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ON CONTRA ρ-CONTINUITY AND ALMOST CONTRA ρ-CONTINUITY WHERE $\rho \in \{L, M, R, S\}$

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Abstract: In the year 2012, the authors introduced the concept of ρ-continuity and almost ρ-continuity between a topological space and a non empty set where $\rho \in \{L, M, R, S\}$. The purpose of this paper is to introduce the concepts of contra ρ-continuity and almost contra ρ-continuity between a topological space and a non empty set.

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1. Introduction

By a multifunction $F : X \to Y$, we mean a point to set correspondence from X into Y with $F(X) \neq \phi$ for all $x \in X$. Any function $f : X \to Y$ induces a multifunction $f^{-1} \circ f : X \to \wp(X)$. It also induces another multifunction $f \circ f^{-1} : Y \to \wp(Y)$ provided f is surjective. The notions of L-continuity, R-continuity and S-continuity of a function $f : X \to Y$ between a topological space and a non-empty set are studied by the authors[3, 4]. Further almost ρ-continuity has been investigated by the authors[5]. In this paper contra ρ-

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continuity and almost ρ-continuity are introduced and their basic properties are studied.

2. Preliminaries

The following definitions and results that are due to the authors Navpreet Singh Noorie and Rajni Bala[2] will be useful in sequel.

Definition 2.1. Let $f : X \to Y$ be any map and E be any subset of X. Then (1) $f^{\#}(E) = \{y \in Y : f^{-1} \subseteq E\}$; (2) $E^{\#} = f^{-1}(f^{\#}(E)$. [2]

Lemma 2.2. *Let E be a subset of X and let $f : X \to Y$ be a function. Then (1) $f^{\#}(E) = Y\ f(X/E)$; (2) $f(E) = Y\ f^{\#}(XE)$. [2]*

The next two lemmas are the consequences of the above Lemma.

Lemma 2.3. *Let E be a subset of X and let $f : X \to Y$ be a function. Then (1) $f^{-1}(f^{\#}(E)) = X/f^{-1}(f(X/E))$; (2) $f^{-1}(f(E)) = X/f^{-1}\left(f^{\#}(X/E)\right)$. [4]*

Lemma 2.4. *Let E be a subset of X and let $f : X \to Y$ be a function. Then (1) $f^{\#}(f^{-1}(E)) = Y/f(f^{-1}(Y/E))$; (2) $f(f^{-1}(E)) = Y/f^{\#}(f^{-1}(Y/E))$. [4]*

For, ρ-continuous, almost ρ-continuous and contra continuous functions, the reader can refer [3, 4, 5, 1].

3. Contra ρ-Continuity, where $\rho \in \{L, M, R, S\}$

Definition 3.1. Let $f : (X, \tau) \to Y$ be a function. Then f is contra L-continuous (resp. contra M-continuous) if $f^{-1}(f(A))$ is open (resp. closed)in X for every closed (resp. open) set A in X.

Definition 3.2. Let $f : X \to (Y, \sigma)$ be a function. Then f is contra R-continuous (resp. contra S-continuous) if $f(f^{-1}(B))$ is open (resp. closed) in Y for every closed (resp. open) set B in Y.

Theorem 3.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be open (resp.closed) and contra continuous. Then f is contra MR-continuous (resp. contra LS-continuous).*

Proof. Let $A \subseteq X$ be open in X. Since f is open (resp. closed), $f(A)$ is open (resp. closed) in Y. Again since f is contra continuous $f^{-1}(f(A))$ is closed (resp. open) in X. Therefore f is contra M-continuous (resp. contra L-continuous). Now let B be a closed (resp. open) subset of Y. Since f is contra continuous, $f^{-1}(B)$ is open (resp. closed) in X. Since f is open (resp. closed) $f(f^{-1}(B))$ is open (resp. closed) in Y. Therefore f is contra R-continuous (resp. contra S-continuous).This shows that f is contra MR-continuous (resp. contra LS-continuous). □

Corollary 3.4. *Let $f : (X, \tau) \to (Y, \sigma)$ be open, closed and contra continuous. Then f is contra ρ-continuous where $\rho \in \{L, M, R, S\}$.*

Proof. Follows from Theorem 3.3. □

Theorem 3.5. *Let $g : Y \to Z$ and $f : X \to Y$ be any two functions. Then the following hold.*

(1) Let f be closed (resp. open) and continuous. If g is contra L-continuous (resp. contra M-continuous) then $g \circ f : X \to Z$ is contra L-continuous (resp. contra M-continuous).

