ON VERTEX DISJOINT CYCLES OF MIDDLE AND TOTAL GRAPH OF K_n AND $K_{m,n}$

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Abstract: Tulgeity $\tau(G)$ of a graph G is the maximum number of vertex disjoint cycles contained in G . In this paper the basic results on tulgeity of a graph have been reviewed and the formula for the tulgeity of the middle and total graph of complete graph and complete bigraph are derived. Also an upper bound for the tulgeity of middle graph of any graph is presented and the graph for which the tulgeity attains its upper bound has been classified.

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1. Introduction

Cycle is a significant feature of a graph. The analysis of cycle plays an important role in the design and development of some graph model. Several problems of finding shortest path, shortest spanning trees, least cost Hamiltonian cycles, etc., of a graph have been studied. We discuss the problems of finding the maximum number of vertex disjoint cycles.

We consider finite, simple, undirected graph $G(V(G), E(G))$ where $V(G)$ and $E(G)$ represent vertex set and edge set of G respectively. p and q denote the

number of vertices and edges of the graph G respectively. For any real number r , $[r]$ and $\lceil r \rceil$ denote the largest integer not exceeding r and the least integer not less than r , respectively. If $\mathcal{C}$ is the collection of cycles of a graph then $V(\mathcal{C})$ denotes the set of vertices belonging to the cycles of $\mathcal{C}$. $\langle S \rangle$ denotes the subgraph of G induced by the set of vertices $S \subseteq V(G)$. The other notations and terminology used in this paper can be found in [5].

Graph can be characterized by a type of configuration or subgraph they possess. In [3], Gary Chartrand et al., characterized several classes of graphs by defining them with the property P_n . A graph G has the property P_n , where n is a positive integer, if G contains no subgraph which is homeomorphic from the complete graph K_{n+1} or the complete bigraph $K\left(\left\lfloor \frac{n+2}{2} \right\rfloor \left\lfloor \frac{n+2}{2} \right\rfloor\right)$. Trivially, a graph G has the property P_2 , if G contains no cycles. They introduced the concept of point partition number and dual point partition number of a graph G . The point partition number of a graph is defined as the minimum number of subsets into which the vertex set of G can be partitioned so that the subgraph induced by each subset has the property P_n . Dual to this concept the dual point partition number of a graph is defined as the maximum number of subsets into which the vertex set of G can be Partitioned so that the subgraph induced by each subset does not possesses the property P_n . The problems of Nordhans - Gaddum type for the dual point partition number are investigated in [6].

In [4] Gary Chartrand et al., defined the term tulgeity of a graph as the dual point partition number for $n = 2$. (i.e) Tulgeity is the maximum number of subsets into which the vertex set of G can be partitioned so that the subgraph induced by each subset contains a cycle. Equivalently the tulgeity of a graph is the maximum number of vertex disjoint cycles in G and it is denoted by $\tau(G)$. Since every cycle contains at least three vertices, an obvious upper bound for tulgeity of any graph G with p vertices is $\tau(G) \leq \lfloor p/3 \rfloor$ and $\tau(G) = \lfloor p/3 \rfloor$ if G is a complete graph with p vertex.

Middle graph $M(G)$ of a graph G is defined with the vertex set $V(G) \cup E(G)$, in which two elements are adjacent if and only if either both are adjacent edges in G or one of the elements is a vertex and the other one is an edge incident to the vertex in G .

Total graph $T(G)$ of a graph G defined with the vertex set $V(G) \cup E(G)$, in which two elements are adjacent if and only if one of the following holds true (i) both are adjacent edges or vertices in G (ii) one is a vertex and other is an edge incident to it in G .

2. Basic Results

We begin by presenting the existing results concerning the tulgeity of a graph.

Theorem 2.1. [4] For any graph G , $\tau(G) = \sum \tau(C) \leq \sum \tau(B)$, where the sums being taken over all components C and blocks B of G , respectively.

Theorem 2.2. [4] For the complete n -partite graph $G = K(p_1, p_2, \dots, p_n)$, $1 \leq p_1 \leq p_2 \leq \dots \leq p_n$ and $\sum p_i = p$, $\tau(G) = \min \left(\left\lceil \frac{1}{2} \sum_{i=1}^{n-1} p_i \right\rceil, \lceil p/3 \rceil \right)$, where $p_0 = 0$.

We have derived the formula to find the tulgeity of the line graph of complete graph and complete bigraph.

