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A STUDY ON BIFURCATIONS OF TRAVELING WAVE SOLUTIONS FOR THE GENERALIZED ZAKHAROV-KUZNETSOV MODIFIED EQUAL WIDTH EQUATION

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Abstract: By using the theory of bifurcations of planar dynamical systems to the generalized Zakharov-Kuznetsov modified equal width equation, the existence of smooth and non-smooth solitary wave, kink and anti-kink wave, smooth and non-smooth periodic wave, and compacton is obtained. Under different regions of parametric spaces, various sufficient conditions to guarantee the existence of the above waves are given. Moreover, some explicit exact parametric representations of traveling wave solutions are determined.

AMS Subject Classification: 34K18, 35C07, 76B25, 34C25
Key Words: generalized Zakharov-Kuznetsov modified equal width equation, solitary wave, periodic wave, compacton

1. Introduction

In 2005, by using the tanh and sine-cosine methods, Wazwaz [11] obtained some exact traveling wave solutions for the following ZK-MEW equation:

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$$u_t + a(u^n)_x + (bu_{xt} + ru_{yy})_x = 0. \tag{1}$$

But, the author did not make further research the bifurcation behavior of traveling wave solutions of equation(1). In this work, we consider the generalized Zakharov-Kuznetsov modified equal width equation of the form:

$$u_t + a(u^n)_x + [b(u^m)_{xt} + r(u^m)_{yy}]_x = 0, n > 1, \tag{2}$$

where a, b and r are real valued constants. This equation is known as the generalized ZK-MEW equation as it is constructed by combining the generalized modified equally width equation with the sense of the ZK equation. When $m = 1$, equation(2) becomes equation(1).

In 2009, Biswas [1] obtained 1-soliton solution for the generalized Zakharov-Kuznetsov modified equal width equation. In 2009, Esen and Kutluay [4] obtained traveling wave solutions for the generalized (2+1) dimensional ZK-MEW equation. Recently, Khalique and Adem [6] studied the (2+1) dimensional Zakharov-Kuznetsov modified equal width equation using Lie group analysis and obtained some exact non-topological soliton solutions, cnoidal wave solutions and the traveling wave solutions. It is clear that the analysis of the general solution is very much difficult and only some exact solutions of ZK-MEW equation are obtained. Therefore, it is very important to study the bifurcation behavior of traveling wave solutions of the generalized Zakharov-Kuznetsov modified equal width equation. By using the bifurcation theory and methods of planar dynamical systems, we obtain the bifurcation behavior of traveling wave solutions of equation(2).

Substituting the traveling wave transformation $u(x, y, t) = \phi(x + y - ct) = \phi(\xi)$ into equation (2), we obtain the ordinary differential equation:

$$-c\phi_\xi + a(\phi^n)_\xi + (r - bc)(\phi^m)_{\xi\xi\xi} = 0, \tag{3}$$

where c is the speed of the traveling wave. Integrating once and neglecting integral constant, the system (3) becomes:

$$-c\phi + a(\phi^n) + (r - bc)(\phi^m)_{\xi\xi} = 0. \tag{4}$$

We have the following traveling wave system:

$$\begin{cases} \frac{d\phi}{d\xi} = z, \\ \frac{dz}{d\xi} = \frac{c\phi - a\phi^n - m(m-1)(r-bc)\phi^{m-2}z^2}{m(r-bc)\phi^{m-1}}. \end{cases} \tag{5}$$

The equation (5) is a planar Hamiltonian system with the Hamiltonian function:

$$H(\phi, z) = m(r - bc)(\phi^{m-1})^2 z^2 - \frac{2c}{m+1}\phi^{m+1} + \frac{2a}{m+n}\phi^{m+n} = h, \text{say}. \tag{6}$$

The system equation (5) is a planar dynamical system depending on the parameters a, b, c, r, m, n. Because the phase orbits defined by the vector fields of equation (5) determine all traveling wave solutions of equation (2), we search for bifurcations of phase portraits of equation (5) in the (ϕ, z) -phase plane as the parameters a, b, c, r, m and n change. It should be noted that to this physical model, only bounded solutions are meaningful, so we shall now concentrate on the bounded solutions of equation (5) which are physically acceptable. It is well known that the solitary wave solutions of equation (2) correspond to homoclinic orbits of equation (5). A kink (or anti-kink) wave solution of equation (2) corresponds to a heteroclinic orbit of equation (5). Similarly, periodic orbits of equation (5) correspond to periodic traveling wave solutions of equation (2). Therefore, to investigate all possible bifurcations of solitary waves, kink waves and periodic waves of equation (2), we should discover all periodic orbits, homoclinic and heteroclinic orbits of equation (5) depending on the system parameters. The bifurcation theory of planar dynamical systems plays an important role in our study [9], [5]-[8].