(2) Let g be open (resp. closed) and continuous. If f is contra R-continuous (resp. contra S-continuous) then $g \circ f : X \to Z$ is contra R-continuous (resp. contra S-continuous).

Proof. Suppose g is contra L-continuous (resp. contra M-continuous. Let f be closed (resp. open) and continuous. Let A be closed (resp. open) in X. Then

$$(g \circ f)^{-1}(g \circ f)(A) = f^{-1}(g^{-1}(g(f(A)))).$$

Since f is closed (resp. open), $f(A)$ is closed (resp. open) in Y. Since g is contra L-continuous, $g^{-1}(g(f(A)))$ is open (resp. closed) in Y. Since f is continuous, $f^{-1}(g^{-1}(g(f(A))))$ is open (resp. closed) in X. Therefore, $g \circ f$ is contra L-continuous (resp. contra M-continuous). This proves (1).

Let $f : X \to Y$ be contra R -continuous (resp. contra S-continuous) and $g : Y \to Z$ be open (resp. closed) and continuous. Let B be closed (resp. open) in Z. Then $(g \circ f)(g \circ f)^{-1}(B) = (g \circ f)(f^{-1}g{-}1(B)) = g(f(f^{-1}(g^{-1}(B))))$. Since g is continuous, $g^{-1}(B)$ is closed (resp. open) in Y. Since f is contra R-continuous (resp. contra S-continuous), $f(f^{-1}(g{-}1(B)))$ is open (resp. closed) in Y. Since g is open (resp. closed), $g(f(f^{-1}(g^{-1}(B))))$ is open (resp. closed) in Z. Therefore, $g \circ f$ is contra R-continuous (resp. contra S-continuous). This proves (2). □

Theorem 3.6. *Let $f : X \to Y$ be a function and A be a subset of X. Then the following hold.*

If $f : X \to Y$ is contra M-continuous (resp. contra L-continuous) and if A is an open (resp. closed) subspace of X then the restriction of f to A is contra M-continuous (resp. contra L-continuous).

Proof. Suppose $f : X \to Y$ is contra M-continuous (resp. contra L-continuous) and if A is an open (resp. closed) subspace of X. Let $h = f|_A$. Then $h = f \circ j$ where j is the inclusion map $j : A \to X$. Since A is open (resp. closed), j is open (resp. closed) and continuous. Since $f : X \to Y$ is contra M-continuous (resp. contra L-continuous), using Theorem 3.5(1), h is contra M-continuous (resp. contra L-continuous). □

Theorem 3.7. *Let $f : X \to Y$ be a function $f(X) \subseteq Z \subseteq Y$. Suppose $h : X \to Z$ is defined by $h(x) = f(x)$ for all $x \in X$. Then the following hold. If $f : X \to Y$ is contra R-continuous (resp. contra S-continuous) and $f(X)$ be open (resp.closed) in Z, then h is contra R-continuous (resp. contra S-continuous).*

Proof. By the Definition of h, we see that $h = j \circ f$ where $j : f(X) \to Z$ is an inclusion map. Suppose $f : X \to Y$ is contra R-continuous (resp.contra S-continuous) and $f(X)$ is open (resp.closed) in Z, that implies the inclusion map j is both open (resp.closed) and continuous. Then by applying Theorem 3.5(2), h is contra R-continuous (resp.contra S-continuous). □

4. Almost Contra ρ-Continuity

Definition 4.1. Let $f : (X, \tau) \to Y$ be a function. Then f is almost contra L-continuous (resp. almost contra M-continuous) if $f^{-1}(f(A))$ is open (resp. closed) in X for every regular closed (resp. open) set A in X.

Definition 4.2. Let $f : X \to (Y, \sigma)$ be a function. Then f is almost contra R-continuous (resp. almost contra S-continuous) if $f(f^{-1}(B))$ is open (resp. closed) in Y for every regular closed (resp. open) set B in Y.

It is clear that contra ρ-continuity $\Rightarrow$ almost contra ρ-continuity.

