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# PACKING CHROMATIC NUMBER OF CERTAIN GRAPHS

Albert William[1], S. Roy[2]
[1,2]Department of Mathematics
Loyola College
Chennai 600 034, INDIA

**Abstract:** The packing chromatic number $\chi_\rho(G)$ of a graph $G$ is the smallest integer $k$ for which there exists a mapping $\Pi : V(G) \longrightarrow \{1, 2, ..., k\}$ such that any two vertices of color $i$ are at distance at least $i + 1$. It is a frequency assignment problem used in wireless networks, which is also called broadcasting coloring. It is proved that packing coloring is NP-complete for general graphs and even for trees. In this paper, we study the packing chromatic number of comb graph, circular ladder, windmill, H-graph and uniform theta graph.

**AMS Subject Classification:** 05C15
**Key Words:** packing chromatic number, comb graph, circular ladder, windmill, $H$-graph, uniform theta graph

## 1. Introduction

Let $G$ be a connected graph and $k$ be an integer, $k \geq 1$. A packing $k$-coloring of a graph $G$ is a mapping $\Pi : V(G) \longrightarrow \{1, 2, ..., k\}$ such that any two vertices of color $i$ are at distance at least $i + 1$. The packing chromatic number $\chi_\rho(G)$ of $G$ is the smallest integer $k$ for which $G$ has packing $k$-coloring [15]. The concept of packing coloring comes from the area of frequency assignment in wireless networks [6] and was introduced by Goddard et al. [10] under the name broadcast coloring. It also has several applications, such as, in resource replacement and biological diversity [3]. The term packing chromatic number

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was introduced by Brešar [3]. The packing coloring is NP-complete for general graphs [10] and even for trees [6]. It is polynomial time solvable for graphs whose treewidth and diameter are both bounded [10] and it also holds for cographs, split graphs [10] and for the class of $(q, q-4)$ graphs, partner limited graphs and for an infinite subclass of lobsters, including caterpillars [1, 7]. Sloper [14] studied a special type of packing coloring, called eccentric coloring and he proved that the infinite 3-regular tree has packing chromatic number 7. For an infinite planar square lattice $\mathbb{Z}^2$, the lower bound is 10 [7] and the upper bound is 17 [11]. The packing coloring of distance graphs was studied by [5, 15]. The packing coloring for the infinite hexagonal lattice $\mathbb{H}$ is $\chi_\rho(\mathbb{H}) = 7$. The infinite planar triangular lattice and the three dimensional square lattice have unbounded packing chromatic number [12].

## 2. Preliminary

In this section, we give some known results of packing coloring.

**Proposition 1.** *[10] $\chi_\rho(C_n) = 4$ when $n$ is not a multiple of 4.*

**Proposition 2.** *[10] Let $H$ be a subgraph of $G$. Then $\chi_\rho(H) \leq \chi_\rho(G)$.*

**Lemma 3.** *[10]* $\chi_\rho(C_n) = \begin{cases} 3 & \text{when } n \text{ is a multiple of 4} \\ 4 & \text{when } n \text{ is not a multiple of 4} \end{cases}$

**Theorem 4.** *[16] Let $G$ be a connected graph and $L(G)$ be the line graph of $G$. Then for a connected graph $G$, $G \equiv L(G)$ if and only if $G$ is a cycle.*

## 3. Main Results

**Definition 5.** [9] The comb graph $P_n \Theta K_1$ is the graph obtained from a path $P_n$ by attaching pendent edge at each vertex of the path and is denoted by $P_n^+$.

**Proposition 6.** *For $n \geq 8$, $\chi_\rho(P_n^+) \leq 5$.*

*Proof.* Let the vertices of path $P_n$ in $P_n^+$ be $u_1, u_2, ..., u_n$ and pendent vertices of $P_n^+$ be $v_1, v_2, ..., v_n$. Using the color-sequences 141315,... for the vertices $u_i$ and 21,... for $v_i$, any comb graph of length $n \geq 8$ can be colored with 5 colors.

