

LOCALLY AND WEAKLY PGPR-CLOSED SETS

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Abstract: In the year 2005, the authors introduced and studied the concept of *pgpr*-closed sets in topological spaces. The purpose of this paper is to introduce locally and weakly *pgpr*-closed sets and investigate their basic properties.

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1. Introduction

The notion of locally closed sets was introduced by Bourbaki. Several mathematicians generalized this notion by replacing open sets with nearly open sets and generalized open sets and/or by replacing closed sets with nearly closed sets and generalized closed sets. In this paper we introduce locally *pgpr*-closed sets and we study their relations with other locally closed sets. Throughout this chapter, X and Y represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a topological space X , $Cl(A)$ and $Int(A)$ denote the closure of A and the interior of A in X

respectively; $X - A$ denotes the complement of A in X . In this paper, A and B denote the subsets of X unless otherwise specified. The pre-closure of a subset A of X is the intersection of all pre-closed sets containing A and is denoted by $pCl(A)$. The pre-interior of a subset A of X is the union of all pre-open sets contained in A and is denoted by $pInt(A)$.

2. Preliminaries

Definition 2.1. [9, 7]

- (1) A is regular open if $A = Int(Cl(A))$ and regular closed if $A = Cl(Int(A))$.
- (2) A is pre-open if $A \subseteq Int(Cl(A))$ and pre-closed if $Cl(Int(A)) \subseteq A$.

Definition 2.2. [6, 8]

- (1) A is generalized closed (briefly g -closed) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .
- (2) B is regular generalized closed (briefly rg -closed) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X .

Definition 2.3. [6] A space X is $T_{1/2}$ if every g -closed is closed.

Definition 2.4. [5, 4] A subset A of (X, τ) is called a

- (1) locally closed set if $A = U \cap F$ where $U \in \tau$ and F is closed in (X, τ) .
- (2) generalized locally closed set (briefly glc -set) if $A = G \cap F$ where G is g -open in (X, τ) and F is g -closed in (X, τ) .
- (3) glc^* -set if there exists a g -open set G and a closed set F of (X, τ) such that $A = G \cap F$.
- (4) glc^{**} -set if there exist an open set U and a g -closed set F of (X, τ) such that $A = U \cap F$.

Definition 2.5. [1] A subset B of a space X is called pre-generalized pre-regular-closed (briefly $pgpr$ -closed) if $pCl(B) \subseteq U$ whenever $B \subseteq U$ and U is rg -open.

Definition 2.6. [1] For a subset A of (X, τ) , $pgpr-Cl(A) = \bigcap \{F : A \subseteq F \text{ and } F \text{ is } pgpr\text{-closed in } X\}$ is called the $pgpr$ -closure of A

Definition 2.7. Let A be a subset of (X, τ) . Then A is called locally *pgpr*-closed if there exist an open set U and a *pgpr*-closed set F of X such that $A = U \cap F$. The collection of all locally *pgpr*-closed sets is denoted by $LPGPRC(X, \tau)$. It is easy to prove that the class of all locally closed sets is contained in the class of all locally *pgpr*-closed sets.

Theorem 2.8. For a topological space (X, τ) ,

$$LC(X, \tau) \subseteq LPGPRC(X, \tau).$$

Theorem 2.9. For a submaximal space (X, τ) ,

$$LPGPRC(X, \tau) \subseteq GLC(X, \tau).$$

Proof. Let $A \in LPGPRC(X, \tau)$. Then there exist an open set U and a *pgpr*-closed set F of X such that $A = U \cap F$. In a submaximal space X , every *pgpr*-closed set is g -closed. Therefore F is g -closed. Since every open set is g -open, it follows that A is an intersection of g -open set U and a g -closed set F of X . Therefore, $A \in GLC(X, \tau)$. This proves the theorem. \square

Theorem 2.10. If X is $T_{1/2}$, then

$$LC(X, \tau) = GLC^{**}(X, \tau) = GLC(X, \tau) = GLC^*(X, \tau) \subseteq LPGPRC(X, \tau).$$

Proof. Let X be $T_{1/2}$. Since every g -closed set is closed in a $T_{1/2}$ space, $LC(X, \tau) = GLC^{**}(X, \tau) = GLC(X, \tau) = GLC^*(X, \tau)$. Also, every closed set is *pgpr*-closed, we have $GLC^*(X, \tau) \subseteq LPGPRC(X, \tau)$. Therefore, $LC(X, \tau) = GLC^{**}(X, \tau) = GLC(X, \tau) = GLC^*(X, \tau) \subseteq LPGPRC(X, \tau)$. \square

Definition 2.11. A space is said to have the *pgpr*-closure preserving property if *pgpr*- $Cl(A)$ is always *pgpr*-closed.

