International Journal of Pure and Applied Mathematics
Volume 87 No. 5 2013, 719-728
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v87i5.5

ijpam.eu

DETOUR DISTANCE PATTERN OF A GRAPH

Kumar Abhishek[1 §], Ashwin Ganesan[2]
[1,2]Amrita School of Engineering
Amrita Vishwa Vidyapeetham
Coimbatore, 641 112, Tamil Nadu, INDIA

Abstract: For a simple connected graph $G = (V, E)$, let $M \subseteq V$ and $u \in V$. The M-*detour distance pattern* of G is the set $f_M(u) = \{D(u, v) : v \in M\}$. If f_M is injective function, then the set M is a *detour distance pattern distinguishing set* (or, *ddpd*- set in short) of G. A graph G is defined as *detour distance pattern distinguishing* (or, *ddpd*-) graph if it admits a *ddpd*-set. The objective of this article is to initiate the study of graphs that admit *marker set* M for which f_M is injective. This article establishes some general results on *ddpd*-graphs.

AMS Subject Classification: 05C22
Key Words: detour distance, detour distance pattern, detour distance pattern distinguishing set

1. Introduction

All the graphs considered here are finite, undirected, connected and simple. For standard graph theory terminology and notations not defined here, we refer [1].

Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. For arbitrary pair of vertices $u, v \in V$, the standard distance $d(u, v)$ is the length of the shortest $u - v$ path and the detour distance $D(u, v)$ is the length of the longest $u - v$ path between u and v in G. As with standard distance, detour distance is known to be a metric on the vertex set of any connected graph. Chartrand *et.al,* [2], were the first to introduce the notion of detour distances

Received: April 9, 2013

§Correspondence author

in graphs and was subsequently explored by many scholars. A variety of results pertaining to the detour distances in graphs are known to us. Before we proceed any further we recall the following results which we require subsequently.

Theorem 1. [2]

1. *For any graph G, $D(u, v) = 1$ if and only if uv is a bridge in G.*

2. *For any graph G, $d(u, v) = D(u, v)$ for every pair of vertices u and v of G if and only if G is a tree.*

The *detour eccentricity* $e_D(G)$ of a vertex v is the detour distance from v to a vertex farthest from v. The *detour radius* of $rad_D(G)$ of a connected graph G is the minimum detour eccentricity among the vertices of G and the *detour diameter* $diam_D(G)$ is the maximum detour eccentricity among the vertices of G. The *detour center* $C_D(G)$ of a graph G is the subgraph of G induced by those vertices of G whose detour eccentricity is $rad)d(G)$.

For a simple connected graph $G = (V, E)$, let $M \subseteq V(G)$ and $u \in V$. The *M-detour distance pattern* of G is the set $f_M(u) = \{D(u, v) : v \in M\}$. If f_M is injective function, then the set M is a *detour distance pattern distinguishing set* (or, *ddpd*- set in short) of G. A graph G is defined as *detour distance pattern distinguishing* (or, *ddpd*-) graph if it admits a *ddpd*-set.

It must be noted that for any graph G, with $|V| \geq 2$, V is not a *ddpd*-set, since for any two vertices $u, v \in V$ with $D(u, v) = diam_D(G) = D$ we have $f_M(u) = f_M(v) = \{0, 1, 2, \ldots, D\}$.

Further, for a given simple connected graph $G = (V, E)$, let $M \subseteq V(G)$ and $u \in V(G)$. The *M-distance pattern* of G is the set $f_M(u) = \{d(u, v) : v \in M\}$. If f_M is injective function, then the set M is a *distance pattern distinguishing set* (or, *dpd*- set in short) of G. A graph G is defined as *distance pattern distinguishing* (or, *dpd*-) graph if it admits a *dpd*-set. The notion of *dpd*-graph was introduced by Germina [3] which was subsequently investigated by many researchers since its inception in 2010.

