International Journal of Pure and Applied Mathematics
Volume 87 No. 5 2013, 729-738
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v87i5.6

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PATH AND STAR RELATED GRAPHS ON EVEN SEQUENTIAL HARMONIOUS, GRACEFUL, ODD GRACEFUL AND FELICITOUS LABELLING

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Abstract: In this paper, we prove that the $D_2(P_n)$ is an even sequential harmonious and graceful graph, $D_2(K_1, n)$ an even sequential harmonious, graceful graph, odd graceful and felicitous graph, $spl(P_n)$ is an even sequential harmonious and odd graceful graph, $spl(K_1, n)$ is an even sequential harmonious and felicitous graph.

AMS Subject Classification: even sequential harmonious, graceful, odd graceful, felicitous shadow graph, split graph
Key Words: 05C78

Received: July 22, 2013

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1. Introduction

All graphs in this paper are finite, simple and undirected. Terms not defined here are used in the sense of Harary [2]. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph. Graph labeling where the vertices are assigned values to certain conditions have been motivated by practical problems. According to Beineke and Hegde graph labeling serves as a frontier between number theory and structure of graphs. For a dynamic survey of various graph labeling problems along with extensive bibliography we refer to Gallian [1]. Graham and Slone [1] introduced the harmonious graphs. We refer to the excellent survey by Gallian [1] for vertices of labeling and graphs. Gayathri et al [6] say that a labeling is an even sequential harmonious labeling if there exists an injection from the vertex set V to $\{0, 1, 2, \cdots, 2q\}$ such that the induced mapping f^* from the edge set E to $\{2, 4, 6, \cdots, 2q\}$ defined by

$$f^*(uv) = \begin{cases} f(u) + f(v) \; if \; f(u) + f(v) \; is \; even \\ f(u) + f(v) + 1 \; if \; f(u) + f(v) \; is \; odd \end{cases}$$

are distinct. A graph G is said to be an even sequential harmonious graph if it admits an even sequential harmonious labeling.

A function f is called graceful labeling of a graph G if $f : V(G) \to \{0, 1, 2, \cdots, q\}$ is injective and induced function $f^* : E(G) \to \{1, 2, \cdots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. The graph which admits graceful labeling is called a graceful graph.

A function f is called odd graceful labeling of a graph G if $f : V(G) \to \{0, 1, 2, \cdots, 2q-1\}$ is injective and induced function $f^* : E(G) \to \{1, 3, \cdots, 2q-1\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. The graph which admits odd graceful labeling is called an odd graceful graph.

A function f is called felicitous labeling of a graph G if $f : V(G) \to \{0, 1, 2, \cdots, q\}$ is injective and induced function $f^* : E(G) \to \{0, 1, 2, \cdots, q-1\}$ defined as $f^*(e = uv) = f(u) + f(v)(mod \; q)$ are distinct. Clearly, a harmonious graph is felicitous. An example of a felicitous graph which is not harmonious is the graph $K_{m,n}$, where $m, n > 1$.

The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbours of the corresponding vertex v' in G''.

For a graph G the split graph is obtained by adding to each vertex v a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G. The resultant graph is denoted as $spl(G)$.

In this paper, we investigate even sequential harmonious labeling, graceful labeling, odd graceful labeling and felicitous labeling of path and star related graphs.

2. Results on an Even Sequential Harmonious

Theorem 2.1. *$D_2(P_n)$ is an even sequential harmonious graph.*

Proof. Consider two copies of P_n. Let $v_1, v_2, \cdots, v_n$ be the vertices of the first copy of P_n and $v_1^{'}, v_2^{'}, \cdots, v_n^{'}$ be the vertices of the second copy of P_n. Let G be the graph $D_2(P_n)$. Then $|V(G)| = 2n$ and $|E(G)| = 4(n-1)$. To define $f : V(G) \to \{0, 1, 2, 3, \cdots, 8(n-1)\}$. we consider the following three cases.
Case(i): $n = 2$ The graphs $D_2(P_2)$ is to be dealt with separately and its even sequential harmonious labeling is shown in figure 1.