Theorem 2.3. [1] $\tau(L(K_n)) = \left\lceil \frac{n(n-1)}{6} \right\rceil$

Theorem 2.4. [1] $\tau(L(K_{m,n})) = \left\lceil \frac{mn}{3} \right\rceil$

Also, we have derived an upper bound for the tulgeity of line graph of any graph and characterized the graphs for which the tulgeity attains its upper bound.

Theorem 2.5. [1] For any graph G , $\tau(L(G)) \leq \sum_i \left\lceil \frac{\deg v_i}{3} \right\rceil$ where $\deg v_i$ denotes the degree of the vertex v_i and the summation taken over all the vertices of G .

Theorem 2.6. [1] If G is a tree and for each pair of vertices (v_i, v_j) with $\deg v_i, \deg v_j > 2$, if there exist a vertex v of degree 2 on $P(v_i, v_j)$ then $\tau(L(G)) \leq \sum_i \left\lceil \frac{\deg v_i}{3} \right\rceil$.

Also we have derived [2] the results to find the tulgeity of line, middle and total graph of Wheel graph, Gear graph and Helm graph.

3. Tulgeity of Middle graph of K_n

Theorem 3.1. The tulgeity of middle graph of K_n ($n \geq 4$), $\tau(M(K_n)) = \left\lceil \frac{n(n+1)}{6} \right\rceil$

Proof. Let $V(K_n) = \{v_i : 1 \leq i \leq n\}$ and $E(K_n) = \{e_{ij} = v_i v_j : 1 \leq$

$i \leq n, i+1 \leq j \leq n\}$. By the definition of middle graph, $V(M(K_n)) = V(K_n) \cup E(K_n)$.

If $n = 4$, then $V(M(K_4)) = \{v_i : 1 \leq i \leq 4\} \cup \{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$ and there exists the collection $\mathbb{C} = \{e_{12}v_2e_{23}, e_{13}v_3e_{34}, e_{14}e_{24}v_4\}$ of disjoint cycles of $M(K_4)$. Hence $\tau(M(K_4)) = \left\lceil \frac{n(n+1)}{6} \right\rceil$ where $n = 4$.

If $n = 5$, then $V(M(K_n)) = \{v_i : 1 \leq i \leq 5\} \cup \{e_{12}, e_{13}, e_{14}, e_{15}, e_{23}, e_{24}, e_{25}, e_{34}, e_{35}, e_{45}\}$ and there exists the collection $\mathbb{C} = \{v_1e_{12}e_{15}, v_2v_{23}e_{25}, v_3e_{13}e_{34}, v_4e_{14}e_{24}, v_5e_{35}e_{45}\}$ of disjoint cycles of $M(K_5)$. Hence

$$\tau(M(K_5)) = \left\lceil \frac{n(n+1)}{6} \right\rceil,$$

where $n = 5$.

Assume $n > 5$. For each $i (1 \leq i \leq n)$, $\langle \{v_i\} \cup \{e_{ij} : i+1 \leq j \leq n\} \rangle$ is a clique (say K'_i) of order $n-i+1$ in $M(K_n)$.

Cliques in $M(K_n)$

Figure 3.1

If $V(K'_l) \cap V(K'_m) \neq \emptyset$ for any $l \neq m$ then there exists a vertex $e_{ik} \in V(K'_l) \cap V(K'_m)$. (ie) $e_{ik} \in \{v_l\} \cup \{e_{lj} : l+1 \leq j \leq n\}$ and $e_{ik} \in \{v_m\} \cup \{e_{mj} : m+1 \leq j \leq n\}$. (ie) $e_{ik} \in \{e_{lj} : l+1 \leq j \leq n\}$ and $e_{ik} \in \{e_{mj} : m+1 \leq j \leq n\}$. (ie) $i = l$ and $i = m \Rightarrow l = m$. This contradiction shows that $V(K'_l) \cap V(K'_m) = \emptyset$ for $l \neq m$. Also $\bigcup_{i=1}^n V(K'_i) = V(M(K_n))$. Hence $\tau(M(K_n)) \geq \sum_{i=1}^n \tau(K'_i)$. Clearly for each $i (1 \leq i \leq n)$, $|V(K'_i)| = n - i + 1$.

case (i) ($n \equiv 0 \pmod{3}$)

For each $i = 3k - 2$, ($1 \leq k \leq \frac{n}{3}$), $|V(K'_i)| = n + 3 - 3k \equiv 0 \pmod{3}$. Hence there exists $\frac{1}{3}|V(K'_i)|$ vertex disjoint 3-cycles in K'_i for each $i = 3k - 2 (1 \leq k \leq$