In this article, we initiate the study of *ddpd*-graph, by changing the standard distance $d(u, v)$ between two vertices u and v in a graph by its detour distance $D(u, v)$.

2. A Few Facts Involving *dpd*-Graphs

It was noted [3] that not every graph is a *dpd*-graph. For example, the complete graph K_n, $n \geq 3$ is does not possess a *dpd*-set, hence is not a *dpd*-graph.

The following results have been established in the recent past in relation to *dpd*-graphs.

Theorem 2. [3] *Let G be any connected graph then, G has a dpd-set of cardinality 1 if and only if it is a path.*

Theorem 3. [4] *Let G be any connected graph. There exists no dpd-set of cardinality 2.*

Theorem 4. [3] *If M be a dpd-set of G then, the induced subgraph $\langle M \rangle$ is disconnected.*

Theorem 5. [3] *If M be a dpd-set of G then, any vertex of G is adjacent to at most two pendant vertices. Further, if G has a vertex with exactly two pendant vertices adjacent to it then, exactly one of them belongs to M.*

The problem of characterizing *dpd*-graph remains to be an open problem since its inception. The following classes of graphs are known to be *dpd*-graph.

Theorem 6. [4] *A cycle C_n of order n is a dpd-graph if and only if $n \geq 7$.*

Theorem 7. [4] *$K_{1,n}$ is a dpd-graph if and only if $n \leq 2$.*

Theorem 8. [5] *A double star $S_{m,n}$ is a dpd-graph if and only if $n, m \leq 2$.*

Theorem 9. [5] *A uniform binary tree is T is a dpd-graph if and only if $O(T) = 2^m - 1$ where $m = 1, 2, 3$.*

Theorem 10. [5] *k-uniform caterpillars are dpd-graphs if and only if $k = 1$.*

Theorem 11. [5] *Every olive tree is a dpd-graph.*

3. *ddpd*-Graphs Versus *dpd*-Graphs

The objective of this section is to do a comparative study of *ddpd*- and *dpd*-graphs. We begin this section with the following result which we state without a proof.

Proposition 12. *If u and v are any two distinct vertices of a cycle C_n of order n then, $D(u, v) + d(u, v) = n$.*

In view of Proposition 12 and Theorem 6 the notion of *ddpd*- and *dpd*-graphs are equivalent for the case of cycles. Further in view of Theorem 1 the notion of *ddpd*- and *dpd*-graphs are equivalent for the case of trees. Also, the notion of *ddpd*- and *dpd*-graphs are equivalent for the case of complete graphs. With the above few instances of equivalence in the notion of *ddpd*- and *dpd*-graphs the reader must not conclude that the two notions are equivalent in general. As the following example illustrates. Let G be the graph as shown in the figure 1.

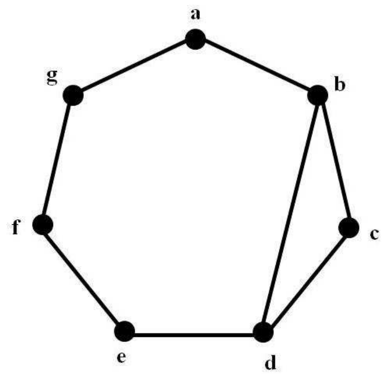

Figure 1: G

Consider the subset $M = \{a, b, d\} \subset V$. The *M-detour distance pattern* of the vertices of G is the following:

$$\begin{aligned} f_M(a) &= \{0,4,6\} \\ f_M(b) &= \{0,5,6\} \\ f_M(c) &= \{5,6\} \\ f_M(d) &= \{0,4,5\} \\ f_M(e) &= \{4,6\} \\ f_M(f) &= \{4,5\} \\ f_M(g) &= \{4,5,6\}. \end{aligned}$$

Clearly f_M is an injective function. Hence M is a *ddpd*- set of G. Note that the induced subgraph $\langle M \rangle$ is connected. Thus the induced subgraph of a *ddpd*-set need not be disconnected, whereas in view of Theorem 3 the induced subgraph $\langle M \rangle$ of a *dpd*-set has to be disconnected.