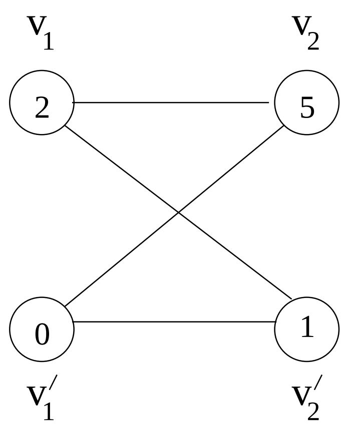

Figure 1: The graph $D_2(P_2)$ and its even sequential harmonious labeling

Case(ii): n is odd, $n \geq 3$
$f(v_1) = 0, f(v_2) = 1, f(v_1^{'}) = 4, f(v_2^{'}) = 3$
$f(v_{2i+1}) = 8i; 1 \leq i \leq \frac{n-1}{2}$
$f(v_{2i+2}) = 8i + 1; 1 \leq i \leq \frac{n-3}{2}$
$f(v_{2i+1}^{'}) = 8i + 4; 1 \leq i \leq \frac{n-1}{2}$
$f(v_{2i+2}^{'}) = 8i + 3; 1 \leq i \leq \frac{n-3}{2}$

Case(iii): n is even, $n \geq 4$
$f(v_1) = 0, f(v_2) = 1, f(v_1^{'}) = 4, f(v_2^{'}) = 3$
$f(v_{2i+1}) = 8i; 1 \leq i \leq \frac{n-2}{2}$
$f(v_{2i+2}) = 8i + 1; 1 \leq i \leq \frac{n-2}{2}$

$f(v'_{2i+1}) = 8i+4; 1 \leq i \leq \frac{n-2}{2}$
$f(v'_{2i+2}) = 8i+3; 1 \leq i \leq \frac{n-2}{2}$

Let A, B, C, D, E, F are denote the edge set.
$A = \{e_i = v_i v_{i+1} / f(e_i) = 8i-6 : 1 \leq i \leq n-1\}$
$B = \{e'_i = v'_i v'_{i+1} / f(e'_i) = 8i : 1 \leq i \leq n-1\}$
$C = \{e_i = v_i v'_{i+1} / f(e_i) = 8i-4 : 1 \leq i \leq n-1;$ i is odd$\}$
$D = \{e_i = v_i v'_{i+1} / f(e_i) = 8i-2 : 1 \leq i \leq n-1;$ i is even$\}$
$E = \{e_i = v'_i v_{i+1} / f(e_i) = 8i-2 : 1 \leq i \leq n-1;$ i is odd$\}$
$F = \{e_i = v'_i v_{i+1} / f(e_i) = 8i-4 : 1 \leq i \leq n-1;$ i is even$\}$

Hence, $D_2(P_n)$ is an even sequential graph. □

Illustration

An even sequential labeling of the graph $D_2(P_{11})$ is shown in the figure 2.

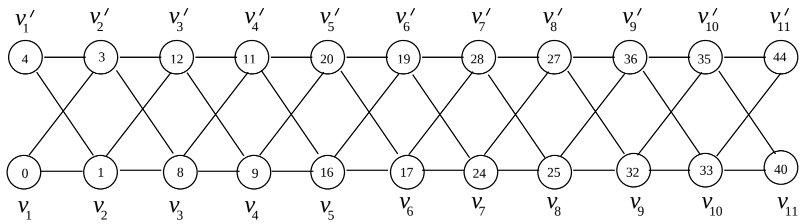

Figure 2: The graph $D_2(P_{11})$ and its even sequential harmonious labeling

Theorem 2.2. *$D_2(K_{1,n})$ is an even sequential harmonious graph.*

Proof. Consider two copies of $K_{1,n}$. Let $v_1, v_2, \cdots, v_n$ be the pendent vertices of the first copy of $K_{1,n}$ and $v'_1, v'_2, \cdots, v'_n$ be the pendent vertices of the second copy of $K_{1,n}$ with v and v' respective apex vertices. Let G be the graph $D_2(K_{1,n})$.Then $|V(G)| = 2n+2$ and $|E(G)| = 4n$. we define $f : V(G) \to \{0, 1, 2, 3, \cdots, 8n\}$ as follows.
$f(v) = 0, f(v') = 2,$
$f(v_i) = 4(i-1)+1; 1 \leq i \leq n$
$f(v'i) = 4(n-1+i)+1; 1 \leq i \leq n$