$\frac{n}{3}$). We find the partition of $V(M(K_n))$ as follows. $V(M(K_n)) = V' \cup V''$ where $V' = \bigcup_{i=1}^{n-6} V(K'_i)$ and $V'' = \bigcup_{i=n-5}^n V(K'_i)$. It is clear that $\tau(M(K_n)) \geq \tau(< V' >) + \tau(< V'' >)$. V' can be expressed as

$$V' = \bigcup_{k=1}^{\frac{n}{3}-2} V(K'_{3k-2}) \bigcup_{k=1}^{\frac{n}{3}-2} V(K'_{3k-1}) \bigcup_{k=1}^{\frac{n}{3}-2} V(K'_{3k}).$$

(ie) $\tau(< V' >) = \sum_{k=1}^{\frac{n}{3}-2} \tau(K'_{3k-2}) + \sum_{k=1}^{\frac{n}{3}-2} \tau(K'_{3k-1}) + \sum_{k=1}^{\frac{n}{3}-2} \tau(K'_{3k})$. For each k ($1 \leq k \leq \frac{n}{3} - 2$), $|V(K'_{3k-2})| = n - 3k + 3 \equiv 0 \pmod{3}$ and hence K'_{3k-2} is a clique with $\frac{1}{3}|V(K'_{3k-2})|$ vertex disjoint 3-cycles. Hence $\tau(K'_{3k-2}) = \frac{1}{3}(n - 3k + 3) = \frac{n}{3} - k + 1$ for each k ($1 \leq k \leq \frac{n}{3} - 2$). $\sum_{k=1}^{\frac{n}{3}-2} \tau(K'_{3k-2}) = \sum_{k=1}^{\frac{n}{3}-2} (\frac{n}{3} - k + 1) = \frac{1}{2} \left(\frac{n}{3} - 2 \right) \left(\frac{n}{3} + 3 \right)$. For each k ($1 \leq k \leq \frac{n}{3} - 2$) $|V(K'_{3k-1})| = n - 3k + 2 \equiv 2 \pmod{3}$ and $|V(K'_{3k})| = n - 3k + 1 \equiv 1 \pmod{3}$. Let $K''_{3k-1} = K'_{3k-1} - \{e_{(3k-1)3k}, e_{(3k-1)(3k+1)}\}$ and $K''_{3k} = K'_{3k} - \{e_{(3k)(3k+1)}\}$ for each k , ($1 \leq k \leq \frac{n}{3} - 2$). It is clear that $|V(K''_{3k-1})| = |V(K''_{3k})| = n - 3k \equiv 0 \pmod{3}$ for each k ($1 \leq k \leq \frac{n}{3} - 2$). Also $< \{e_{(3k-1)3k}, e_{(3k-1)(3k+1)}, e_{3k(3k+1)}\} >$ is a 3 cycle for each k ($1 \leq k \leq \frac{n}{3} - 2$). Hence $\sum_{k=1}^{\frac{n}{3}-2} \tau(K'_{3k-1}) + \sum_{k=1}^{\frac{n}{3}-2} \tau(K'_{3k}) = \sum_{k=1}^{\frac{n}{3}-2} \left(\frac{1}{3}|V(K''_{3k-1})| + \frac{1}{3}|V(K''_{3k})| + 1 \right) = \sum_{k=1}^{\frac{n}{3}-2} \left(\frac{1}{3}(n - 3k) + \frac{1}{3}(n - 3k) + 1 \right) = \left(\frac{n}{3} - 2 \right) \left(\frac{n}{3} + 2 \right)$. Since $V'' = \bigcup_{i=n-5}^5 V(K'_i)$, $|V''| = 6 + 5 + \cdots + 1 = 21 \equiv 0 \pmod{3}$. We find the following collection of seven disjoint 3-cycles of $< V'' >$

$$\begin{aligned} \mathbb{C}_i = \{ & v_{n-5}e_{(n-5)(n-4)}e_{(n-5)(n-3)}, e_{(n-5)(n-2)}v_{n-2}e_{(n-2)(n-1)}, e_{(n-5)(n-1)} \\ & v_{n-1}e_{(n-4)(n-1)}, e_{(n-5)n}e_{(n-4)n}e_{(n-3)n}, e_{(n-4)(n-3)}v_{n-4}e_{(n-4)(n-2)}, e_{(n-3)(n-2)} \\ & v_{n-3}e_{(n-3)(n-1)}, e_{(n-2)n}v_n e_{(n-1)n} \}. \end{aligned}$$