Theorem 4.3. *Let X be a topological space. If A is a regular closed (resp. regular open) subspace of X, the inclusion function $j : A \to X$ is almost contra L-continuous and almost contra R-continuous (resp. almost contra M-continuous and almost contra S-continuous).*

Proof. Let $j : A \to X$ be the inclusion function. Let $U \subseteq X$ be regular closed (resp. regular open) in X. Then $j(j^{-1}(U)) = j(U \cap A) = U \cap A$ which is open (resp. closed) in X. Hence j is almost contra R-continuous (resp. almost contra S-continuous). Now let $U \subseteq A$ be regular closed (resp.regular open) in A. Then $j^{-1}(j(U)) = j^{-1}(U) = U$ which is open (resp. closed) in A. Hence j is almost contra L-continuous (resp. almost contra M-continuous).This shows that j is almost contra LR-continuous. □

Theorem 4.4. *Let $g : Y \to Z$ and $f : X \to Y$ be any two functions. Then the following hold.*

(1) If g is almost contra L-continuous (resp. almost contra M-continuous) and let f be closed (resp. open) and continuous then $g \circ f$ is almost contra L-continuous (resp. almost contra M-continuous).

(2) If g is open (resp. closed) and almost continuous and f is contra R-continuous (resp. contra S-continuous), then $g \circ f$ is almost contra R-continuous (resp. almost contra S-continuous).

Proof. Suppose g is almost contra L-continuous (resp. almost contra M-continuous) and f is regular closed (resp. regular open) and continuous. Let A be regular closed (resp. regular open) in X. Then $(g \circ f)^{-1}(g \circ f)(A) = f^{-1}(g^{-1}(g(f(A))))$. Since f is regular closed (resp. regular open), $f(A)$ is regular closed (resp. regular open) in Y. Since g is almost contra L-continuous (resp. almost contra M-continuous), $g^{-1}(g(f(A)))$ is open (resp. closed) in Y. Since f is continuous, $f^{-1}(g^{-1}(g(f(A))))$ is open (resp. closed) in X. Therefore, $g \circ f$ is almost contra L-continuous (resp. almost contra M-continuous).This proves (1).

Let $f : X \to Y$ be contra R-continuous (resp. contra S-continuous) and $g : Y \to Z$ be open (resp. closed) and continuous. Let B be regular closed (resp. regular open) in Z. Then

$$(g \circ f)(g \circ f)^{-1}(B) = (g \circ f)(f^{-1}g^{-1}(B)) = g(f(f^{-1}(g^{-1}(B)))).$$

Since g is almost continuous, g^{-1}(B) is closed (resp. open) in Y. Since f is contra R-continuous (resp. contra S-continuous), $ff^{-1}(g^{-1}(B))$ is open (resp. closed) in Y. Since g is open (resp. closed) $g(f(f^{-1}(g^{-1}(B))))$ is open (resp. closed) in Z. Therefore, $g \circ f$ is almost contra R-continuous(almost contra S-continuous). This proves (2). □

We establish the pasting Lemmas for contra R-continuous, contra S-continuous, almost contra R-continuous and almost contra S-continuous functions.

Theorem 4.5. *Let $X = A \cup B$. Let $f : A \to (Y, \sigma)$ and $g : B \to (Y, \sigma)$ be contra R-continuous (resp. contra S-continuous) functions. If $f(x) = g(x)$ for every $x \in A \cap B$, the function $h : X \to Y$ defined by*

$$h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$$

is contra R-continuous (resp. contra S-continuous).

Proof. Let C be a open (resp. closed) set in Y. Now

$$\begin{aligned} h \circ h^{-1} &= h(f^{-1}(c) \cup g^{-1}(c)) \\ &= h(f^{-1}(c)) \cup h(g^{-1}(c)) \\ &= f(f^{-1}(c)) \cup g(g^{-1}(c)). \end{aligned}$$

Since f is contra R-continuous (resp. contra S-continuous), $f(f^{-1}(C))$ is open (resp. closed) in Y and since g is contra R-continuous (resp. contra S-continuous), $g(g^{-1}(C))$ is open (resp. closed) in Y. Therefore, $h \circ h^{-1}(C)$ is also open (resp. closed) in Y. This shows that h is contra R-continuous (resp. contra S-continuous). □

Theorem 4.6. *Let $X = A \cup B$. Let $f : A \to (Y, \sigma)$ and $g : B \to (Y, \sigma)$ be almost contra R-continuous (resp. almost contra S-continuous)functions. If $f(x) = g(x)$ for every $x \in A \cap B$, the function $h : X \to Y$ defined by*

$$h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$$

is almost contra R-continuous (resp. almost contra S-continuous).