One can verify that the graph G as shown in the figure 1 is not a *dpd*-graph.

Thus, G is a *ddpd*-graph which is not *dpd*-graph. With the discussion of foregoing paragraph in mind it is quite natural to ask the following:

Open Problem 13. Characterize the ddpd-graphs for which the induced subgraph $\langle M \rangle$ is connected.

Open Problem 14. Characterize the graphs which are both ddpd- and dpd-graph.

4. Some General Results on *DDPD*-Graphs

We begin this section by establishing some of the introductory results pertaining to *ddpd*-graphs.

Proposition 15. *The set of all vertices in the detour diametral path of a graph G cannot form a ddpd-set.*

Proof. Let $P_n = u_1, u_2, u_3, \ldots, u_n$ be an arbitrary detour path. If possible let $M = \{u_1, u_2, u_3, \ldots, u_n\}$ be a *ddpd*-set of G. Then

$$f_M(u_1) = \{0, 1, 2, \ldots, diam_D(G)\} = f_M(u_n).$$

A contradiction to the fact that M is a *ddpd*-set of G. □

Using a similar argument the following is immediate

Proposition 16. *The set of all vertices in the detour center of a graph G cannot form a ddpd-set.*

Our next result provides a necessary condition for a graph to admit *ddpd*-set of cardinality 3.

Proposition 17. *If M be a ddpd-set of G such that $|M| = 3$ then, the detour distances between any pair of vertices in M must be distinct.*

Proof. Assume to the contrary, let $M = \{v_1, v_2, v_3\}$ be a ddpd -set of G.
Case 1: $D(v_1, v_2) = D(v_2, v_3) = D(v_3, v_1) = k$. In this case

$$\begin{aligned} f_M(v_1) &= \{D(v_1, v_1), D(v_1, v_2), D(v_1, v_3)\} \\ f_M(v_2) &= \{D(v_2, v_1), D(v_2, v_2), D(v_2, v_3)\} \\ f_M(v_3) &= \{D(v_3, v_1), D(v_3, v_2), D(v_3, v_3)\} \end{aligned}$$

Since detour distance is a metric for any connected graph, thus $f_M(v_1) = \{0, k\} = f_M(v_2) = f_M(v_3)$. A contradiction to the fact that M is a *ddpd*-set of G.

Case 2: $D(v_1, v_2) = D(v_2, v_3) = k \neq D(v_3, v_1) = l$. In this case

$$f_M(v_1) = \{D(v_1, v_1), D(v_1, v_2), D(v_1, v_3)\}$$

$$f_M(v_3) = \{D(v_3, v_1), D(v_3, v_2), D(v_3, v_3)\}$$

Since detour distance is a metric for any connected graph, thus $f_M(v_1) = \{0, k, l\} = f_M(v_3)$. A contradiction to the fact that M is a *ddpd*-set of G.

In each of the cases we get a contradiction to the fact that M is a *ddpd*-set of G. Hence the proof follows. □

We now proceed to characterize the graphs admitting a *ddpd*-set M of cardinality 1, but before that we prove the following.

Lemma 18. *Let G be a graph admitting a ddpd-set M of cardinality 1, say $M = \{x\}$, then x is a pendant vertex of G.*

Proof. Let G be a graph admitting a *ddpd*-set M of cardinality 1, say $M = \{x\}$. We claim that x is a pendant vertex of G. The proof is by contradiction. Suppose that the set of neighbors of x, denoted by $N(x)$, contains at least two elements.

Case 1: Suppose that for all distinct pairs of elements y, z in $N(x)$, there exists no $y{-}z$ path in $G{-}x$. Then clearly $f_M(y) = f_M(z) = \{1\}$, a contradiction to the fact that M is a *ddpd*-set of G.