Hence $\tau(< V'' >) = 7$.

$$\begin{aligned} \tau(M(K_n)) &\geq \tau(< V' >) + \tau(< V'' >) \\ &= \frac{\left(\frac{n}{3} - 2 \right) \left(\frac{n}{3} + 3 \right)}{2} + \left(\frac{n}{3} - 2 \right) \left(\frac{n}{3} + 2 \right) + 7 \end{aligned}$$

$$= \frac{n(n+1)}{6}.$$

case (ii) ($n \equiv 1 \pmod{3}$)

Set $V' = V(K'_1) \bigcup_{i=2}^{n-6} V(K'_i)$, $V'' = \bigcup_{i=n-5}^n V(K'_i)$ so that $V(M(K')) = V' \cup V''$.

$$V' = V(K'_1) \bigcup_{i=2}^{n-6} V(K'_i) = V(K'_1) \bigcup_{k=1}^{\frac{n-7}{3}} V(K'_{3k-1}) \bigcup_{k=1}^{\frac{n-7}{3}} V(K'_{3k}) \bigcup_{k=1}^{\frac{n-7}{3}} V(K'_{3k+1}).$$

Since $|V(K'_1)| = n \equiv 1 \pmod{3}$, K'_1 has $\frac{n-1}{3}$ vertex disjoint 3-cycles. For each k ($1 \leq k \leq \frac{n-7}{3}$), $|V(K'_{3k-1})| = n - 3k + 2 \equiv 0 \pmod{3}$ and hence $\tau(K'_{3k-1}) = \frac{1}{3}(n - 3k + 2)$, ($1 \leq k \leq \frac{n-7}{3}$). Hence

$$\sum_{k=1}^{\frac{n-7}{3}} \tau(K'_{3k-1}) = \frac{1}{3} \sum_{k=1}^{\frac{n-7}{3}} (n - 3k + 2) = \frac{(n-7)(n+8)}{18}.$$

For each k ($1 \leq k \leq \frac{n-7}{3}$),

$$|V(K'_{3k})| = n - 3k + 1 \equiv 2 \pmod{3}$$

and

$$|V(K'_{3k+1})| = n - 3k - 1 + 1 = n - 3k \equiv 1 \pmod{3}.$$

Let $K''_{3k} = K'_{3k} - \{e_{(3k)(3k+1)}, e_{(3k)(3k+2)}\}$ and $K''_{3k+1} = K'_{3k+1} - \{e_{(3k+1)(3k+2)}\}$ for each k ($1 \leq k \leq \frac{n-7}{3}$). It is clear that $|V(K''_{3k})| = n - 3k - 1 \equiv 0 \pmod{3}$ and $|V(K''_{3k+1})| = n - 3k - 1 \equiv 0 \pmod{3}$ for each k ($1 \leq k \leq \frac{n-7}{3}$). Also for each k ($1 \leq k \leq \frac{n-7}{3}$), $\langle \{e_{(3k)(3k+1)}, e_{(3k)(3k+2)}, e_{(3k+1)(3k+2)}\} \rangle$ is a 3-cycle. Hence $\sum_{k=1}^{\frac{n-7}{3}} \tau(K'_{3k}) + \sum_{k=1}^{\frac{n-7}{3}} \tau(K'_{3k+1}) = \frac{(n-7)(n+8)}{18}$. Since $|V''| = \sum_{i=n-5}^n |V(K'_i)| = \sum_{i=n-5}^n (n - i + 1) = 21 \equiv 0 \pmod{3}$. There exists the following collection $\mathbb{C}$ of vertex disjoint cycles in $\langle V'' \rangle$

$$\begin{aligned} \mathbb{C} = \{ & v_{n-5} e_{(n-5)(n-3)} e_{(n-5)(n-2)}, e_{(n-5)(n-4)} v_{n-4} e_{(n-4)(n-2)}, \\ & e_{(n-5)(n-1)} e_{(n-4)(n-1)} e_{(n-3)(n-1)}, e_{(n-5)n} e_{(n-4)n} e_{(n-1)n}, e_{(n-4)(n-3)} v_{n-3} e_{(n-3)n}, \\ & e_{(n-3)(n-2)} v_{n-2} e_{(n-2)n}, e_{(n-2)(n-1)} v_{n-1} e_{(n-1)n} \}. \end{aligned}$$