Proof. Let C be a regular open (resp. closed) set in Y. Now

$$\begin{aligned} h \circ h^{-1} &= h(f^{-1}(c) \cup g^{-1}(c)) \\ &= h(f^{-1}(c)) \cup h(g^{-1}(c)) \\ &= f(f^{-1}(c)) \cup g(g^{-1}(c)) \end{aligned}$$

Since f is almost contra R-continuous (resp. almost contra S-continuous), $f(f^{-1}(C))$ is open (resp. closed) in Y and since g is almost contra R-continuous (resp. almost contra S-continuous), $g(g^{-1}(C))$ is open (resp. closed) in Y. Therefore, $h \circ h^{-1}(C)$ is also open (resp. closed) in Y. This shows that h is almost contra R-continuous (resp. almost contra S-continuous). □

5. Characterizations

In this section, we characterize contra ρ-continuity and almost contra ρ-continuity functions by the hash function $f^{\#}$ of $f : X \to Y$.

Theorem 5.1. *The function $f : X \to Y$ is contra L-continuous if and only if $f^{-1}(f^{\#}(A))$ is closed in X for every open subset G of X.*

Proof. Suppose f is contra L-continuous. Let G be open in X. Then $A = X/G$ is closed in X. By Lemma 2.3(1) $f^{-1}(f^{\#}(G)) = X/f^{-1}(f(A))$. Since f is contra L- continuous, and since A is closed in X, $f^{-1}(f(A))$ is open in X. Hence $f^{-1}(f^{\#}(G))$ is closed in X.

Conversely, assume that $f^{-1}(f^{\#}(G))$ is closed in X for every open subset G of X. Let A be closed in X. By Lemma 2.3(2), $f^{-1}(f(A)) = X/f^{-1}(f^{\#}(G))$ where $G = X/A$. By our assumption, $f^{-1}(f^{\#}(G))$ is closed and hence $f^{-1}(f(G))$ is open in X. Therefore f is contra L-continuous. □

Theorem 5.2. *The function $f : X \to Y$ is contra M-continuous if and only if $f^{-1}(f^{\#}(A))$ is open in X for every closed subset A of X.*

Proof. Suppose f is contra M-continuous. Let A be closed in X. Then $G = X/A$ is open in X. By Lemma 2.3(1), $f^{-1}(f^{\#}(A)) = X/f^{-1}(f(G))$. Since f is contra M-continuous and since G is open in X, $f^{-1}(f(G))$ is closed in X. Hence $f^{-1}(f^{\#}(A))$ is open in X.

Conversely, assume that $f^{-1}(f^{\#}(A))$ is open in X for every closed subset G of X. Let G be open in X. By Lemma 2.3(2), $f^{-1}(f(G)) = X/f^{-1}(f^{\#}(A))$ where $A = X/G$. By our assumption, $f^{-1}(f^{\#}(A))$ is open and hence $f^{-1}(f(G))$ is closed in X. Therefore f is contra M-continuous. □

Theorem 5.3. *The function $f : X \to Y$ is contra R-continuous if and only if $f^{\#}(f^{-1}(G))$ is closed in Y for every open subset G of Y.*

Proof. Suppose f is contra R-continuous. Let G be open in Y. Then $A = Y/G$ is closed in Y. By Lemma 2.4(1), $f^{\#}(f^{-1}(G)) = Y/f(f^{-1}(A))$. Since f is contra R-continuous and since A is closed in Y, $f(f^{-1}(A))$ is open in Y. Hence $f^{\#}(f^{-1}(A))$ is closed in Y.

Conversely, assume that $f^{\#}(f^{-1}(A))$ is closed in Y for every open subset G of Y. Let A be closed in Y. By Lemma 2.4(2), $f(f^{-1}(A)) = Y/f^{\#}(f^{-1}(G))$ where $G = Y/A$. By our assumption, $f(f^{-1}(G))$ is closed and hence $f(f^{-1}(A))$ is open in Y. Therefore f is contra R-continuous. □

Theorem 5.4. *The function $f : X \to Y$ is contra S-continuous if and only if $f^{\#}(f^{-1}(A))$ is open in Y for every closed subset A of Y.*

Proof. Suppose f is contra S-continuous. Let A be open in Y. Then $G = Y/A$ is open in Y. By Lemma 2.4(1), $f^{\#}(f^{-1}(A)) = X/f(f^{-1}(G))$. Since f is contra S-continuous and since G is open in Y, $ff^{-1}(G))$ is closed in Y. Hence $f^{\#}(f^{-1}(A))$ is open in Y.

