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ON THE ACHROMATIC NUMBER OF SILICATE NETWORKS

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Abstract: The silicates are the largest, the most interesting and the most complicated class of minerals by far. The basic chemical unit of silicates is the (SiO_4) tetrahedron. A silicate sheet is a ring of tetrahedrons which are linked by shared oxygen nodes to other rings in a two dimensional plane that produces a sheet-like structure. We consider the silicate sheet as a fixed interconnection parallel architecture and call it a silicate network. The achromatic number for a graph $G= (V, E)$ is the largest integer m such that there is a partition of V into disjoint independent sets $(V_1, ..., V_m)$ satisfying the condition that for each pair of distinct sets V_i , V_j , $V_i \cup V_j$ is not an independent set in G. In this paper, we determine an approximation algorithm for the achromatic number of Silicate Network which is NP complete even for trees.

AMS Subject Classification: 05C15, 05C85
Key Words: silicate networks, achromatic number, approximation algorithms, NP-completeness, graph algorithms

1. Introduction

A *proper* coloring of a graph G is an assignment of colors to the vertices of G such that adjacent vertices are assigned different colors. A proper coloring of a graph G is said to be *complete* if for every pair of colors i and j there are adjacent vertices u and v colored i and j, respectively. The *achromatic number*

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of the graph G is the largest number m such that G has a complete coloring with m colors. Thus the achromatic number for a graph $G = (V, E)$ is the largest integer m such that there is a partition of V into disjoint independent sets $(V_1, ..., V_m)$ such that for each pair of distinct sets V_i , V_j , $V_i \cup V_j$ is not an independent set in G.

Graph coloring problem is expected to have wide variety of applications such as scheduling, frequency assignment in cellular networks, timetabling, crew assignment etc. Scheduling problems with inter-processor communication delays are recent problems arising with the development of new message-passing architectures whose number of processors is increasing more and more. Small Communication Time task systems show that the achromatic number of the co-comparability graph is an upper bound on the minimum number of processors [17].

Multiprocessor interconnection networks are often required to connect thousands of homogeneously replicated processor-memory pairs, each of which is called a *processing node*. Instead of using a shared memory, all synchronization and communication between processing nodes for program execution is often done via message passing. Design and use of multiprocessor interconnection networks have recently drawn considerable attention due to the availability of inexpensive, powerful microprocessors and memory chips. The homogeneity of processing nodes and the interconnection network is very important because it allows for cost/performance benefits from the inexpensive replication of multiprocessor components [18].

The silicates are the largest, the most interesting and the most complicated class of minerals by far. The basic chemical unit of silicates is the (SiO_4) tetrahedron. Silicates are obtained by fusing metal oxides or metal carbonates with sand. Essentially all the silicates contain SiO_4 tetrahedra. In chemistry, the corner vertices of SiO_4 tetrahedron represent oxygen ions and the center vertex represents the silicon ion. In graph theory, we call the corner vertices as oxygen nodes and the center vertex as silicon node [18]. See Figure 1.

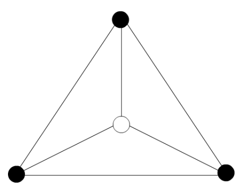

Figure 1: SiO_4 tetrahedra where the corner vertices represent oxygen ions and the center vertex the silicon ion

The minerals are obtained by successively fusing oxygen nodes of two tetrahedra of different silicates. The different types of silicate structure arise from the ways in which these tetrahedra are arranged: they may exist as separate unlinked entities, as linked finite arrays, as 1-dimensional chains, as 2-dimensional sheets or as 3-dimensional frameworks [18].

2. Overview of the Paper

The achromatic number was introduced by Harary, Hedetniemi and Prins [10, 11]. The survey articles by Hughes and MacGillivray [12] and Edwards [5] contain huge collection of references of research papers related to achromatic problem.

Computing achromatic number of a general graph was proved to be NP-complete by Yannakakis and Gavril [21]. A simple proof of this fact appears in [9]. Farber et al. [6] show that the problem is NP- hard on bipartite graphs. It was further proved that the achromatic number problem remains NP- complete even for connected graphs which are both interval graphs and cographs simultaneously [1]. Cairnie and Edwards [2], Manlove and McDiarmid [15] show that the problem is NP- hard even on trees. Further it is polynomially solvable for paths, cycles [5], bipartite graphs [14] and union of paths [16].