Case 2: Suppose that there exists two distinct elements y, z in $N(x)$ such that there is a $y - z$ path in $G - x$. Then, amongst all pairs $y, z \in N(x)$, find a pair a, b having an $a - b$ path P of maximal length in $G - x$. Then $f_M(a) = f_M(b) = \{1 + |E(P)|\}$, a contradiction to the fact that M is a *ddpd*-set of G.

Thus, x is a pendant vertex of G.

□

Theorem 19. *A graph G has a ddpd-set of cardinality 1 if and only if it is a path.*

Proof. Let G be a graph admitting a *ddpd*-set M of cardinality 1, say $M = \{x\}$. In view of Lemma 18, x is a pendant vertex of G. We will now show that G is a path. Assume to the contrary, then there exists a vertex in $V(G)$ of degree three or more. Let w be the unique vertex of degree three or more that has the smallest distance to x. Thus, G contains an $x - w$ path $x, x_1, x_2, \ldots, x_k, w$, with w having at least three neighbors x_k, y and z say.

Case 1: Suppose G has no cycles. Then, $f_M(y) = \{D(x, y)\} = \{D(x, w) + D(w, y)\}$ and $f_M(z) = \{D(x, z)\} = \{D(x, w) + D(w, z)\}$. But $D(w, y) = D(w, z) = 1$, G being a graph devoid of cycles. Hence $f_M(y) = \{D(x, w) + 1\} = f_M(z)$, a contradiction to the fact that M is a *ddpd* set.

Case 2: Suppose G contains a cycle.

Subcase (2.1): Suppose the degree of w is exactly 3. If there exists no cycle in G containing the vertices y and z, then $f_M(y) = f_M(z) = k+2$, a contradiction. If there exists a cycle in G containing the vertices y and z, then consider the set of all $a-b$ paths in $G-w$, where a, b vary over $N(w) - x_k$. Let P denote a path of maximal length in this set, and suppose this path is from vertex c to vertex d. Then $f_M(c) = f_M(d)$, a contradiction.

Subcase (2.2): Suppose the degree of w is at least 4. If there exists no pairs of distinct vertices $a, b \in N(w) - x_k$ that lie in a cycle of G, then $f_M(y) = f_M(z)$, a contradiction. Now suppose there exist a cycle in G that contains two vertices of $N(w) - x_k$. Amongst all pairs $a, b \in N(w) - x_k$, pick a pair that has the longest $a-b$ path in $G-w$. Suppose this pair is $\{c, d\}$. Then, $f_M(c) = f_M(d)$, a contradiction.

□

Having characterized the graphs admitting a *ddpd*-set M of cardinality 1, we now proceed to show that there exists no graphs with *ddpd*-set M of cardinality 2.

Theorem 20. *There exists no ddpd-set of cardinality 2.*

Proof. Let $M = \{u, v\}$ be a *ddpd*-set of a graph G. Since detour distance is a metric for any connected graph, thus for any two vertices $u, v \in V$, we have $D(u,v) = D(v,u)$, which implies that $f_M(u) = f_M(v)$. A contradiction to the fact that M is a *ddpd*-set of G. □

Theorem 21. *If M be a ddpd-set of G then, any vertex of G is adjacent to at most two pendant vertices. Further, if G has a vertex with exactly two pendant vertices adjacent to it then, exactly one of them belongs to M.*

Proof. Let $M = \{u_1, u_2, u_3, \ldots, u_k\}$ be a *ddpd*-set of G and $v \in V$ such that it is adjacent to at least three pendant vertices x_1, x_2, x_3. Then,

$$f_M(x_i) = \{D(x_i, v_1), D(x_i, v_2), \ldots, D(x_i, v_k)\}, \ 1 \le i \le 3 \tag{1}$$

Case 1: If $x_1, x_2, x_3 \notin M$, then for any $v_j \in M$, $1 \le j \le k$, we have

$$D(x_i, v_j) = D(x_i, v) + D(v, v_j), \ 1 \le i \le 3$$

But x_1v, x_2v, x_3v are bridges in G and in view of Theorem 1 $D(x_i, v) = 1$, $1 \le i \le 3$. Hence, we have

$$D(x_i, v_j) = 1 + D(v, v_j), \ 1 \le i \le 3 \tag{2}$$

By 1 and 2 we get $f_M(x_1) = f_M(x_2) = f_M(x_3)$. Which is a contradiction to the fact that M is a *ddpd*-set in G.