Hence $\langle V'' \rangle = 7$. Therefore

$$\begin{aligned}\tau(M(K_n)) &\geq \tau(K'_1) + \sum_{k=1}^{\frac{n-7}{3}} \tau(K'_{3k-1}) + \sum_{k=1}^{\frac{n-7}{3}} \tau(K'_{3k}) + \tau(\langle V'' \rangle) \\ &= \frac{n-1}{3} + \frac{(n-7)(n+8)}{18} + \frac{(n-7)(n+5)}{9} + 7 \\ &= \frac{n(n+1)}{6} - \frac{1}{3} \\ &= \left\lfloor \frac{n(n+1)}{6} \right\rfloor\end{aligned}$$

case (iii) ($n \equiv 2 \pmod{3}$). In this case we set $V' = \bigcup_{i=1}^{n-5} V(K'_i)$ and

$V'' = \bigcup_{i=n-4}^n V(K'_i)$ so that $V' \cup V'' = V$ and $V' \cap V'' = \emptyset$. Now $V' = \bigcup_{i=1}^{n-5} V(K'_i)$

$$= \bigcup_{k=1}^{\frac{n-5}{3}} V(K'_{3k-2}) \bigcup_{k=1}^{\frac{n-5}{3}} V(K'_{3k-1}) \bigcup_{k=1}^{\frac{n-5}{3}} V(K'_{3k})$$

$$(\text{ie.}) \quad \tau(\langle V' \rangle) = \sum_{k=1}^{\frac{n-5}{3}} \tau(K'_{3k-2}) + \sum_{k=1}^{\frac{n-5}{3}} \tau(K'_{3k-1}) + \sum_{k=1}^{\frac{n-5}{3}} \tau(K'_{3k})$$

For each k ($1 \leq k \leq \frac{n-5}{3}$), $|V(K'_{3k-2})| = n - 3k + 2 + 1 = n - 3k + 3 \equiv 2 \pmod{3}$ and $|V(K'_{3k-1})| = n - 3k + 1 + 1 = n - 3k + 2 \equiv 1 \pmod{3}$. Let $K''_{3k-2} = K'_{3k-2} - \{e_{(3k-2)(3k-1)}, e_{(3k-2)(3k)}\}$ and $K''_{3k-1} = K'_{3k-1} - \{e_{(3k-1)(3k)}\}$. It is clear that $|V(K''_{3k-2})| = |V(K''_{3k-1})| = n - 3k + 1 \equiv 0 \pmod{3}$ for each k ($1 \leq k \leq \frac{n-5}{3}$). Also $\langle \{e_{(3k-2)(3k-1)}, e_{(3k-2)(3k)}, e_{(3k-1)(3k)}\} \rangle$ is a 3-cycle. $|V(K'_{3k})| = n - 3k + 1 \equiv 0 \pmod{3}$ and hence $\tau(K'_{3k}) = \frac{1}{3}(n - 3k + 1)$ for each k ($1 \leq k \leq \frac{n-5}{3}$). Hence

$$\begin{aligned}\tau(\langle V' \rangle) &= \sum_{k=1}^{\frac{n-5}{3}} \tau(K'_{3k-2}) + \sum_{k=1}^{\frac{n-5}{3}} \tau(K'_{3k-1}) + \sum_{k=1}^{\frac{n-5}{3}} \tau(K'_{3k}) \\ &= \sum_{k=1}^{\frac{n-5}{3}} \left(\frac{1}{3}|V(K''_{3k-2})| + \frac{1}{3}|V(K''_{3k-1})| + 1 \right) + \sum_{k=1}^{\frac{n-5}{3}} \frac{1}{3}|V(K_{3k})| \\ &= \sum_{k=1}^{\frac{n-5}{3}} \left(\frac{1}{3}(n - 3k + 1) + \frac{1}{3}(n - 3k + 1) + 1 \right) + \sum_{k=1}^{\frac{n-5}{3}} \frac{1}{3}(n - 3k + 1) \\ &= \frac{(n-5)(n+6)}{6}\end{aligned}$$