Theorem 2.12. Suppose X has the *pgpr*-closure preserving property and let A be a subset of (X, τ) . Then $A \in LPGPRC(X, \tau)$ if and only if $A = U \cap pgpr - Cl(A)$ for some open set U .

Proof. Let $A \in LPGPRC(X, \tau)$. Then $A = U \cap F$ where U is open and F is *pgpr*-closed. By Definition 2.11, *pgpr*- $Cl(A)$ is *pgpr*-closed in X , $A \subseteq F$ implies *pgpr*- $Cl(A) \subseteq F$. Now, $A = A \cap pgpr - Cl(A) = U \cap F \cap pgpr - Cl(A) = U \cap pgpr - Cl(A)$. Therefore, $A = U \cap pgpr - Cl(A)$ for some open set U . Conversely, assume that $A = U \cap pgpr - Cl(A)$ for some open set U . By Definition 2.11, *pgpr*- $Cl(A)$ is *pgpr*-closed in X . Therefore, $A \in LPGPRC(X, \tau)$. This proves the theorem. \square

Definition 2.13. Let A be a subset of (X, τ) . Then A is called *pgpr*-locally closed if there exist a *pgpr*-open set U and a *pgpr*-closed set F of X such that $A = U \cap F$.

Definition 2.14. Let A be a subset of (X, τ) . Then A is called *pgpr**-locally closed if there exist a *pgpr*-open set U and a closed set F of X such that $A = U \cap F$. The collection of all *pgpr*-locally closed sets *PGPRLC* sets of (X, τ) will be denoted by $PGPRLC(X, \tau)$. Similarly the collection of all *pgpr**-locally closed sets of (X, τ) will be denoted by $PGPRLC^*(X, \tau)$.

Proposition 2.15. For a topological space (X, τ) ,

$$LC(X, \tau) \subseteq PGPRLC^*(X, \tau) \subseteq PGPRLC(X, \tau).$$

Proof. Follows from the fact that every open set is *pgpr*-open and every closed set is *pgpr*-closed. $PGPRLC^*(X, \tau)$ and $LPGPRC(X, \tau)$ are independent of each other as seen in the next example. \square

Example 2.16. Let $X = \{a, b, c\}$ and $\tau_2 = \{\phi, \{a\}, \{a, b\}, X\}$. Then $\{a, c\}$ is *pgpr**-locally closed set but not a locally *pgpr*-closed set. Let $X = \{a, b, c, d\}$ and $\tau_8 = \{\phi, \{a\}, \{a, b, c\}, X\}$. Then $\{b, c\}$ is a locally *pgpr*-closed set but not a *pgpr**-locally closed set.

Theorem 2.17. For a sub maximal space (X, τ) , $PGPRLC^*(X, \tau) \subseteq PGPRLC(X, \tau) \subseteq GLC(X, \tau)$.

Proof. Follows from the fact that in a sub maximal space every *pgpr*-closed set is *g*-closed. \square

Theorem 2.18. If X is $T_{1/2}$, then

$$(1) \quad GLC^{**}(X, \tau) = GLC(X, \tau) = GLC^*(X, \tau) \subseteq PGPRLC(X, \tau).$$

$$(2) \quad GLC^{**}(X, \tau) = GLC(X, \tau) = GLC^*(X, \tau) \subseteq PGPRLC^*(X, \tau).$$

Proof. Let X be $T_{1/2}$. Since every *g*-closed set is closed in a $T_{1/2}$ space, $LC(X, \tau) = GLC^{**}(X, \tau) = GLC(X, \tau) = GLC^*(X, \tau)$. Since every closed set is *pgpr*-closed, we have $GLC^*(X, \tau) \subseteq LPGPRC(X, \tau)$. Therefore, $LC(X, \tau) = GLC^{**}(X, \tau) = GLC(X, \tau) = GLC^*(X, \tau) \subseteq PGPRLC(X, \tau)$. This proves (1). The proof for (2) is similar. \square

Theorem 2.19. Suppose a space X has the *pgpr*-closure preserving property. Then the following are equivalent.

$$(1) \quad A \in PGPRLC(X, \tau)$$

(2) $A = U \cap pgpr - Cl(A)$ for some $pgpr$ -open set U .