Case 2: If $x_1, x_2, x_3 \in M$. Without loss of generality let us assume that $v_1 = x_1$, $v_2 = x_2$, and $v_3 = x_3$. Since x_1, x_2, x_3 are pendant vertices we have the following

$$D(v_i, v_j) = \begin{cases} 0, & \text{if } 1 \leq i, j \leq 3 \text{ and } i = j; \\ 2, & \text{if } 1 \leq i, j \leq 3 \text{ and } i \neq j; \\ 1 + D(v, v_j), & \text{if } 1 \leq i \leq 3,\ i \neq j \text{ and } 4 \leq j \leq k. \end{cases} \tag{3}$$

By 1 and 3 we get $f_M(x_1) = f_M(x_2) = f_M(x_3)$. Which is a contradiction to the fact that M is a *ddpd*-set in G.

Case 3: If $x_1, x_2 \in M$ and $x_3 \notin M$. Without loss of generality let us assume that $v_1 = x_1$, $v_2 = x_2$. Since x_1, x_2 are pendant vertices we have the following

$$D(v_i, v_j) = \begin{cases} 0, & \text{if } 1 \leq i, j \leq 2 \text{ and } i = j; \\ 2, & \text{if } 1 \leq i, j \leq 2 \text{ and } i \neq j; \\ 1 + D(v, v_j), & \text{if } 1 \leq i \leq 2,\ i \neq j \text{ and } 3 \leq j \leq k. \end{cases} \tag{4}$$

By 1 and 4 we get $f_M(x_1) = f_M(x_2)$. Which is a contradiction to the fact that M is a *ddpd*-set in G.

Case 4: If $x_1, x_2 \notin M$ and $x_3 \in M$. Without loss of generality let us assume that $v_1 = x_3$. Since x_1, x_2, x_3 are pendant vertices we have the following

$$D(v_i, v_j) = \begin{cases} 2, & \text{if } 1 \leq i \leq 2 \text{ and } j = 1; \\ 1 + D(v, v_j), & \text{if } 1 \leq i \leq 2, \text{ and } 2 \leq j \leq k. \end{cases} \tag{5}$$

By 1 and 5 we get $f_M(x_1) = f_M(x_2)$. Which is a contradiction to the fact that M is a *ddpd*-set in G.

In each of the cases we get a contradiction to the fact that M is a *ddpd*-set of G. Hence the proof follows. □

Theorem 22. *Complete bipartite graphs $K_{X,Y}$ possess a ddpd set M if and only if either $|X| = |Y| = 1$ or $|X| = 1, |Y| = 2$.*

Proof. Let $K_{X,Y}$ such that either $|X| = |Y| \neq 1$ or $|X| \neq 1, |Y| \neq 2$ possess a *ddpd*-set M. In view of Theorem 19 and Theorem 20, $|M| \geq 3$. Let $M = \{u_1, u_2, \ldots u_k\}$, where $k \geq 3$.

Case 1: Suppose $K_{X,Y}$ contains a hamiltonian cycle. Since $K_{X,Y}$ contains a hamiltonian cycle therefore $|X| = |Y|$.

Subcase (1.1): Let $M \subseteq X$ and $M \cap Y = \emptyset$. Let $u_i, u_j \in M$ then $f_M(u_i) = \{0, 2(|X| + |Y|) - 2\} = f_M(u_j)$ a contradiction.