$V'' = \bigcup_{i=n-4}^n V(K'_i)$ and hence $|V''| = \sum_{i=n-4}^n (n-i+1) = 15 \equiv 0 \pmod{3}$ and the following is the collection of disjoint 3-cycles of $\langle V'' \rangle$.
 $\mathbb{C} = \{v_{n-4}e_{(n-4)(n-3)}e_{(n-4)n}, v_{n-3}e_{(n-3)(n-2)}e_{(n-3)n}, v_{n-2}e_{(n-4)(n-2)}e_{(n-2)(n-1)}, v_{n-1}e_{(n-4)(n-1)}e_{(n-3)(n-1)}, v_n e_{(n-2)n}e_{(n-1)n}\}$.
Hence $\langle V'' \rangle = 5$ and

$$\begin{aligned}\tau(M(K_n)) &\geq \tau(\langle V' \rangle) + \tau(\langle V'' \rangle) \\ &= \frac{(n-5)(n+6)}{6} + 5 \\ &= \frac{n(n+1)}{6}.\end{aligned}$$

Hence for all the cases $\tau(M(K_n)) \geq \left\lceil \frac{n(n+1)}{6} \right\rceil$. Since $|V(M(K_n))| = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$, $\tau(M(K_n)) \leq \left\lceil \frac{n(n+1)}{6} \right\rceil$. Therefore $\tau(M(K_n)) = \left\lceil \frac{n(n+1)}{6} \right\rceil$. $\square$

Since $V(T(K_n)) = V(M(K_n))$ and $\tau(M(K_n)) = \frac{1}{3}|V(M(K_n))|$ it can be concluded that $\tau(T(K_n)) = \tau(M(K_n))$. Each vertex v_i of a graph G with degree $\deg v_i$ induce a clique of order $\deg v_i + 1$ in $M(G)$. (ie.) At each vertex v_i of G , we count $\left\lceil \frac{\deg v_i + 1}{3} \right\rceil$ vertex disjoint cycles in $M(G)$.

This argument yields an upper bound for the tulgeity of middle graph of any graph.

Theorem 3.2. For any graph G , $\tau(M(G)) \leq \sum_i \left\lceil \frac{\deg v_i + 1}{3} \right\rceil$

Theorem 3.3. If T is a tree and for each pair of vertices (v_i, v_j) with $\deg v_i, \deg v_j > 2$ if there exists a vertex v of degree 2 on $P(v_i, v_j)$ then $\tau(M(T)) = \sum_i \left\lceil \frac{\deg v_i + 1}{3} \right\rceil$

Proof. Let T be a tree satisfying the given condition. Each vertex v_i of T of degree > 2 induce a clique of order $\deg v_i + 1$ in $M(T)$. Since for each pair of vertices (v_i, v_j) of T with degree > 2 , there exists a vertex v of degree 2 on $P(v_i, v_j)$, there exists a bridge joining the corresponding cliques in $M(T)$ and hence $\tau(M(T)) = \sum_i \left\lceil \frac{\deg v_i + 1}{3} \right\rceil$ $\square$

4. Tulgeity of Middle graph of $K_{m,n}$

Theorem 4.1. $\tau(M(K_{m,n})) = \left\lceil \frac{m+n+mn}{3} \right\rceil$ for any $m, n \geq 4$

Proof. Let $V(K_{m,n}) = \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\}$ and $E(K_{m,n}) = \{e_{ij} = u_i v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. It is clear that $e_{ij} \neq e_{ji}$ for any $i \neq j$. By the definition of middle graph, $M(K_{m,n})$ has the vertex set $V(M(K_{m,n})) = V(K_{m,n}) \cup E(K_{m,n})$.

Cliques in $M(K_{m,n})$

Figure 4.1

For each i ($1 \leq i \leq m$), u_i is adjacent with all the vertices of $\{e_{ij} : 1 \leq j \leq n\}$ and for each j ($1 \leq j \leq n$), v_j is adjacent with each vertex of $\{e_{ij} : 1 \leq i \leq m\}$. Also $e_{ij}e_{kl} \in E(M(K_{m,n}))$ if and only if either $i = k$ or $j = l$.

For each i , the set of vertices $\{e_{ij} : 1 \leq j \leq n\} \cup \{u_i\}$ of $M(K_{m,n})$ ($1 \leq i \leq m$) induces a clique of order $n+1$ (say k_i'). Similarly for each j , the set of vertices $\{e_{ij} : 1 \leq i \leq m\} \cup \{v_j\}$ of $M(K_{m,n})$ ($1 \leq j \leq n$) induces a clique of order $m+1$ (say k_j''). Also $\langle \{u_i : 1 \leq i \leq m\} \rangle$ and $\langle \{v_j : 1 \leq j \leq n\} \rangle$ are totally disconnected in $M(K_{m,n})$.

