DETOUR DISTANCE PATTERN OF A GRAPH

Kumar Abhishek$^1$§, Ashwin Ganesan$^2$
$^{1,2}$Amrita School of Engineering
Amrita Vishwa Vidyapeetham
Coimbatore, 641 112, Tamil Nadu, INDIA

Abstract: For a simple connected graph $G = (V, E)$, let $M \subseteq V$ and $u \in V$. The $M$-detour distance pattern of $G$ is the set $f_M(u) = \{D(u, v) : v \in M\}$. If $f_M$ is injective function, then the set $M$ is a detour distance pattern distinguishing set (or, $ddpd$- set in short) of $G$. A graph $G$ is defined as detour distance pattern distinguishing (or, $ddpd$-) graph if it admits a $ddpd$-set. The objective of this article is to initiate the study of graphs that admit marker set $M$ for which $f_M$ is injective. This article establishes some general results on $ddpd$-graphs.

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1. Introduction

All the graphs considered here are finite, undirected, connected and simple. For standard graph theory terminology and notations not defined here, we refer [1].

Specifically, let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. For arbitrary pair of vertices $u, v \in V$, the standard distance $d(u, v)$ is the length of the shortest $u - v$ path and the detour distance $D(u, v)$ is the length of the longest $u - v$ path between $u$ and $v$ in $G$. As with standard distance, detour distance is known to be a metric on the vertex set of any connected graph. Chartrand et.al, [2], were the first to introduce the notion of detour distances.

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§Correspondence author
in graphs and was subsequently explored by many scholars. A variety of results pertaining to the detour distances in graphs are known to us. Before we proceed any further we recall the following results which we require subsequently.

**Theorem 1.** [2]

1. For any graph $G$, $D(u, v) = 1$ if and only if $uv$ is a bridge in $G$.

2. For any graph $G$, $d(u, v) = D(u, v)$ for every pair of vertices $u$ and $v$ of $G$ if and only if $G$ is a tree.

The detour eccentricity $e_D(G)$ of a vertex $v$ is the detour distance from $v$ to a vertex farthest from $v$. The detour radius of $rad_D(G)$ of a connected graph $G$ is the minimum detour eccentricity among the vertices of $G$ and the detour diameter $diam_D(G)$ is the maximum detour eccentricity among the vertices of $G$. The detour center $C_D(G)$ of a graph $G$ is the subgraph of $G$ induced by those vertices of $G$ whose detour eccentricity is $rad_D(G)$.

For a simple connected graph $G = (V, E)$, let $M \subseteq V(G)$ and $u \in V$. The $M$-detour distance pattern of $G$ is the set $f_M(u) = \{D(u, v) : v \in M\}$. If $f_M$ is injective function, then the set $M$ is a detour distance pattern distinguishing set (or, $ddpd$- set in short) of $G$. A graph $G$ is defined as detour distance pattern distinguishing (or, $ddpd$-) graph if it admits a $ddpd$-set.

It must be noted that for any graph $G$, with $|V| \geq 2$, $V$ is not a $ddpd$-set, since for any two vertices $u, v \in V$ with $D(u, v) = diam_D(G) = D$ we have $f_M(u) = f_M(v) = \{0, 1, 2, \ldots, D\}$.

Further, for a given simple connected graph $G = (V, E)$, let $M \subseteq V(G)$ and $u \in V(G)$. The $M$-distance pattern of $G$ is the set $f_M(u) = \{d(u, v) : v \in M\}$. If $f_M$ is injective function, then the set $M$ is a distance pattern distinguishing set (or, $dpd$- set in short) of $G$. A graph $G$ is defined as distance pattern distinguishing (or, $dpd$-) graph if it admits a $dpd$-set. The notion of $dpd$-graph was introduced by Germina [3] which was subsequently investigated by many researchers since its inception in 2010.

In this article, we initiate the study of $ddpd$-graph, by changing the standard distance $d(u, v)$ between two vertices $u$ and $v$ in a graph by its detour distance $D(u, v)$.