Let A, B, C, D are denote the edge set.
$A = \{e_i = vv_i / f(e_i) = 4i - 2 : 1 \leq i \leq n\}$
$B = \{e_i^{'} = v^{'}v_i / f(e_i^{'}) = 4i : 1 \leq i \leq n\}$
$C = \{e_i = vv_i^{'} / f(e_i) = 4(n+i) - 2 : 1 \leq i \leq n\}$
$D = \{e_i = v^{'}v_i^{'} / f(e_i) = 4(n+i) : 1 \leq i \leq n\}$
Hence, $D_2(K_{1,n})$ is an even sequential graph. □

Theorem 2.3. *$spl(P_n)$ is an even sequential graph.*

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices corresponding to $v_1, v_2, \cdots, v_n$ of P_n which are added to obtain $spl(P_n)$. Let G be the graph $spl(P_n)$.Then $|V(G)| = 2n$ and $|E(G)| = 3(n-1)$. we define $f : V(G) \to \{0, 1, 2, 3, \cdots, 6(n-1)\}$. we consider the following cases.

Case(i): n is even, $n \geq 2$
$f(v_1) = 0, f(v_2) = 1,$
$f(v_{2i+1}) = 6i; 1 \leq i \leq \frac{n-2}{2}$
$f(v_{2i+2}) = 6i + 1; 1 \leq i \leq \frac{n-2}{2}$
$f(u_i) = 3i - 1; 1 \leq i \leq n$

Case(ii): n is odd, $n \geq 3$
$f(v_1) = 0, f(v_2) = 1,$
$f(v_{2i+1}) = 6i; 1 \leq i \leq \lceil \frac{n-2}{2} \rceil$
$f(v_{2i+2}) = 6i + 1; 1 \leq i \leq \frac{n-3}{2}$
$f(ui) = 3i1; 1 \leq i \leq n$
Let A, B, C, D, E are denote the edge set.
$A = \{e_i = v_iv_{i+1} / f(e_i) = 6i - 4 : 1 \leq i \leq n-1\}$.
$B = \{e_i = v_iu_{i+1} / f(e_i) = 6i : 1 \leq i \leq n-1;\ i \text{ is odd}\}$
$C = \{e_i = v_iu_{i+1} / f(e_i) = 6i - 2 : 1 \leq i \leq n-1;\ i \text{ is even}\}$
$D = \{e_i = u_iv_{i+1} / f(e_i) = 6i : 1 \leq i \leq n-1;\ i \text{ is even }\}$
$E = \{e_i = u_iv_{i+1} / f(e_i) = 6i - 2 : 1 \leq i \leq n-1;\ i \text{ is odd}\}$
Hence, $spl(P_n)$ is an even sequential graph. □

Illustration

An even sequential labeling of the graph $spl(P_{10})$ is shown in the figure 3.

Theorem 2.4. *The $spl(K_{1,n})$ is an even sequential harmonious graph.*

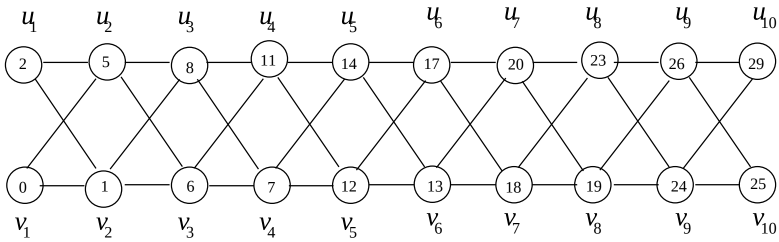

Figure 3: The graph $spl(P_{10})$ and its even sequential harmonious labeling

Proof. Let $v_1, v_2, \cdots, v_n$ be the pendent vertices and v be the apex vertex of $K_{1,n}$ and $u, u_1, u_2, \cdots, u_n$ are added vertices corresponding to v, v_1. Let $v, v_1, v_2, \cdots, v_n$ to obtain $spl(K_{1,n})$. Let G be the graph $spl(K_{1,n})$. Then $|V(G)| = 2n+2$ and $|E(G)| = 3n$. we define $f : V(G) \to \{0, 1, 2, 3, \cdots, 6n\}$
$f(v) = 0, f(u) = 2,$
$f(v_i) = 2n + 4i - 3; 1 \leq i \leq n$
$f(u_i) = 2i - 1; 1 \leq i \leq n$