Vertex disjoint 3-cycles of $M(K_{m,n})$ for the case of $4 \leq m, n \leq 5$ is shown in the figure.

Disjoint 3-cycles in $M(K_{5,5})$

Figure 4.2

Let $n \equiv k \pmod{3}$ and $m \equiv l \pmod{3}$ and $m, n \geq 6$. We choose the partition of $V(M(K_{m,n}))$ such as $V = V' \cup V'' \cup V'''$ where

$$\begin{aligned}
 V' &= \{v_j : 1 \leq j \leq n-2\} \bigcup_{i=1}^2 \{u_i, e_{ij} : 1 \leq j \leq n\} \\
 &= \{u_1, u_2\} \bigcup_{j=1}^n \{v_j, e_{ij} : 1 \leq i \leq 2\} - \{v_{n-1}, v_n\} \\
 V'' &= \bigcup_{i=3}^m \{u_i, e_{ij} : 1 \leq j \leq n-k-4\} \\
 V''' &= \bigcup_{i=3}^m \{e_{ij} : n-k-3 \leq j \leq n\} \cup \{v_{n-1}, v_n\}
 \end{aligned}$$

Here $\tau(M(K_{m,n})) \geq \tau(<V'>) + \tau(<V''>) + \tau(<V'''>)$

Since for each j ($1 \leq j \leq n-2$), $<\{v_j, e_{ij} : 1 \leq i \leq 2\}>$ is a 3-cycle, there exists $n-2$ disjoint 3-cycle in $<V'>$ and also $<V'>$ two 3-cycles $<\{u_1, e_{1(n-1)}, e_{1n}\}>$ and $<\{u_2, e_{2(n-1)}, e_{2n}\}>$. Hence $\tau(<V'>) = n$. Since for each i ($3 \leq i \leq m$), $<\{u_i, e_{ij} : 1 \leq j \leq n-k-4\}>$ is a subgraph of the clique K'_i , there exists $\frac{1}{3}(n-k-3)$ disjoint 3-cycles in $<\{u_i, e_{ij} : 1 \leq j \leq n-k-4\}>$.

(ie) $\tau(<V''>) = \sum_{i=3}^m \frac{1}{3}(n-k-3) = \frac{1}{3}(m-2)(n-k-3)$. Now we find the

collection of vertex disjoint 3-cycles of $\langle V''' \rangle$ as follows

$$\mathbb{C} = \{ \langle \{v_{n-1}, e_{3(n-1)}, e_{4(n-1)}\} \rangle, \langle \{v_n, e_{(m-1)n}, e_{mn}\} \rangle \}$$

$$\mathbb{C}' = \{ \langle \{e_{i(n-3)}, e_{i(n-2)}, e_{i(n-1)}\} \rangle : 5 \leq i \leq m \}$$

$$\mathbb{C}'' = \left\{ \langle \{e_{(5+l+i)n}, e_{(5+l+i+1)n}, e_{(5+l+i+2)n}\} \rangle : i = 3p, 0 \leq p \leq \frac{m-l-9}{3} \right\}.$$

It is clear that the cycles of $\mathbb{C}$, $\mathbb{C}'$ and $\mathbb{C}''$ are vertex disjoint and $|\mathbb{C}| = 2$, $|\mathbb{C}'| = m-4$, $|\mathbb{C}''| = \frac{m-l-9}{3} + 1$. If $V(\mathbb{C})$ denotes the set of vertices belonging to the cycles of $\mathbb{C}$ then we find the disjoint cycles of $V''' - (V(\mathbb{C}) \cup V(\mathbb{C}') \cup V(\mathbb{C}''))$ (say V^{iv}) in the following cases.

Since $K_{m,n} \approx K_{n,m}$, it suffices to consider only the following six cases to find $\tau(\langle V''' \rangle)$.

Case (i) ($k = 0, l = 0$)

In this case $V^{iv} = \{e_{3(n-3)}, e_{3(n-2)}, e_{3n}, e_{4(n-3)}, e_{4(n-2)}, e_{4n}\}$ and hence there exists two disjoint 3-cycles in $\langle V^{iv} \rangle$. Hence $\tau(V''') = |\mathbb{C}| + |\mathbb{C}'| + |\mathbb{C}''| + 2 = \frac{4m-6}{3}$.

$$\text{Therefore } \tau(M(K_{m,n})) \geq n + \frac{1}{3}(m-2)(n-3) + \frac{4m-6}{3} = \left\lceil \frac{m+n+mn}{3} \right\rceil.$$