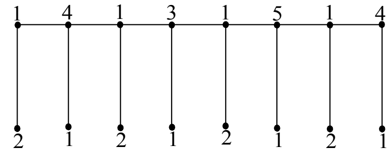


Figure 1: Comb graph $\chi_\rho(P_8^+) = 5$

□

**Definition 7.** [4] The circular ladder graph $CL_n$ is the graph Cartesian product $L_n \times K_2$, where $K_2$ is the complete graph on two nodes and $C_n$ is the cycle graph on $n$ nodes. The graph $CL_n$ consists of two cycles namely top and bottom cycle. Figure 2 depicts the circular ladder graph $CL_6$.

**Theorem 8.** *Let $CL_n, n \geq 6$ be a circular ladder. Then $\chi_\rho(CL_n) \leq 5$ for $n \equiv 0$ mod 6.*

*Proof.* Let the vertices in the top cycle of circular ladder be $u_1, u_2, ..., u_n$ and the vertices in bottom cycle be $v_1, v_2, ..., v_n$. Coloring the vertices $u_i$, $1 \leq i \leq n$, with the color sequence 314121,... and vertices $v_i$,$1 \leq i \leq n$, with the color sequence 121315,... will color the circular ladder of length $n \equiv 0$ *mod* 6 with 5 colors.

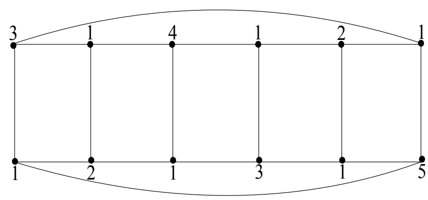


Figure 2:Circular ladder $CL_6$

□

**Definition 9.** [13] An $H$ graph $H(r)$ is a 3-regular graph with vertex set $\{(i,j) : 1 \leq i \leq 3, 1 \leq j \leq n$, where $n$ is the number of vertices in each row of $H(r)$ and edge set $\{((i,j),(i,j+1)), i = 1,3\} \cup \{((2,j),(2,j+1)) : jodd1 \leq j \leq n-1\} \cup \{((1,1),(1,n))),((3,1),(3,n))\} \cup ((i,j),(i+1,j)), i = 1,2, 1 \leq j \leq n.$

**Theorem 10.** *Let $H(r), r \geq 4$ be a $H$ graph. Then $\chi_\rho(H(r)) \leq 5$ for $r$ is even.*

*Proof.* Coloring the vertices $(1,j) : 1 \leq j \leq n$ with the color sequence 2131,...,and the vertices $(2,j) : 1 \leq j \leq n$ with color sequence 1415,... and vertices $(3,j) : 1 \leq j \leq n$, with the color sequence 3121,... will color the $H$ graph with 5 colors.

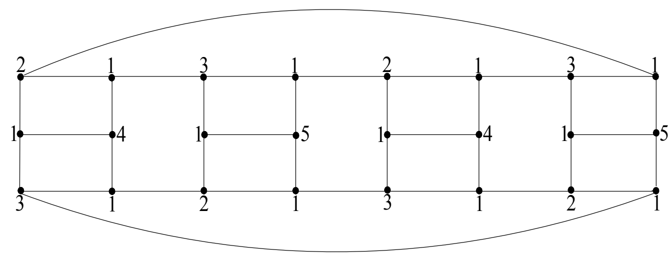


Figure 3: $\chi_\rho(H(4)) = 5$

□

**Definition 11.** [9] The windmill graph $C_n^m$ is the family of graphs consisting of $m$ copies of $C_n$ with a vertex in common.

**Theorem 12.** $\chi_\rho(C_n^m) = \begin{cases} 3, & \textit{when } n \textit{ is a multiple of 4} \\ 4, & \textit{when } n \textit{ is not a multiple of 4} \end{cases}$

In this proof, $m$, the number of copies of cycle does not play any role but $n$, which is important in giving the length of the cycle $C_n$. So that, we consider the following cases using with $n$.