Conversely, assume that $f^{\#}(f^{-1}(A))$ is open in Y for every closed subset A of Y. Let G be open in Y. By Lemma 2.4(2), $f(f^{-1}(G)) = X/f^{\#}(f^{-1}(A))$ where $A = Y/G$. By our assumption, $f^{\#}(f^{-1}(A))$ is open in Y and hence $f(f^{-1}(G))$ is closed in Y. Therefore f is contra S-continuous. □

Theorem 5.5. *The function $f : X \to Y$ is almost contra L-continuous if and only if $f^{-1}(f^{\#}(A))$ is closed in X for every open subset G of X.*

Proof. Suppose f is almost contra L-continuous. Let G be regular open in X. Then $A = X/G$ is regular closed in X. By Lemma 2.3(1), $f^{-1}(f^{\#}(G)) = X/f^{-1}(f(A))$. Since f is almost contra L-continuous and since A is regular closed in X, $f^{-1}(f(A))$ is open in X. Hence $f^{-1}(f^{\#}(G))$ is closed in X.

Conversely, assume that $f^{-1}(f^{\#}(G))$ is closed in X for every open subset G of X. Let A be regular closed in X. By Lemma 2.3(2), $f^{-1}(f(A)) = X/f^{-1}(f^{\#}(G))$ where $G = X/A$. By our assumption, $f^{-1}(f^{\#}(G))$ is closed and hence $f^{-1}(f(G))$ is open in X. Therefore f is almost contra L-continuous. □

Theorem 5.6. *The function $f : X \to Y$ is almost contra M-continuous if and only if $f^{-1}(f^{\#}(A))$ is open in X for every regular closed subset A of X.*

Proof. Suppose f is almost contra M-continuous. Let A be regular closed in X. Then $G = X/A$ is regular open in X. By Lemma 2.3(1), $f^{-1}(f^{\#}(A)) = X/f^{-1}(f(G))$. Since f is almost contra M-continuous and since G is regular open in X, $f^{-1}(f(G))$ is closed in X. Hence $f^{-1}(f^{\#}(A))$ is open in X.

Conversely, assume that $f^{-1}(f^{\#}(A))$ is open in X for every regular closed subset A of X. Let G be regular open in X. By Lemma 2.3(2), $f^{-1}(f(G)) = X/f^{-1}(f^{\#}(A))$ where $A = X/G$. By our assumption, $f^{-1}(f^{\#}(A))$ is open in X. And hence $f^{-1}(f(G))$ is closed in X. Therefore f is almost contra M-continous. □

Theorem 5.7. *The function $f : X \to Y$ is almost contra R-continuous if and only if $f^{\#}(f^{-1}(G))$ is closed in Y for every regular open subset G of Y.*

Proof. Suppose f is almost contra R-continuous. Let G be regular open in Y. Then $A = Y/G$ is regular closed in Y. By Lemma 2.4(1), $f^{\#}(f^{-1}(G)) = Y/f(f^{-1}(A))$. Since f is almost contra R-continuous and since A is regular closed in Y, $f(f^{-1}(A))$ is open in Y. Hence $f^{\#}(f^{-1}(A))$ is closed in Y. Conversely assume that $f^{\#}(f^{-1}(G))$ is closed in Y for every regular open subset G of Y. Let A be regular closed in Y. By Lemma 2.4(2), $f(f^{-1}(A)) = Y/f^{\#}(f^{-1}(G))$ where $G = Y/A$. By our assumption , $f^{\#}(f^{-1}(G))$ is closed and hence $f(f^{-1}(A))$ is open in Y. Therefore f is almost contra R-continuous. □

Theorem 5.8. *The function $f : X \to Y$ is almost contra S-continuous if and only if $f^{\#}(f^{-1}(A))$ is open in Y for every regular closed subset A of Y.*

Proof. Suppose f is almost contra S-continuous. Let A be regular closed in Y. Then $G = Y/A$ is regular open in Y. By Lemma 2.3(1), $f^{\#}(f^{-1}(A)) = X/f(f^{-1}(G))$. Since f is almost contra S-continuous and since G is regular open in Y, $f(f^{-1}(G))$ is closed in Y. Hence $f^{\#}(f^{-1}(A))$ is open in Y.

Conversely assume that $f^{\#}(f^{-1}(A))$ is open in Y for every regular closed subset A of Y. Let G be regular open in Y. By Lemma 2.3(2), $f(f^{-1}(G)) = X/f^{\#}f^{-1}(A)$ where $A = Y/G$. By our assumption, $f^{\#}(f^{-1}(A))$ is open in Y and hence $f(f^{-1}(G))$ is closed in Y. Therefore f is almost contra S-continuous. □

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