Since achromatic optimization problem is NP-hard, most of the research studies related to achromatic problem focus on approximation algorithms. An *approximation algorithm* for a problem, loosely speaking, is an algorithm that runs in polynomial time and produces an *approximate solution* to the problem. We say that an algorithm is α-approximation algorithm for a maximization problem if it always delivers a solution whose value is at least a factor $\frac{1}{\alpha}$ of the optimum. The parameter α is called the *approximation ratio* [7].

Let n denote the number of vertices in the input graph G and let $\psi(G)$ be the achromatic number of G. It is conjectured in [3] that the achromatic number on general graphs admits an $O(\sqrt{n})$ approximation. Chaudhary and Vishwanathan [4] realize an algorithm for trees with a constant approximation ratio 7. For general graphs an algorithm that approximates the achromatic number within ratio of $O(\frac{(n.loglogn)}{(logn)})$ is given in [13].

It is stated in [5] that "for achromatic numbers, there appear to be only a few results on special graphs apart from those for paths and cycles". Geller and Kronk [8] proved that there is almost optimal coloring for families of paths and cycles [5, 12]. This result was extended to bounded trees [2]. Also they generalized this result to all families of bounded degree graphs. Roichman, gives the achromatic number of Hypercubes. Indra et al. found Approximation

Algorithm for the achromatic number of mesh like topologies [23], Butterfly and Benes network [22]. Also [24] gives for Circulant networks. In this paper, we determine an approximation algorithm for the achromatic number of silicate networks.

3. Main Results

Let $G = (V, E)$ be a simple graph. A *block* of the graph G is a maximal connected subgraph of G that has no cut-vertex. If G itself is connected and has no cut-vertex, then G is a block. The block-cut-point graph of G is a path.

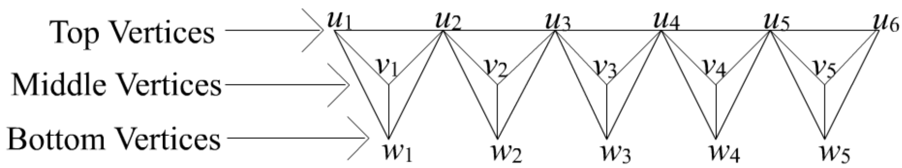

Figure 2: $5K_4$-chain

A K_4-*chain* is a connected graph in which all blocks are complete graph on 4 vertices and the block-cut-point graph is a path P. It is also obtained from a path $u_1, u_2, ..., u_n$ namely the top vertices by joining u_i and $u_{(i+1)}$ to the new vertices namely the middle vertex v_i and the bottom vertex w_i for $i = 1, 2, ..., n-1$. See Figure 2.

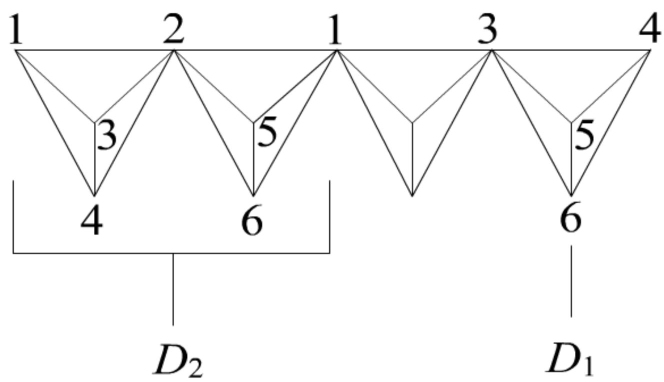

Figure 3: Achromatic labeling of $4K_4$-chain

Algorithm Achromatic tK_4-Chain

Theorem 3.1. *For $t \geq 1$, $\psi(tK_4 - Chain) \geq 2k + 2$.*

Proof. Let k be an integer such that $\frac{(k(k+1))}{2} \leq t \leq \frac{(k+1)(k+2)}{2}$. Partition the set of $tK_4's$ into $D_k \cup D_{(k-1)} \cup ... \cup D_1$ where each D_i contains i number of $K_4's$, $1 \leq i \leq k$. For $0 \leq j \leq k-1$, label $D_{(k-j)}$ as follows: Label the top vertices of each $K_4's$ in $D_{(k-j)}$ alternatively as $2j+1$ and $2j+2$ beginning with $2j+1$ from left to right. The remaining $2(k-j)$ vertices in the $K_4's$ of $D_{(k-j)}$ are labeled $2j+3$,$2j+4$,...,$2j+(2(k-j)+2)$ from middle vertices to bottom vertices beginning with $2j+3$. Here $2j+2(k-j)+2 = 2k+2$. See Figure 3. Since each K_4 is a complete graph on four vertices in every $D_{(k-j)}$, vertices labeled $2j+1$ and $2j+2$ are adjacent and they in turn are adjacent to vertices labeled $2j+3$,...,$2k+2$. Thus the labeling of the vertices gives an achromatic labeling of tK_4-Chain and hence ψ $(tK_4 - Chain) \geq 2k+2$. This completes the proof.