Let A, B, C are denote edge set.
$A = \{e_i = vu_i/f(e_i) = 2i : 1 \leq i \leq n\}$
$B = \{e_i = vv_i/f(e_i) = 2n + 4i - 2 : 1 \leq i \leq n\}$
$C = \{e_i = v_iu/f(e_i) = 2n + 4i : 1 \leq i \leq n\}$
Hence, $spl(K_{1,n})$ is an even sequential graph. □

3. Illustration

An even sequential labeling of the graph $spl(K_{1,5})$ is shown in the figure 4.

4. Results on Graceful Labeling

Theorem 4.1. *$D_2(P_n)$ is a graceful graph.*

Proof. Consider two copies of P_n. Let $v_1, v_2, \cdots, v_n$ be the vertices of the first copy of P_n and $v_1', v_2', \cdots, v_n'$ be the vertices of the second copy of P_n. Let G be the graph of $D_2(P_n)$. Then $|V(G)| = 2n$ and $|E(G)| = 4(n-1)$. We define $f : V(G) \to \{0, 1, 2, 3, \cdots, 4(n-1)\}$

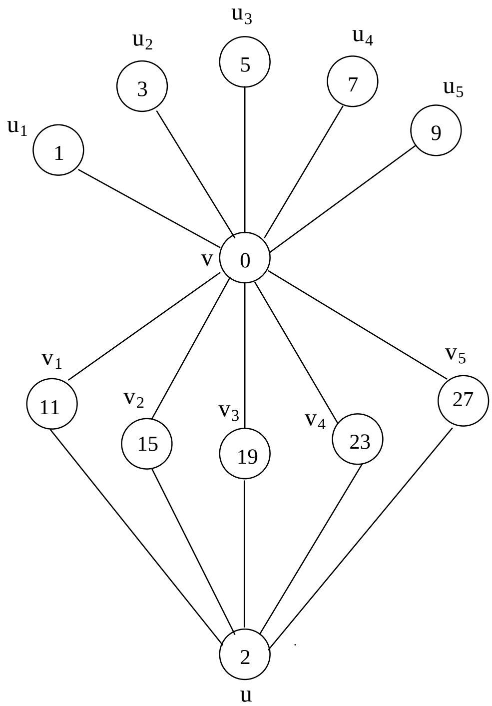

Figure 4: The graph $spl(K_{1,5})$ and its even sequential harmonious labeling

$f(v_i) = 2(i-1); 1 \leq i \leq n, i$ is odd .
$f(v_i) = 4(n - \frac{i}{2}); 1 \leq i \leq n, i$ is even
$f(u_i) = 2i; 1 \leq i \leq n, i$ is odd.
$f(u_i) = 2(2n-i) - 1; 1 \leq i \leq n, i$ is even
Let A, B, C, D, E, F are denote the edge set.
$A = \{e_i = v_i v_{i+1} / f(e_i) = 4(n-i) : 1 \leq i \leq n-1\}$
$B = \{e_i = u_i u_{i+1} / f(e_i) = 4(n-i) - 3 : 1 \leq i \leq n-1\}$
$C = \{e_i = v_i u_{i+1} / f(e_i) = 4(n-i) - 1 : 1 \leq i \leq n-1,\ i \text{ is odd}\}$
$D = \{e_i = v_i u_{i+1} / f(e_i) = 4(n-i) - 2 : 1 \leq i \leq n-1;\ i \text{ is even}\}$
$E = \{e_i = u_i v_{i+1} / f(e_i) = 4(n-i) - 2 : 1 \leq i \leq n-1;\ i \text{ is odd}\}$
$F = \{e_i = u_i v_{i+1} / f(e_i) = 4(n-i) - 1 : 1 \leq i \leq n-1;\ i \text{ is even}\}$
Hence $D_2(P_n)$ is a graceful graph. □