### 2. A Few Facts Involving $dpd$-Graphs

It was noted [3] that not every graph is a $dpd$-graph. For example, the complete graph $K_n$, $n \geq 3$ is does not possess a $dpd$-set, hence is not a $dpd$-graph.
The following results have been established in the recent past in relation to \(dpd\)-graphs.

**Theorem 2.** [3] Let \(G\) be any connected graph then, \(G\) has a \(dpd\)-set of cardinality 1 if and only if it is a path.

**Theorem 3.** [4] Let \(G\) be any connected graph. There exists no \(dpd\)-set of cardinality 2.

**Theorem 4.** [3] If \(M\) be a \(dpd\)-set of \(G\) then, the induced subgraph \(\langle M \rangle\) is disconnected.

**Theorem 5.** [3] If \(M\) be a \(dpd\)-set of \(G\) then, any vertex of \(G\) is adjacent to at most two pendant vertices. Further, if \(G\) has a vertex with exactly two pendant vertices adjacent to it then, exactly one of them belongs to \(M\).

The problem of characterizing \(dpd\)-graph remains to be an open problem since its inception. The following classes of graphs are known to be \(dpd\)-graph.

**Theorem 6.** [4] A cycle \(C_n\) of order \(n\) is a \(dpd\)-graph if and only if \(n \geq 7\).

**Theorem 7.** [4] \(K_{1,n}\) is a \(dpd\)-graph if and only if \(n \leq 2\).

**Theorem 8.** [5] A double star \(S_{m,n}\) is a \(dpd\)-graph if and only if \(n, m \leq 2\).

**Theorem 9.** [5] A uniform binary tree is \(T\) is a \(dpd\)-graph if and only if \(O(T) = 2^m - 1\) where \(m = 1, 2, 3\).

**Theorem 10.** [5] \(k\)-uniform caterpillars are \(dpd\)-graphs if and only if \(k = 1\).

**Theorem 11.** [5] Every olive tree is a \(dpd\)-graph.

### 3. \(ddpd\)-Graphs Versus \(dpd\)-Graphs

The objective of this section is to do a comparative study of \(ddpd\)- and \(dpd\)-graphs. We begin this section with the following result which we state without a proof.

**Proposition 12.** If \(u\) and \(v\) are any two distinct vertices of a cycle \(C_n\) of order \(n\) then, \(D(u, v) + d(u, v) = n\).

In view of Proposition 12 and Theorem 6 the notion of \(ddpd\)- and \(dpd\)-graphs are equivalent for the case of cycles. Further in view of Theorem 1 the notion of \(ddpd\)- and \(dpd\)-graphs are equivalent for the case of trees. Also, the notion of \(ddpd\)- and \(dpd\)-graphs are equivalent for the case of complete graphs. With the
above few instances of equivalence in the notion of \emph{ddpd-} and \emph{dpd}-graphs the reader must not conclude that the two notions are equivalent in general. As the following example illustrates. Let $G$ be the graph as shown in the figure 1.

![Figure 1: $G$](image)

Consider the subset $M = \{a, b, d\} \subset V$. The $M$-\emph{detour distance pattern} of the vertices of $G$ is the following:

$$
\begin{align*}
    f_M(a) &= \{0, 4, 6\} \\
    f_M(b) &= \{0, 5, 6\} \\
    f_M(c) &= \{5, 6\} \\
    f_M(d) &= \{0, 4, 5\} \\
    f_M(e) &= \{4, 6\} \\
    f_M(f) &= \{4, 5\} \\
    f_M(g) &= \{4, 5, 6\}.
\end{align*}
$$

Clearly $f_M$ is an injective function. Hence $M$ is a \emph{ddpd-} set of $G$. Note that the induced subgraph $\langle M \rangle$ is connected. Thus the induced subgraph of a \emph{ddpd-} set need not be disconnected, whereas in view of Theorem 3 the induced subgraph $\langle M \rangle$ of a \emph{dpd-} set has to be disconnected.

