau} H_\alpha \tag{22}$$

If each path of a hypersurface F^{n-1} with respect to the induced connection is also a path of the enveloping space F^n then F^{n-1} is called a hyperplane of the first kind. A hyperplane of the first kind is characterized by $H_\alpha = 0$ [2]. Hence from (22) we have

Theorem 4.1. *If $b_i(x)$ be a vector field in a Finsler space F^n satisfying (21) then a hypersurface F^{n-1} is a hyperplane of the first kind if and only if the hypersurface F^{*n-1} is a hyperplane of the first kind.*

Next contracting (13) by $B^i_\alpha N^{*j} N^{*k}$ and paying attention to (19), $M_\alpha = C_{ijk} B^i_\alpha N^j N^k$, $a_i N^i = 0$, $h_{jk} N^j N^k = 1$ and $h_{ij} B^i_\alpha N^j = 0$, we get

$$M^*_\alpha = M_\alpha - \frac{1}{2(e^\sigma L + \beta)} a_i B^i_\alpha \tag{23}$$

From (5), (20), (21), (22) and (23) we have

$$H^*_{\alpha\beta} = \sqrt{\tau}\{H_{\alpha\beta} - \frac{1}{2(e^\sigma L + \beta)} a_i B^i_\alpha H_\beta\} \tag{24}$$

If each h-path of a hypersurface F^{n-1} with respect to the induced connection is also h-path of the enveloping space F^n, then F^{n-1} is called a hyperplane of the second kind. A hyperplane of the second kind is characterized by $H_{\alpha\beta} = 0$ [2]. Since $H_{\alpha\beta} = 0$ implies that $H_\alpha = 0$ from (22) and (24) we have the following:

Theorem 4.2. *If $b_i(x)$ be a vector field in a Finsler space F^n satisfying (21) then a hypersurface F^{n-1} is a hyperplane of the second kind if and only if the hypersurface F^{*n-1} is a hyperplane of the second kind.*

Finally contracting (13) by $B^i_\alpha B^j_\beta N^{*k}$ and paying attention to (6), (19), $a_i N^i = 0$ and $h_{ij} B^i_\alpha N^j = 0$, we have

$$M^*_{\alpha\beta} = \sqrt{\tau} M_{\alpha\beta} \tag{25}$$

If the unit normal vector of F^{n-1} is parallel along each curve of F^{n-1} then F^{n-1} is called a hyperplane of the third kind. A hyperplane of the third kind is characterized by $H_{\alpha\beta} = 0$, $M_{\alpha\beta} = 0$ [2]. From (24) and (25) we have

Theorem 4.3. *If $b_i(x)$ be a vector field in a Finsler space F^n satisfying (21) then a hypersurface F^{n-1} is a hyperplane of the third kind if and only if the hypersurface F^{*n-1} is a hyperplane of the third kind.*

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