Theorem 4.2. *$D_2(K_{1,n})$ is a graceful graph.*

Proof. Consider two copies of $K_{1,n}$. Let $v_1, v_2, \cdots, v_n$ be the pendent vertices of the first copy of $K_{1,n}$ and $v_1^{'}, v_2^{'}, \cdots, v_n^{'}$ be the pendent vertices of the second copy of $K_{1,n}$ with v and $v^{'}$ respective apex vertices. Let G be the graph $D_2(K_{1,n})$. Then $|V(G)| = 2n + 2$ and $|E(G)| = 4n$ we define $f : V(G) \to \{0, 1, 2, 3, \cdots, 4n\}$ as follows.
$f(v) = 0, f(v') = 1,$
$f(v_i) = 2n + 2i; 1 \leq i \leq n. f(v_i^{'}) = 2i; 1 \leq i \leq n.$
Let A, B, C, D are edge set.
$A = \{e_i = vv_i / f(e_i) = 2n + 2i : 1 \leq i \leq n\}$
$B = \{e_i = v^{'}v_i / f(e_i) = 2n + 2i - 1 : 1 \leq i \leq n\}$
$C = \{e_i = vv_i^{'} / f(e_i) = 2i : 1 \leq i \leq n\}$
$D = \{e_i = v^{'}v_i^{'} / f(e_i) = 2i - 1 : 1 \leq i \leq n\}$
Hence, $D_2(K_{1,n})$ is a graceful graph. □

5. Results on Odd Graceful Labeling

Theorem 5.1. *$D_2(K_{1,n})$ is a odd graceful graph.*

Proof. Consider two copies of $K_{1,n}$. Let $v_1, v_2, \cdots, v_n$ be the pendent vertices of the first copy of $K_{1,n}$ and $v_1^{'}, v_2^{'}, \cdots, v_n^{'}$ be the pendent vertices of the second copy of $K_{1,n}$ with v and $v^{'}$ respective apex vertices. Let $G = D_2(K_{1,n})$.Then $|V(G)| = 2n + 2$ and $|E(G)| = 4n$.
we define $f : V(G) \to \{0, 1, 2, 3, \cdots, 8n - 1\}$ as follows.
$f(v) = 0, f(v^{'}) = 2,$
$f(v_i) = 4(n + i) - 1; 1 \leq i \leq n$
$f(v_i^{'}) = 4i - 1; 1 \leq i \leq n$
LetA, B, C, D are edge set.
$A = \{e_i = vv_i / e_i = 4(n + i) - 1 : 1 \leq i \leq n\}$
$B = \{e_i = v^{'}v_i / e_i = 4(n + i) - 3 : 1 \leq i \leq n\}$
$C = \{e_i = vv_i^{'} / e_i = 4i - 1 : 1 \leq i \leq n\}$
$D = \{e_i = v^{'}v_i^{'} / e_i = 4i - 3 : 1 \leq i \leq n\}$
Hence, $D_2(K_{1,n})$ is a odd graceful graph. □

Theorem 5.2. *$spl(P_n)$ is odd graceful graph.*

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices corresponding to $v_1, v_2, \cdots, v_n$ of P_n which are added to obtain $spl(P_n)$. Let G be the graph of $spl(P_n)$. Then $|V(G)| = 2n$ and $|E(G)| = 3(n - 1)$.
we define $f : V(G) \to \{0, 1, 2, 3, \cdots, 6n - 7\}$.