One can verify that the graph $G$ as shown in the figure 1 is not a \emph{dpd-} graph.

Thus, $G$ is a \emph{ddpd-} graph which is not \emph{dpd-} graph. With the discussion of foregoing paragraph in mind it is quite natural to ask the following:

\textbf{Open Problem 13.} Characterize the \emph{ddpd-} graphs for which the induced subgraph $\langle M \rangle$ is connected.
Open Problem 14. Characterize the graphs which are both ddpd- and dpd-graph.

4. Some General Results on DDPD-Graphs

We begin this section by establishing some of the introductory results pertaining to ddpd-graphs.

Proposition 15. The set of all vertices in the detour diametral path of a graph $G$ cannot form a ddpd-set.

Proof. Let $P_n = u_1, u_2, u_3, \ldots, u_n$ be an arbitrary detour path. If possible let $M = \{u_1, u_2, u_3, \ldots, u_n\}$ be a ddpd-set of $G$. Then

$$f_M(u_1) = \{0, 1, 2, \ldots, \text{diam} D(G)\} = f_M(u_n).$$

A contradiction to the fact that $M$ is a ddpd-set of $G$. \hfill \Box

Using a similar argument the following is immediate

Proposition 16. The set of all vertices in the detour center of a graph $G$ cannot form a ddpd-set.

Our next result provides a necessary condition for a graph to admit ddpd-set of cardinality 3.

Proposition 17. If $M$ be a ddpd-set of $G$ such that $|M| = 3$ then, the detour distances between any pair of vertices in $M$ must be distinct.

Proof. Assume to the contrary, let $M = \{v_1, v_2, v_3\}$ be a ddpd-set of $G$.

Case 1: $D(v_1, v_2) = D(v_2, v_3) = D(v_3, v_1) = k$. In this case

$$f_M(v_1) = \{D(v_1, v_1), D(v_1, v_2), D(v_1, v_3)\}$$

Since detour distance is a metric for any connected graph, thus $f_M(v_1) = \{0, k\} = f_M(v_2) = f_M(v_3)$. A contradiction to the fact that $M$ is a ddpd-set of $G$.

Case 2: $D(v_1, v_2) = D(v_2, v_3) = k \neq D(v_3, v_1) = l$. In this case

$$f_M(v_1) = \{D(v_1, v_1), D(v_1, v_2), D(v_1, v_3)\}$$
\[ f_M(v_3) = \{ D(v_3, v_1), D(v_3, v_2), D(v_3, v_3) \} \]

Since detour distance is a metric for any connected graph, thus \( f_M(v_1) = \{ 0, k, l \} = f_M(v_3) \). A contradiction to the fact that \( M \) is a \textit{ddpd}-set of \( G \).

In each of the cases we get a contradiction to the fact that \( M \) is a \textit{ddpd}-set of \( G \). Hence the proof follows. \qed

We now proceed to characterize the graphs admitting a \textit{ddpd}-set \( M \) of cardinality 1, but before that we prove the following.

**Lemma 18.** Let \( G \) be a graph admitting a \textit{ddpd}-set \( M \) of cardinality 1, say \( M = \{ x \} \), then \( x \) is a pendant vertex of \( G \).

**Proof.** Let \( G \) be a graph admitting a \textit{ddpd}-set \( M \) of cardinality 1, say \( M = \{ x \} \). We claim that \( x \) is a pendant vertex of \( G \). The proof is by contradiction. Suppose that the set of neighbors of \( x \), denoted by \( N(x) \), contains at least two elements.

**Case 1:** Suppose that for all distinct pairs of elements \( y, z \) in \( N(x) \), there exists no \( y-z \) path in \( G-x \). Then clearly \( f_M(y) = f_M(z) = \{ 1 \} \), a contradiction to the fact that \( M \) is a \textit{ddpd}-set of \( G \).

