PACKING CHROMATIC NUMBER OF CERTAIN GRAPHS

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Abstract: The packing chromatic number $\chi_{\rho}(G)$ of a graph $G$ is the smallest integer $k$ for which there exists a mapping $\Pi : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that any two vertices of color $i$ are at distance at least $i + 1$. It is a frequency assignment problem used in wireless networks, which is also called broadcasting coloring. It is proved that packing coloring is NP-complete for general graphs and even for trees. In this paper, we study the packing chromatic number of comb graph, circular ladder, windmill, H-graph and uniform theta graph.

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1. Introduction

Let $G$ be a connected graph and $k$ be an integer, $k \geq 1$. A packing $k$-coloring of a graph $G$ is a mapping $\Pi : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that any two vertices of color $i$ are at distance at least $i + 1$. The packing chromatic number $\chi_{\rho}(G)$ of $G$ is the smallest integer $k$ for which $G$ has packing $k$-coloring [15]. The concept of packing coloring comes from the area of frequency assignment in wireless networks [6] and was introduced by Goddard et al. [10] under the name broadcast coloring. It also has several applications, such as, in resource replacement and biological diversity [3]. The term packing chromatic number
was introduced by Brešar [3]. The packing coloring is NP-complete for general graphs [10] and even for trees [6]. It is polynomial time solvable for graphs whose treewidth and diameter are both bounded [10] and it also holds for cographs, split graphs [10] and for the class of \((q, q - 4)\) graphs, partner limited graphs and for an infinite subclass of lobsters, including caterpillars [1, 7]. Sloper [14] studied a special type of packing coloring, called eccentric coloring and he proved that the infinite 3-regular tree has packing chromatic number 7. For an infinite planar square lattice \(\mathbb{Z}^2\), the lower bound is 10 [7] and the upper bound is 17 [11]. The packing coloring of distance graphs was studied by [5, 15]. The packing coloring for the infinite hexagonal lattice \(\mathbb{H}\) is \(\chi_\rho(\mathbb{H}) = 7\). The infinite planar triangular lattice and the three dimensional square lattice have unbounded packing chromatic number [12].

2. Preliminary

In this section, we give some known results of packing coloring.

**Proposition 1.** [10] \(\chi_\rho(C_n) = 4\) when \(n\) is not a multiple of 4.

**Proposition 2.** [10] Let \(H\) be a subgraph of \(G\). Then \(\chi_\rho(H) \leq \chi_\rho(G)\).

**Lemma 3.** [10] \(\chi_\rho(C_n) = \begin{cases} 3 & \text{when } n \text{ is a multiple of } 4 \\ 4 & \text{when } n \text{ is not a multiple of } 4 \end{cases}\)

**Theorem 4.** [16] Let \(G\) be a connected graph and \(L(G)\) be the line graph of \(G\). Then for a connected graph \(G, G \equiv L(G)\) if and only if \(G\) is a cycle.

3. Main Results

**Definition 5.** [9] The comb graph \(P_n \Theta K_1\) is the graph obtained from a path \(P_n\) by attaching pendent edges at each vertex of the path and is denoted by \(P_n^+\).

**Proposition 6.** For \(n \geq 8\), \(\chi_\rho(P_n^+) \leq 5\).
Definition 7. [4] The circular ladder graph $CL_n$ is the graph Cartesian product $L_n \times K_2$, where $K_2$ is the complete graph on two nodes and $C_n$ is the cycle graph on $n$ nodes. The graph $CL_n$ consists of two cycles namely top and bottom cycle. Figure 2 depicts the circular ladder graph $CL_6$.

Theorem 8. Let $CL_n$, $n \geq 6$ be a circular ladder. Then $\chi_p(CL_n) \leq 5$ for $n \equiv 0 \mod 6$.

Proof. Let the vertices in the top cycle of circular ladder be $u_1, u_2, ..., u_n$ and the vertices in bottom cycle be $v_1, v_2, ..., v_n$. Coloring the vertices $u_i$, $1 \leq i \leq n$, with the color sequence $314121,...$ and vertices $v_i, 1 \leq i \leq n$, with the color sequence $121315,...$ will color the circular ladder of length $n \equiv 0 \mod 6$ with 5 colors.

Definition 9. [13] An $H$ graph $H(r)$ is a 3-regular graph with vertex set $\{(i, j) : 1 \leq i \leq 3, 1 \leq j \leq n\}$, where $n$ is the number of vertices in each row of $H(r)$ and edge set $\{((i, j), (i, j + 1)) \cup (((2, j), (2, j + 1)) : j \text{odd}\} \cup \{((1, 1), (1, n)), ((3, 1), (3, n))\} \cup \{(i, j), (i + 1, j)) \cup \{((1, 1), (1, n)), ((3, 1), (3, n))\} \cup \{(i, j), (i + 1, j)) \cup \{((1, 1), (1, n)), ((3, 1), (3, n))\}$.

Theorem 10. Let $H(r), r \geq 4$ be a $H$ graph. Then $\chi_p(H(r)) \leq 5$ for $r$ is even.
Proof. Coloring the vertices \((1, j) : 1 \leq j \leq n\) with the color sequence 2131,..., and the vertices \((2, j) : 1 \leq j \leq n\) with color sequence 1415,... and vertices \((3, j) : 1 \leq j \leq n\), with the color sequence 3121,... will color the \(H\) graph with 5 colors.

![Figure 3: \(\chi_\rho(H(4)) = 5\)](image)

Definition 11. \cite{9} The windmill graph \(C_n^m\) is the family of graphs consisting of \(m\) copies of \(C_n\) with a vertex in common.