$f(v_1) = 2, f(u_1) = 0,$
$f(v_i) = 3(2n - i) - 1; 2 \leq i \leq n, i$ is even
$f(v_i) = 3(i - 1); 2 \leq i \leq n, i$ is odd
$f(u_i) = 3(2n - i - 1); 2 \leq i \leq n, i$ is even
$f(u_i) = 3i + 1; 2 \leq i \leq n, i$ is odd
Let $e_1 = u_1 v_2 = 6n - 7$ and $e_1^{'} = v_1 u_2 = 6n - 11$.
Let A, B, C, D, E denote the edge set.
$A = \{e_i = v_i v_{i+1} / f(e_i) = 6(n - i) - 1 : 2 \leq i \leq n - 1\}$
$B = \{e_i = v_i u_{i+1} / f(e_i) = 6n - 6i - 5 : 2 \leq i \leq n - 1,\ i\ is\ even\}$
$C = \{e_i = v_i u_{i+1} / f(e_i) = 3(n + 6 - 2i) : 2 \leq i \leq n - 1,\ i\ is\ odd\}$
$D = \{e_i = u_i v_{i+1} / f(e_i) = 6(n - i) - 3 : 2 \leq i \leq n - 1, i\ is\ even\}$
$E = \{e_i = u_i v_{i+1} / f(e_i) = 3(n - 2i) + 16 : 2 \leq i \leq n - 1, i\ is\ odd\}$
Hence, $spl(P_n)$ is a odd graceful graph. □

6. Results on Felicitous Labeling

Theorem 6.1. *$D_2(K_{1,n})$ is felicitous graph.*

Proof. Consider two copies of $K_{1,n}$. Let $v_1, v_2, \cdots, v_n$ be the pendent vertices of the first copy of $K_{1,n}$ and $v_1^{'}, v_2^{'}, \cdots, v_n^{'}$ be the pendent vertices of the second copy of $K_{1,n}$ with v and $v^{'}$ respective apex vertices. Let G be the graph $D_2(K_{1,n})$. Then $|V(G)| = 2n + 2, |E(G)| = 4n$. we define $f : V(G) \to \{0, 1, 2, 3, \cdots, 4n\}$ as follows.
$f(v) = 0, f(v^{'}) = 1,$
$f(v_i) = 2(n + i); 1 \leq i \leq n$
$f(v_i^{'}) = 2i; 1 \leq i \leq n$
Let A, B, C, D are edge set.
$A = \{e_i = vv_i / f(e_i) = 2(n + i)(mod\ q) : 1 \leq i \leq n\}$
$B = \{e_i = v^{'} v_i / f(e_i) = (2n + 2i + 1)(mod\ q) : 1 \leq i \leq n\}$
$C = \{e_i = vv_i^{'} / f(e_i) = 2i(mod\ q) : 1 \leq i \leq n\}$
$D = \{e_i = v^{'} v_i^{'} / f(e_i) = (2i + 1)(mod\ q) : 1 \leq i \leq n\}$
Hence, $D_2(K_{1,n})$ is a felicitous labeling. □

Theorem 6.2. *$spl(K_{1,n})$ is a felicitous graph.*

Proof. Let $v_1, v_2, \cdots, v_n$ be the pendent vertices and v be the apex vertex of $K_{1,n}$ and $u, u_1, u_2, \cdots, u_n$ are added vertices corresponding to v, v_1. Let $v, v_1, v_2, \cdots, v_n$ to obtain $spl(K_{1,n})$. Let G be the graph of $spl(K_{1,n})$. Then $|V(G)| = 2n + 2$ and $|E(G)| = 3n$.

we define $f : V(G) \rightarrow \{0, 1, 2, 3, \cdots, 3n\}$
$f(v) = 0, f(u) = 1,$
$f(v_i) = i + 1; 1 \leq i \leq n. f(u_i) = n + 2i; 1 \leq i \leq n.$
Let A, B, C denote edge set.
$A = \{e_i = vu_i/f(e_i) = (i + 1)(mod\ q) : 1 \leq i \leq n\}.$
$B = \{e_i = v_iv/f(e_i) = (n + 2i)(mod\ q) : 1 \leq i \leq n\}.$
$C = \{e_i = v_iu/f(e_i) = (n + 2i + 1)(mod\ q) : 1 \leq i \leq n\}.$
Hence, $spl(K_{1,n})$ is a felicitous graph. □

Conclusion: Shadow of any graph is an Eulerian graph.
Acknowledgement: The authors are thankful to the anonymous referee for valuable comments and kind suggestions towards improvement of our results.

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