**Case 2:** Suppose that there exists two distinct elements \( y, z \) in \( N(x) \) such that there is a \( y-z \) path in \( G-x \). Then, amongst all pairs \( y, z \in N(x) \), find a pair \( a, b \) having an \( a-b \) path \( P \) of maximal length in \( G-x \). Then \( f_M(a) = f_M(b) = \{ 1 + |E(P)| \} \), a contradiction to the fact that \( M \) is a \textit{ddpd}-set of \( G \).

Thus, \( x \) is a pendant vertex of \( G \). \qed

**Theorem 19.** A graph \( G \) has a \textit{ddpd}-set of cardinality 1 if and only if it is a path.

**Proof.** Let \( G \) be a graph admitting a \textit{ddpd}-set \( M \) of cardinality 1, say \( M = \{ x \} \). In view of Lemma 18, \( x \) is a pendant vertex of \( G \). We will now show that \( G \) is a path. Assume to the contrary, then there exists a vertex in \( V(G) \) of degree three or more. Let \( w \) be the unique vertex of degree three or more that has the smallest distance to \( x \). Thus, \( G \) contains an \( x-w \) path \( x, x_1, x_2, \ldots, x_k, w \), with \( w \) having at least three neighbors \( x_k, y, z \) say.

**Case 1:** Suppose \( G \) has no cycles. Then, \( f_M(y) = \{ D(x, y) \} = \{ D(x, w) + D(w, y) \} \) and \( f_M(z) = \{ D(x, z) \} = \{ D(x, w) + D(w, z) \} \). But \( D(w, y) = D(w, z) = 1, G \) being a graph devoid of cycles. Hence \( f_M(y) = \{ D(x, w) + 1 \} = f_M(z) \), a contradiction to the fact that \( M \) is a \textit{ddpd} set.
Case 2: Suppose \( G \) contains a cycle.

**Subcase (2.1):** Suppose the degree of \( w \) is exactly 3. If there exists no cycle in \( G \) containing the vertices \( y \) and \( z \), then \( f_M(y) = f_M(z) = k + 2 \), a contradiction. If there exists a cycle in \( G \) containing the vertices \( y \) and \( z \), then consider the set of all \( a - b \) paths in \( G - w \), where \( a, b \) vary over \( N(w) - x_k \). Let \( P \) denote a path of maximal length in this set, and suppose this path is from vertex \( c \) to vertex \( d \). Then \( f_M(c) = f_M(d) \), a contradiction.

**Subcase (2.2):** Suppose the degree of \( w \) is at least 4. If there exists no pairs of distinct vertices \( a, b \in N(w) - x_k \) that lie in a cycle of \( G \), then \( f_M(y) = f_M(z) \), a contradiction. Now suppose there exist a cycle in \( G \) that contains two vertices of \( N(w) - x_k \). Amongst all pairs \( a, b \in N(w) - x_k \), pick a pair that has the longest \( a - b \) path in \( G - w \). Suppose this pair is \( \{c, d\} \). Then, \( f_M(c) = f_M(d) \), a contradiction.

Having characterized the graphs admitting a \( ddpd \)-set \( M \) of cardinality 1, we now proceed to show that there exists no graphs with \( ddpd \)-set \( M \) of cardinality 2.

**Theorem 20.** There exists no \( ddpd \)-set of cardinality 2.

*Proof.* Let \( M = \{u, v\} \) be a \( ddpd \)-set of a graph \( G \). Since detour distance is a metric for any connected graph, thus for any two vertices \( u, v \in V \), we have \( D(u, v) = D(v, u) \), which implies that \( f_M(u) = f_M(v) \). A contradiction to the fact that \( M \) is a \( ddpd \)-set of \( G \). \( \Box \)

**Theorem 21.** If \( M \) be a \( ddpd \)-set of \( G \) then, any vertex of \( G \) is adjacent to at most two pendant vertices. Further, if \( G \) has a vertex with exactly two pendant vertices adjacent to it then, exactly one of them belongs to \( M \).