□

4. Achromatic Number For Silicate Networks

A silicate sheet is a ring of tetrahedrons which are linked by shared oxygen nodes to other rings in a two dimensional plane that produces a sheet-like structure. Silicates are obtained by fusing metal oxides or metal carbonates with sand. Essentially all the silicates contain SiO_4 tetrahedra. A silicate network can be constructed in different ways [18]. We consider the construction of a silicate network from a honeycomb network. For instance, the graph in Figure 4 is $HC(2)$. The parameter n of $HC(n)$ is called the dimension of $HC(n)$.

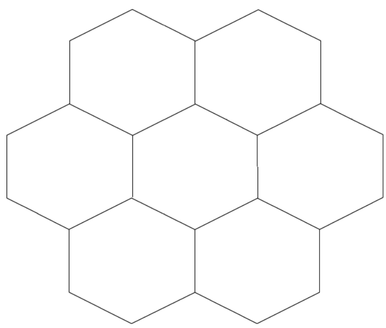

Figure 4: A honeycomb network $HC(2)$

Consider a honeycomb network $HC(n)$ of dimension n. Place silicon ions on all the vertices of $HC(n)$. Subdivide each edge of $HC(n)$ once. Place oxygen ions on the new vertices. Introduce $6n$ new pendant edges one each at the 2-degree silicon ions of $HC(n)$ and place oxygen ions at the pendent vertices. See Figure 5(a).

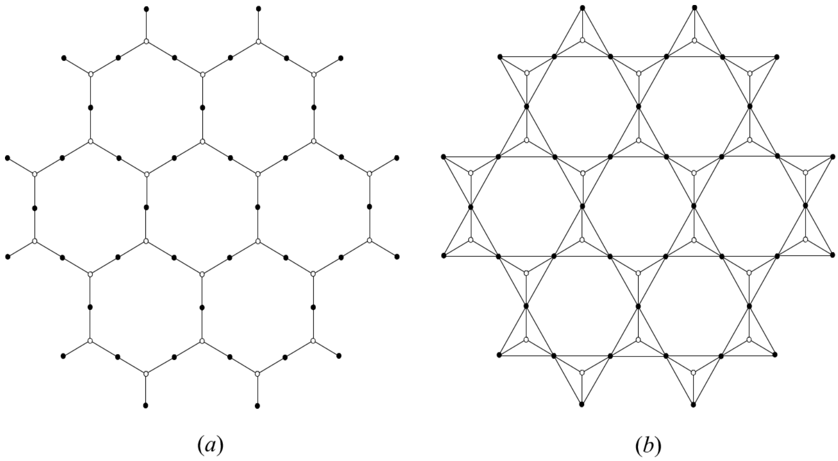

Figure 5: Silicate network construction and boundary nodes

With every silicon ion associate the three adjacent oxygen ions and form a tetrahedron as in Figure 5(b). The resulting network is a silicate network of dimension n, denoted $SL(n)$. The diameter of $SL(n)$ is $4n$. The graph in Figure 5(b) is a silicate network of dimension two. The 3-degree oxygen nodes of silicates are called *boundary nodes*. The number of nodes in $SL(n)$ is $15n^2+3n$ and the number of edges of $SL(n)$ is $36n^2$.

Our strategy is to identify suitable induced independent subgraphs of the Silicate network.

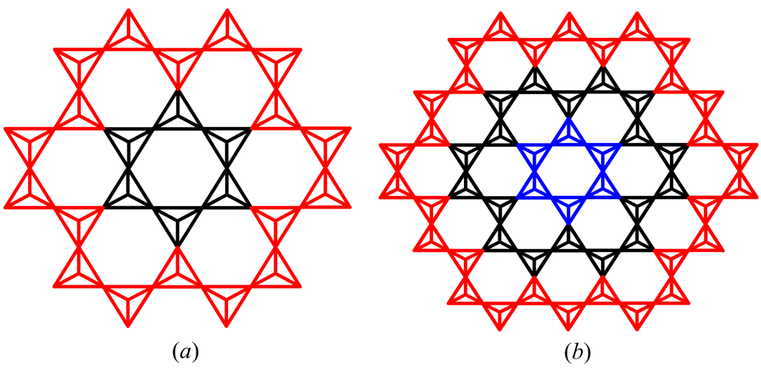

Figure: 6(a) Silicate network $SL(2)$ and 6(b) Silicate network $SL(3)$

Lemma 4.1. *Let $SL(r)$ be a silicate network of dimension r. Then $SL(r)$ network contains a disjoint union of the induced circular tK_4-Chain of the following type.*

(i) When $n = 2t, t \geq 1$, we get one $18, 42, ..., (24t - 6)$ layers of tK_4-Chains. See Figure 6(a).