*Proof.* **Case 1:** $n \equiv 0 \ mod \ 4$. Let the vertex of degree $2m$ be $v_x$. Color each cycle of $(C_n^m)$ using the sequence 2131,... starting from vertex $v_x$. This colors $(C_n^m)$ with 3 colors.

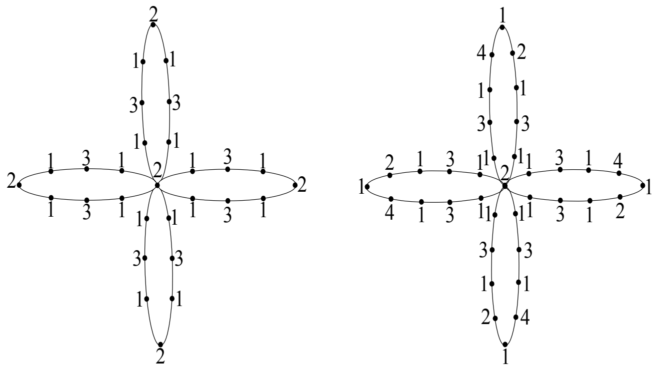

Figure 4: Windmill graph $\chi_\rho(C_8^4)$=3 and $\chi_\rho(C_{10}^4)$=4

**Case 2:** $n \equiv$ 1,2,3 *mod* 4. Let any one of vertices of each cycle $C_n$ which is at distance 2 to $v_x$ be $v_y$. Color each cycle of $(C_n^m)$ using the sequence 3121,... starting from vertex $v_y$ with an adjustment at the very end [1]:

$$n = 4r + 1 : 3121, 3121, ..., 31214$$
$$n = 4r + 2 : 3121, 3121, ..., 312141$$
$$n = 4r + 1 : 3121, 3121, ..., 3121412$$

This colors $(C_n^m)$ with 4 colors. By Proposition [2] and lemma [3], $\chi_\rho(C_n^m) = 4$. □

**Definition 13.** [2] A generalized theta graph $\theta(s_1, s_2, ..., s_n)$ consists of a pair of end vertices joined by $n$ internally disjoint paths of lengths $> 1$, where $s_1, s_2, ..., s_n$ denote the number of internal vertices in the respective paths. we call the end vertices as North pole (N)and South pole (S). A path between the North Pole and South Pole is called as a *longitude* and is denoted by $L$. Figure 5 shows a theta graph with four longitudes.

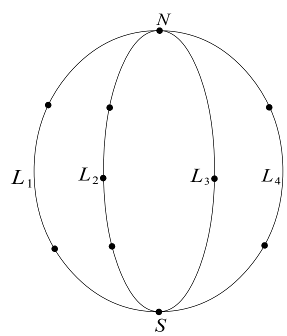


Figure 5: A generalized theta graph $\theta(2, 3, 1, 2)$

**Theorem 14.** *Let $G = \theta(s_1, s_2, ..., s_n)$ be a generalized theta graph with $l$ longitudes $L_1, L_2, ..., L_l, l \geq 3$ and $s_i \geq 3$ for all $i$. Let $s_i + 1 \equiv 0$ mod 4,$1 \leq i \leq n-1$ and $s_n \equiv$ 0,1,2 mod 4. Then $\chi_\rho(G) = 4$.*

*Proof.* Let the vertices of longitudes $L_l$ from south to north pole be $t_1, t_2, ...,$ $t_n$. For the upper bound, color the body of longitudes $L_i$,$1 \leq i \leq l-1$ with color-sequence 3121,...from south to north pole and for $L_l$, change the color of a vertices $t_n, t_{n-1}$ and $t_{n-2}$ to 4 when $s_l \equiv$ 0,1,2 *mod* 4 respectively. By proposition [2], $\chi_\rho(G = \theta(s_1, s_2, ..., s_n)) = 4$.