*Proof.* Let \( M = \{u_1, u_2, u_3, \ldots, u_k\} \) be a \( ddpd \)-set of \( G \) and \( v \in V \) such that it is adjacent to at least three pendant vertices \( x_1, x_2, x_3 \). Then,

\[
    f_M(x_i) = \{D(x_i, v_1), D(x_i, v_2), \ldots, D(x_i, v_k)\}, \quad 1 \leq i \leq 3 \tag{1}
\]

**Case 1:** If \( x_1, x_2, x_3 \notin M \), then for any \( v_j \in M \), \( 1 \leq j \leq k \), we have

\[
    D(x_i, v_j) = D(x_i, v) + D(v, v_j), \quad 1 \leq i \leq 3
\]

But \( x_1v, x_2v, x_3v \) are bridges in \( G \) and in view of Theorem 1 \( D(x_i, v) = 1 \), \( 1 \leq i \leq 3 \). Hence, we have

\[
    D(x_i, v_j) = 1 + D(v, v_j), \quad 1 \leq i \leq 3 \tag{2}
\]
By 1 and 2 we get \( f_M(x_1) = f_M(x_2) = f_M(x_3) \). Which is a contradiction to the fact that \( M \) is a \( ddpd \)-set in \( G \).

**Case 2:** If \( x_1, x_2, x_3 \in M \). Without loss of generality let us assume that \( v_1 = x_1, v_2 = x_2, \) and \( v_3 = x_3 \). Since \( x_1, x_2, x_3 \) are pendant vertices we have the following

\[
D(v_i, v_j) = \begin{cases} 
0, & \text{if } 1 \leq i, j \leq 3 \text{ and } i = j; \\
2, & \text{if } 1 \leq i, j \leq 3 \text{ and } i \neq j; \\
1 + D(v, v_j), & \text{if } 1 \leq i \leq 3, i \neq j \text{ and } 4 \leq j \leq k.
\end{cases}
\]

By 1 and 3 we get \( f_M(x_1) = f_M(x_2) = f_M(x_3) \). Which is a contradiction to the fact that \( M \) is a \( ddpd \)-set in \( G \).

**Case 3:** If \( x_1, x_2 \in M \) and \( x_3 \notin M \). Without loss of generality let us assume that \( v_1 = x_1, v_2 = x_2 \). Since \( x_1, x_2 \) are pendant vertices we have the following

\[
D(v_i, v_j) = \begin{cases} 
0, & \text{if } 1 \leq i, j \leq 2 \text{ and } i = j; \\
2, & \text{if } 1 \leq i, j \leq 2 \text{ and } i \neq j; \\
1 + D(v, v_j), & \text{if } 1 \leq i \leq 2, i \neq j \text{ and } 3 \leq j \leq k.
\end{cases}
\]

By 1 and 4 we get \( f_M(x_1) = f_M(x_2) \). Which is a contradiction to the fact that \( M \) is a \( ddpd \)-set in \( G \).

**Case 4:** If \( x_1, x_2 \notin M \) and \( x_3 \in M \). Without loss of generality let us assume that \( v_1 = x_3 \). Since \( x_1, x_2, x_3 \) are pendant vertices we have the following

\[
D(v_i, v_j) = \begin{cases} 
2, & \text{if } 1 \leq i \leq 2 \text{ and } j = 1; \\
1 + D(v, v_j), & \text{if } 1 \leq i \leq 2, \text{ and } 2 \leq j \leq k.
\end{cases}
\]

By 1 and 5 we get \( f_M(x_1) = f_M(x_2) \). Which is a contradiction to the fact that \( M \) is a \( ddpd \)-set in \( G \).