Theorem 12. \(\chi_\rho(C_n^m) = \begin{cases} 3, & \text{when } n \text{ is a multiple of 4} \\ 4, & \text{when } n \text{ is not a multiple of 4} \end{cases}\)

In this proof, \(m\), the number of copies of cycle does not play any role but \(n\), which is important in giving the length of the cycle \(C_n\). So that, we consider the following cases using with \(n\).

Proof. Case 1: \(n \equiv 0 \mod 4\). Let the vertex of degree \(2m\) be \(v_x\). Color each cycle of \((C_n^m)\) using the sequence 2131,... starting from vertex \(v_x\). This colors \((C_n^m)\) with 3 colors.
Case 2: $n \equiv 1, 2, 3 \mod 4$. Let any one of vertices of each cycle $C_n$ which is at distance 2 to $v_x$ be $v_y$. Color each cycle of $(C_n^m)$ using the sequence 3121,... starting from vertex $v_y$ with an adjustment at the very end [1]:

- $n = 4r + 1 : 3121, 3121, ..., 31214$
- $n = 4r + 2 : 3121, 3121, ..., 312141$
- $n = 4r + 1 : 3121, 3121, ..., 3121412$

This colors $(C_n^m)$ with 4 colors. By Proposition [2] and lemma [3], $\chi_\rho(C_n^m) = 4$.

Definition 13. [2] A generalized theta graph $\theta(s_1, s_2, ..., s_n)$ consists of a pair of end vertices joined by $n$ internally disjoint paths of lengths > 1, where $s_1, s_2, ..., s_n$ denote the number of internal vertices in the respective paths. We call the end vertices as North pole (N) and South pole (S). A path between the North Pole and South Pole is called a longitude and is denoted by $L$. Figure 5 shows a theta graph with four longitudes.

Figure 5: A generalized theta graph $\theta(2, 3, 1, 2)$
Theorem 14. Let $G = \theta(s_1, s_2, ..., s_n)$ be a generalized theta graph with $l$ longitudes $L_1, L_2, ..., L_l$, $l \geq 3$ and $s_i \geq 3$ for all $i$. Let $s_i + 1 \equiv 0 \mod 4, 1 \leq i \leq n - 1$ and $s_n \equiv 0,1,2 \mod 4$. Then $\chi_\rho(G) = 4$.

Proof. Let the vertices of longitudes $L_l$ from south to north pole be $t_1, t_2, ..., t_n$. For the upper bound, color the body of longitudes $L_i, 1 \leq i \leq l - 1$ with color-sequence 3121,... from south to north pole and for $L_l$, change the color of a vertices $t_n, t_{n-1}$ and $t_{n-2}$ to 4 when $s_l \equiv 0,1,2 \mod 4$ respectively. By proposition [2], $\chi_\rho(G = \theta(s_1, s_2, ..., s_n)) = 4$.

Figure 6: Generalized theta graphs with (a) $s_n \equiv 0 \mod 4$ (b)$s_n \equiv 1 \mod 4$ (c) $s_n \equiv 2 \mod 4$

\[\square\]

Definition 15. [2] A generalized theta graph $G$ with $l$ longitudes $L_1, L_2, ..., L_l$ is said to be quasi-uniform if $|L_1| = |L_2| = \cdots = |L_{l-1}| \geq |L_l|$.

Theorem 16. Let $G$ be a quasi uniform theta graph with $l$ longitudes $L_1, L_2, ..., L_l$, $l \geq 3$ and $s_i \geq 7$ for all $i$. Let $s_i + 1 \equiv 0 \mod 4, 1 \leq i \leq n - 1$ and $s_n \equiv 0,1,2 \mod 4$. Then $\chi_\rho(G) = 4$.

Definition 17. [2] A (two-terminal) series-parallel graph is defined recursively as follows:

1. A graph $G$ of a single edge is a series-parallel graph. The ends $v_s$ and $v_t$ of the edges are called the terminals of $G$ and are denoted by $v_s(G)$ and $v_t(G)$.

2. Let $G_1$ be a series-parallel graph with terminals $v_s(G_1)$ and $v_t(G_1)$ and let $G_2$ be series-parallel graph with terminals $v_s(G_2)$ and $v_t(G_2)$.

A graph obtained from $G_1$ and $G_2$ by identifying vertex $v_t(G_1)$ with vertex $v_s(G_2)$ is a series-parallel graph whose terminals are $v_s(G) = v_s(G_1)$ and
\( v_t(G) = v_t(G_2) \). Such a connection is called a series connection and \( G \) is denoted by \( G = G_1 \cdot G_2 \). See Figure 7.

\[ \begin{array}{c}
V_\delta(G) = V_\delta(G_1) \\
G_1 \\
V_t(G) = V_t(G_2) \\
G_2
\end{array} \]

Figure 7: Series connection \( G = G_1 \cdot G_2 \)

**Theorem 18.** Let \( G_1, G_2, \ldots, G_n \) be a quasi uniform theta graphs and each \( G_i \) has \( l \) longitudes \( L_1, L_2, \ldots, L_l \), \( l \geq 3 \) and \( s_i \geq 7 \) for all \( i \). Let \( s_i + 1 \equiv 0 \mod 4 \), \( 1 \leq i \leq n - 1 \) and \( s_n \equiv 0,1,2 \mod 4 \). Then \( \chi_\rho(G_1 \cdot G_2 \cdot \ldots \cdot G_n) = 4 \).

**Theorem 19.** For every connected graph \( G \), \( \chi_\rho(G) = \chi_\rho(L(G)) \) if and only if \( G \) is a cycle.

**Proof.** It is a direct Result from theorem [4]. \( \square \)

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References