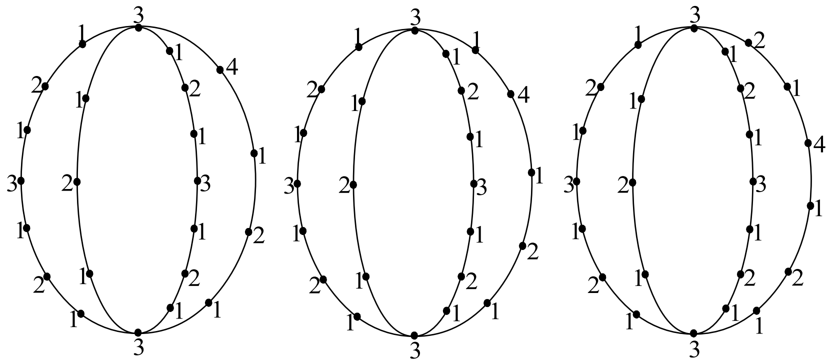

Figure 6: Generalized theta graphs with (a) $s_n \equiv 0$ *mod* 4 (b)$s_n \equiv 1$ *mod* 4 (c) $s_n \equiv 2$ *mod* 4

□

**Definition 15.** [2] A generalized theta graph $G$ with $l$ longitudes $L_1, L_2, ...,$ $L_l$ is said to be quasi-uniform if $|L_1| = |L_2| = \cdots = |L_{l-1}| \geq |L_l|$.

**Theorem 16.** *Let $G$ be a quasi uniform theta graph with $l$ longitudes $L_1, L_2, ..., L_l$, $l \geq 3$ and $s_i \geq 7$ for all $i$. Let $s_i + 1 \equiv 0$ mod 4, $1 \leq i \leq n-1$ and $s_n \equiv$ 0,1,2 mod 4. Then $\chi_\rho(G) = 4$.*

**Definition 17.** [2] A (two-terminal) series-parallel graph is defined recursively as follows:

1. A graph $G$ of a single edge is a series-parallel graph. The ends $v_s$ and $v_t$ of the edges are called the terminals of $G$ and are denoted by $v_s(G)$ and $v_t(G)$.
2. Let $G_1$ be a series-parallel graph with terminals $v_s(G_1)$ and $v_t(G_1)$ and let $G_2$ be series-parallel graph with terminals $v_s(G_2)$ and $v_t(G_2)$.

A graph obtained from $G_1$ and $G_2$ by identifying vertex $v_t(G_1)$ with vertex $v_s(G_2)$ is a series-parallel graph whose terminals are $v_s(G) = v_s(G_1)$ and

$v_t(G) = v_t(G_2)$. Such a connection is called a series connection and $G$ is denoted by $G = G_1 \bullet G_2$. See Figure 7.

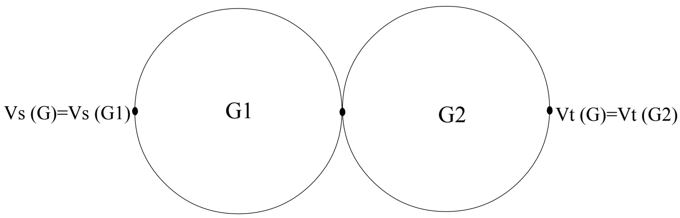


Figure 7: Series connection $G = G_1 \bullet G_2$

**Theorem 18.** *Let $G_1$,$G_2$,...,$G_n$ be a quasi uniform theta graphs and each $G_i$ has $l$ longitudes $L_1, L_2, ..., L_l$, $l \geq 3$ and $s_i \geq 7$ for all $i$. Let $s_i + 1 \equiv 0$ mod 4, $1 \leq i \leq n-1$ and $s_n \equiv$ 0,1,2 mod 4. Then $\chi_\rho(G_1 \bullet G_2 \bullet ... \bullet G_n) = 4$.*

**Theorem 19.** *For every connected graph $G$,$\chi_\rho(G) = \chi_\rho(L(G))$ if and only if $G$ is a cycle.*

*Proof.* It is a direct Result from theorem [4]. □

## 4. Acknowledgments

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