In each of the cases we get a contradiction to the fact that \( M \) is a \( ddpd \)-set of \( G \). Hence the proof follows. \( \square \)

**Theorem 22.** Complete bipartite graphs \( K_{X,Y} \) possess a \( ddpd \) set \( M \) if and only if either \( |X| = |Y| = 1 \) or \( |X| = 1, |Y| = 2 \).

**Proof.** Let \( K_{X,Y} \) such that either \( |X| = |Y| \neq 1 \) or \( |X| \neq 1, |Y| \neq 2 \) possess a \( ddpd \)-set \( M \). In view of Theorem 19 and Theorem 20, \( |M| \geq 3 \). Let \( M = \{u_1, u_2, \ldots, u_k\} \), where \( k \geq 3 \).

**Case 1:** Suppose \( K_{X,Y} \) contains a hamiltonian cycle. Since \( K_{X,Y} \) contains a hamiltonian cycle therefore \( |X| = |Y| \).

**Subcase (1.1):** Let \( M \subseteq X \) and \( M \cap Y = \emptyset \). Let \( u_i, u_j \in M \) then \( f_M(u_i) = \{0, 2(\left|X\right| + \left|Y\right|) - 2\} = f_M(u_j) \) a contradiction.
Subcase (1.2): Let $M \cap X \neq \emptyset$ and $M \cap Y \neq \emptyset$. Let $u_i, u_j \in M \cap X$ and $v \in M \cap Y$ then $f_M(u_i) = \{0, 2(|X| + |Y|) - 1, 2(|X| + |Y|) - 2\} = f_M(u_j)$ a contradiction.

Case 2: Suppose $K_{X,Y}$ does not contain a hamiltonian cycle. Since $K_{X,Y}$ does not contain a hamiltonian cycle therefore $|X| \neq |Y|$, without loss of generality let $|X| > |Y|$.

Subcase (2.1): Let $M \subseteq X$ and $M \cap Y = \emptyset$. Let $u_i, u_j \in M$ then $f_M(u_i) = \{0, 2|Y|\} = f_M(u_j)$ a contradiction.

Subcase (2.2): Let $M \subseteq Y$ and $M \cap X = \emptyset$. Let $u_i, u_j \in M$ then $f_M(u_i) = \{0, 2|X|\} = f_M(u_j)$ a contradiction.

Subcase (2.3): Let $M \cap X \neq \emptyset$ and $M \cap Y \neq \emptyset$. Let $u_i, u_j \in M \cap X$ and $v \in M \cap Y$ then $f_M(u_i) = \{0, 2|Y|\} = f_M(u_j)$ a contradiction.

Hence if $K_{X,Y}$ is such that either $|X| = |Y| \neq 1$ or $|X| \neq 1, |Y| \neq 2$ then $K_{X,Y}$ does not possess a ddpd-set $M$. To see that $K_{X,Y}$ such that either $|X| = |Y| = 1$ or $|X| = 1, |Y| = 2$ possess a ddpd-set $M$ note that $K_{X,Y}$ for $|X| = |Y| = 1$ is $P_2$ and $K_{X,Y}$ for $|X| = 1, |Y| = 2$ is a $P_3$ and $P_n$ are known to be ddpd-graphs.

A beautiful class of binary trees is the class of Fibonacci trees [6], [7]. Fibonacci trees of order $n$ has $F_n$ terminal vertices., where $\{F_n\}$ are the Fibonacci numbers $F_0 = 1 = F_1, F_n = F_{n-1} + F_{n-2}$, and is defined inductively as follows: If $n = 1$ or 2, the Fibonacci tree of order $n$ is simply the root only. If $n \geq 3$ the left subtree of the Fibonacci tree of order $n$ is the Fibonacci tree of order $n - 1$; and the right subtree is the Fibonacci tree of order $n - 2$. The Fibonacci tree of order $n$ will be denoted by $T_n$ for brevity.

We conclude this section by proposing the following which we strongly believe it to be true.

Conjecture 23. Fibonacci trees $T_n$ are ddpd-graph for all values on $n$.

References