Subcase (1.2): Let $M \cap X \neq \emptyset$ and $M \cap Y \neq \emptyset$. Let $u_i, u_j \in M \cap X$ and $v \in M \cap Y$ then $f_M(u_i) = \{0, 2(|X| + |Y|) - 1, 2(|X| + |Y|) - 2\} = f_M(u_j)$ a contradiction.

Case 2: Suppose $K_{X,Y}$ does not contain a hamiltonian cycle. Since $K_{X,Y}$ does not contain a hamiltonian cycle therefore $|X| \neq |Y|$, without loss of generality let $|X| > |Y|$.

Subcase (2.1): Let $M \subseteq X$ and $M \cap Y = \emptyset$. Let $u_i, u_j \in M$ then $f_M(u_i) = \{0, 2|Y|\} = f_M(u_j)$ a contradiction.

Subcase (2.2): Let $M \subseteq Y$ and $M \cap X = \emptyset$. Let $u_i, u_j \in M$ then $f_M(u_i) = \{0, 2|X|\} = f_M(u_j)$ a contradiction.

Subcase (2.3): Let $M \cap X \neq \emptyset$ and $M \cap Y \neq \emptyset$. Let $u_i, u_j \in M \cap X$ and $v \in M \cap Y$ then $f_M(u_i) = \{0, 2|Y|, 2|Y| - 1\} = f_M(u_j)$ a contradiction.

Hence if $K_{X,Y}$ is such that either $|X| = |Y| \neq 1$ or $|X| \neq 1, |Y| \neq 2$ then $K_{X,Y}$ does not possess a *ddpd*-set M. To see that $K_{X,Y}$ such that either $|X| = |Y| = 1$ or $|X| = 1, |Y| = 2$ possess a *ddpd*-set M note that $K_{X,Y}$ for $|X| = |Y| = 1$ is P_2 and $K_{X,Y}$ for $|X| = 1, |Y| = 2$ is a P_3 and P_n are known to be *ddpd*-graphs. □

A beautiful class of binary trees is the class of Fibonacci trees [6], [7]. Fibonacci trees of order n has F_n terminal vertices., where $\{F_n\}$ are the Fibonacci numbers $F_0 = 1 = F_1, F_n = F_{n-1} + F_{n-2}$, and is defined inductively as follows: If $n = 1$ or 2, the Fibonacci tree of order n is simply the root only. If $n \geq 3$ the left subtree of the Fibonacci tree of order n is the Fibonacci tree of order $n - 1$; and the right subtree is the Fibonacci tree of order $n - 2$. The Fibonacci tree of order n will be denoted by T_n for brevity.

We conclude this section by proposing the following which we strongly believe it to be true.

Conjecture 23. *Fibonacci trees T_n are ddpd-graph for all values on n.*

References

[1] F. Harary, **Graph Theory**, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1969.

[2] Gary Chartrand, Garry L. Johns, Songlin Tian, Detour Distance in Graphs, *Annals of Discrete Mathematics*, **55** (1993), 127-136.

[3] K.A. Germina, Distance-patterns of vertices in a graph, *International Mathematical Forum*, **5**, No. 34 (2010), 1697-170.

[4] K.A. Germina, Alphy Joseph, Sona Jose, Distance neighbourhood pattern matrices, *European Journal of Pure and Applied Mathematics*, **3**, No. 4 (2010), 748-764.

[5] K.A. Germina, Sona Jose, Distance neighbourhood pattern matrices of trees, *International Mathematical Fourm*, **6**, No. 12 (2011), 591-604.

[6] R.P. Grimaldi, Properties of Fibonacci trees, In: *Proceedings of the Twenty-Second Southeastern Conference on Combinatorics, Graph Theory, and Computing*, Baton Rouge, LA, 1991, **84**, 21-32.

[7] D.E. Knuth, *The art of computer programming*, Volume 3, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. (1973); Sorting and searching, Addison-Wesley Series in Computer Science and Information Processing.