Case(ii) ($k = 0, l = 1$)

In this case $V^{iv} = \bigcup_{i=1}^4 \{e_{i(n-3)}, e_{i(n-2)}, e_{in}\} \cup \{e_{5n}\}$. Since $\langle \{e_{i(n-3)}, e_{i(n-2)}, e_{in}\} \rangle$ is a 3-cycle for each i ($3 \leq i \leq 4$), $\tau(\langle V''' \rangle) = 2$. Hence $\tau(\langle V^{iv} \rangle) = \frac{4m-7}{3}$.

$$\text{Therefore } \tau(M(K_{m,n})) \geq n + \frac{1}{3}(m-2)(n-3) + \frac{4m-7}{3} = \left\lceil \frac{m+n+mn}{3} \right\rceil.$$

Case(iii) ($k = 0, l = 2$)

In this case $V^{iv} = \bigcup_{i=3}^4 \{e_{i(n-3)}, e_{i(n-2)}, e_{in}\} \cup \{e_{5n}, e_{6n}\}$ and hence $\langle V^{iv} \rangle$ has 2-vertex disjoint 3-cycles. Hence $\tau(\langle V''' \rangle) = \frac{4m-8}{3}$ and $\tau(\langle V'' \rangle) = \frac{1}{3}(m-2)(n-3)$. Hence $\tau(M(K_{m,n})) \geq n + \frac{1}{3}(m-2)(n-3) + \frac{4m-8}{3} = \left\lceil \frac{m+n+mn}{3} \right\rceil$.

Case(iv) ($k = 1, l = 1$)

In this case $V^{iv} = \{e_{i(n-4)} : 5 \leq i \leq m\} \bigcup_{i=3}^4 \{e_{i(n-4)}, e_{i(n-3)}, e_{i(n-2)}\} \cup \{e_{3n}, e_{4n}, e_{5n}\}$.

It is clear that $\tau(< V^{iv} >) = \frac{m-4}{3} + 2 + 1 = \frac{m+5}{3}$ and $\tau(< V''' >) = \frac{5m-8}{3}$.

$\tau(< V'' >) = \frac{1}{3}(m-2)(n-4)$ and hence $\tau(M(K_{m,n})) \geq n + \frac{1}{3}(m-2)(n-4) + \frac{5m-8}{3} = \left\lceil \frac{m+n+mn}{3} \right\rceil$.

Case(v) ($k=1, l=2$)

In this case $V^{iv} = \{e_{i(n-4)} : 4 \leq i \leq m\} \bigcup_{i=4}^5 \{e_{i(n-3)}, e_{i(n-2)}, e_{in}\} \cup \{e_{5n}, e_{6n}, e_{7n}\}$.

$\tau(< V^{iv} >) = \frac{m-3}{3} + 2 + 1 = \frac{m+6}{3}$. $\tau(< V''' >) = \frac{5m-8}{3}$ and $\tau(< V'' >) = \frac{1}{3}(m-2)(n-4)$.

Hence $\tau(M(K_{m,n})) \geq n + \frac{1}{3}(m-2)(n-4) + \frac{5m-8}{3} = \left\lceil \frac{m+n+mn}{3} \right\rceil$.

Case(vi) ($k=2, l=2$)

In this case $V^{iv} = \bigcup_{i=3}^m \{e_{i(n-5)}, e_{i(n-4)}\} \bigcup_{i=3}^4 \{e_{i(n-3)}, e_{i(n-2)}, e_{in}\} \cup \{e_{5n}, e_{6n}\}$.

$< \{e_{i(n-5)} : 3 \leq i \leq m\} >$ and $< \{e_{i(n-4)} : 3 \leq i \leq m\} >$ are the cliques of order $m-2$ and hence there exists $\frac{2}{3}(m-2)$ disjoint 3-cycles and $\{e_{i(n-3)}, e_{i(n-2)}, e_{in}\}$ ($3 \leq i \leq 4$) induces 2 more disjoint 3-cycles. Hence

$$\tau(< V^{iv} >) = \frac{2}{3}(m-2) + 2$$

and

$$\tau(< V''' >) = 2m - 4,$$

$$\tau(< V'' >) = \frac{1}{3}(n-2-3)(m-2) = \frac{1}{3}(m-2)(n-5).$$

Therefore

$$\tau(M(K_{m,n})) \geq n + \frac{1}{3}(m-2)(n-5) + 2m - 4 = \left\lceil \frac{m+n+mn}{3} \right\rceil.$$

□

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