(ii) When $n = 2t + 1, t \geq 1$, we get one $6, 30, ..., (24t - 18)$ layers of tK_4-Chains. See Figure 6(b).

We now give an algorithm achromatic tK_4- Chain to $SL(r)$ network of dimension r.

Theorem 4.2. *Let $SL(r)$ be a silicate network of dimension r. Then $\psi(SL(r)) \geq 2k + 2$.*

Proof. By Theorem 3.1, from the given $\frac{(k(k+1))}{2}$ K_4 - chain, group k adjacent $K_4's$ as D_k, the next $(k-1)K_4's$ as D_{k-1} and so on and the last K_4 as D_1. Since each tK_4-Chain is in the circular arrangement, labeling of the first top vertex of D_k and the end top vertex of D_1 in each tK_4-Chain is the same. For $0 \leq i \leq k-1$, label $D_{(k-i)}$ as follows:

Label the top vertices alternately as $2i+1$ and $2i+2$ beginning with $2i+1$ from left to right. Label the middle vertices and bottom vertices with consecutive numbers $2i+3$, $2i+4$,..., $2k+2$ beginning with $2i+3$ from middle vertices to the bottom vertices. See Figure 7.

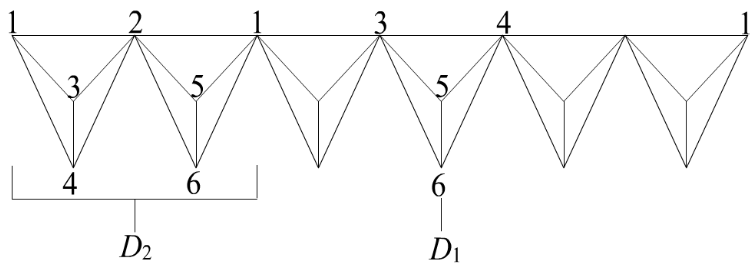

Figure 7: Achromatic labeling of $6K_4$-chain nodes from $SL(1)$ network

Proof of correctness: As the top vertices in each D_i receive two labels alternately; none of the adjacent top vertices in D_i receive the same label. Again the middle and bottom labels used in different D_i's are different. Hence the labeling is proper. Since each K_4 is a complete graph on four vertices, it is easy to verify that the labeling of the vertices induces an achromatic labeling yielding the achromatic number for $\frac{(k(k+1))}{2}$ K_4 -chain to be at least $2k+2$. □

The following theorem is straight forward as the number of edges of $SL(n)$ is $36n^2$.

Theorem 4.3. *Let $SL(n)$ be a Silicate network of dimension of n, $n \geq 2$. Then* $\psi(SL(n)) \leq (\frac{1+\sqrt{(1+288n^2)}}{2})$

Theorem 4.2 and Theorem 4.3 imply the following result.

Theorem 4.4. *Let $SL(n)$ be the silicate network of dimension n. Then:* $\lceil \frac{11}{20}(\frac{1+\sqrt{(1+288n^2)}}{2})\rceil \leq \psi(SL(n)) \leq (\frac{1+\sqrt{(1+288n^2)}}{2})$

Since $SL(n)$ has $15n^2 + 3n$ vertices we have the following result.

Theorem 4.5. *There is an $O(1)$-approximation algorithm to determine the achromatic number of silicate networks on n vertices.*

Proof. A lower bound for the achromatic number of $SL(n)$ is

$$\lceil \frac{11}{20}(\frac{1+\sqrt{(1+288n^2)}}{2})\rceil$$

and the expected achromatic number for silicate network is $(\frac{1+\sqrt{(1+288n^2)}}{2})$. The ratio of the two numbers is of $O(1)$ and hence there is an $O(1)$-approximation algorithm to determine the achromatic number of $SL(n)$. □

5. Conclusion

In this paper we have obtained approximation algorithm to determine the achromatic, number of Silicate networks. We have already obtained the same for Mesh of trees, Hex Derived Networks, H-graphs, extended toroid graphs, also Base-4 Hypercube, Petersen graphs, ladders, combs and Enhanced Hyper Petersen network[20]. Finding efficient approximation algorithms to determine achromatic number for other interconnection networks is quite challenging.

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