Proof. (1) \Rightarrow (2): Let $A \in PGPRLC(X, \tau)$. Then there exist a $pgpr$ -open subset U and a $pgpr$ -closed subset F such that $A = U \cap F$. Since $A \subseteq U$ and since $A \subseteq pgpr - Cl(A)$, $A \subseteq U \cap pgpr - Cl(A)$. By Definition 2.11, $pgpr - Cl(A)$ is $pgpr$ -closed, $pgpr - Cl(A) \subseteq F$ and hence $U \cap pgpr - Cl(A) \subseteq U \cap F = A$. Therefore, $A = U \cap pgpr - Cl(A)$. This proves (2).

(2) \Rightarrow (1): By Definition 2.11, $pgpr - Cl(A)$ is $pgpr$ -closed. Therefore, $A = U \cap pgpr - Cl(A) \in PGPRLC(X, \tau)$. □

Theorem 2.20. For a subset A of (X, τ) , then the following are equivalent

- (1) $A \in PGLC^*(X, \tau)$
- (2) $A = U \cap Cl(A)$ for some $pgpr$ -open set U .
- (3) $A \cup (X - Cl(A))$ is $pgpr$ -open.
- (4) $Cl(A) - A$ is $pgpr$ -closed.

Proof. (1) \Rightarrow (2) : Let $A \in PGPRLC^*(X, \tau)$. Then there exists a $pgpr$ -open set U and a closed set F such that $A = U \cap F$. Since $A \subseteq U$ and $A \subseteq Cl(A)$, we have $A \subseteq U \cap Cl(A)$. Since $Cl(A) \subseteq F$, $U \cap Cl(A) \subseteq U \cap F = A$ which implies $A = U \cap Cl(A)$.

(2) \Rightarrow (1) : Since U is $pgpr$ -open and $Cl(A)$ is closed, $U \cap Cl(A) \in PGPRLC^*(X, \tau)$.

(2) \Rightarrow (3) : Let $A = U \cap Cl(A)$ for some $pgpr$ -open set U . Then we prove that $A \cup (X - Cl(A))$ is $pgpr$ -open. $A \cup (X - Cl(A)) = U \cap Cl(A) \cup X - Cl(A) = U \cap Cl(A) \cup (X - Cl(A)) = U \cap X = U$ which is $pgpr$ -open. Thus, $A \cup (X - Cl(A))$ is $pgpr$ -open.

(3) \Rightarrow (4) : Let $U = A \cup (X - Cl(A))$. Then U is $pgpr$ -open. This implies that $X - U$ is $pgpr$ -closed and $X - U = X - (A \cup (X - Cl(A))) = Cl(A) \cap (X - A) = Cl(A) - A$. Thus, $Cl(A) - A$ is $pgpr$ -closed.

(4) \Rightarrow (2) : Let $F = Cl(A) - A$. Then F is $pgpr$ -closed by the assumption and $X - F = X \cup (X - (Cl(A) - A)) = A \cup (X - Cl(A))$. But $X - F$ is $pgpr$ -open, this shows that $A \cup X - Cl(A)$ is $pgpr$ -open. □

Definition 2.21. A subset A of X is called weakly $pgpr$ -closed if

$$pCl(pInt(A)) \subset G \text{ whenever } A \subset G \text{ and } G \text{ is } rg\text{-open in } X.$$

Every $pgpr$ -closed set is weakly $pgpr$ -closed. Examples can be constructed to show that the converse is not true.

Theorem 2.22. *If a subset A of X is weakly $pgpr$ -closed then $pCl(pInt(A)) - A$ contains no non-empty rg -closed set.*

Proof. Suppose A is weakly $pgpr$ -closed in X . Let F be a rg -closed set such that $F \subseteq pCl(pInt(A)) - A$. Since $X - F$ is rg -open and $A \subseteq X - F$, from the Definition 2.11, $pCl(pInt(A)) \subseteq X - F$. That is, $F \subseteq X - (pCl(pInt(A)))$ that implies $F \subseteq pCl(pInt(A)) \cap (X - (pCl(pInt(A)))) = \phi$. This proves the theorem. \square

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