We notice particularly is that the right hand side of the second equation in equation (5) is discontinuous when $\phi = 0$ for $m > 1$. Thus, it is clear that there exists a singular straight line $\phi = 0$ in the sense that $z_\xi = \phi_{\xi\xi}$ has no definition in equation (5) when $\phi = 0$. This implies that the smooth system (2) sometimes has non-smooth traveling wave solutions. The authors [7]-[10] had already studied this phenomenon and pointed out that the existence of a singular straight line for a traveling wave equation is the original reason why traveling waves lose their smoothness.

The remaining part of this paper is organized as following. In Section 2, we consider bifurcations of phase portraits of the planar Hamiltonian system equation (5). In Section 3, we obtain some explicit exact parametric representations for traveling wave solutions of equation(2). In Section 4, we discuss the existence of smooth and non-smooth solitary wave solutions, kink and anti-kink wave solutions, smooth and non-smooth periodic wave solutions, and compacton of equation(2).

2. Bifurcations of Phase Portraits of Equation (5)

In this section, we shall study the bifurcations of phase portraits of the Hamiltonian system equation (5). Since it is difficult to study equation (5) directly with a singular point, we consider the transformation $d\xi = m(r - bc)\phi^{m-1} d\tau$

and reduce the singular system (5) to the regular system:

$$\begin{cases} \frac{d\phi}{d\tau} = m(r-bc)\phi^{m-1}z, \\ \frac{dz}{d\tau} = c\phi - a\phi^n - m(m-1)(r-bc)\phi^{m-2}z^2. \end{cases} \tag{7}$$

which has the same topological phase portraits as equation (5) except for the straight line $\phi = 0$. For a fixed h, equation (6) will determine a set of invariant curves of equation(7), which will contain different branches of curves. For different values of h , equation (6) will define different families of orbits of equation(7) with different dynamical behaviors. To investigate the bifurcation of phase portraits of equation (7) we find the equilibrium points of the system. To find the equilibrium points of the system, we need to find all zeros of the equation $f(\phi) = 0$, where $f(\phi) = c\phi - a\phi^n$. It is clear that on the (ϕ, z)-phase plane, the abscissas of equilibrium points of system equation(5) on the ϕ-axis will be the zeros of $f(\phi)$.

Let $E_i(\phi_i, 0)$ be an equilibrium point of equation (7) where $f(\phi_i) = 0$. When n is an odd integer $(n > 1)$ and $ac > 0$, there exist three equilibrium points at $E_0(0,0), E_1(\phi_1, 0)$, and $E_2(\phi_2, 0)$, where $\phi_1 = (c/a)^{\frac{1}{n-1}}$ and $\phi_2 = -(c/a)^{\frac{1}{n-1}}$. When n is an even integer $(n > 1)$ and $ac > 0$ or $ac < 0$, there exist two equilibrium points at $E_0(0,0)$, and $E_3(\phi_3, 0)$, where $\phi_3 = (c/a)^{\frac{1}{n-1}}$. Let $M(\phi_i, 0)$ be the coefficient matrix of the linearized system of equation (7) at an equilibrium point $E_i(\phi_i, 0)$. We have

$$J = detM(\phi_i, 0) = -m(r-bc)\phi_i^{m-1}f'(\phi_i), \text{ where} f'(\phi_i) = c - an\phi_i^{n-1}. \tag{8}$$

By the theory of planar dynamical systems [3]-[13], we know that if $J(\phi_i, 0) < 0$, the equilibrium point $E_i(\phi_i, 0)$ of th Hamiltonian system will be a saddle point; if $J(\phi_i, 0) > 0$, the equilibrium point $E_i(\phi_i, 0)$ will be a center; when $J(\phi_i, 0) = 0$ and the Poincaré index of the equilibrium point be 0, then the equilibrium point $E_i(\phi_i, 0)$ will be a cusp.

By using the above facts, we obtain the following phase portraits of equation (5) shown in figures 1-10.

3. Exact Traveling Wave Solutions of Equation (2)

By using the traveling wave system (5) and the Hamiltonian (6) with $h = 0$ to do calculations, we obtain the following explicit exact parametric representations.

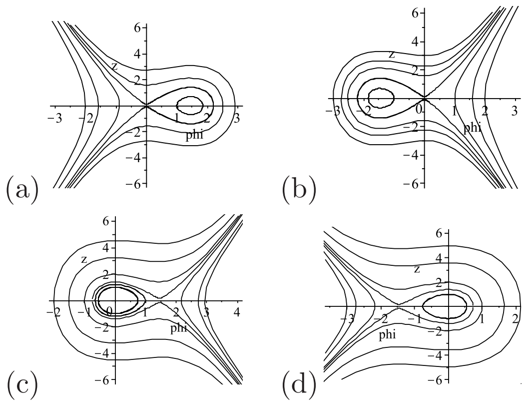

Figure 1: The phase portraits of equation (5) for $m = 1, n = 2$, (a)$ac > 0; a(r - bc) > 0$, (b)$ac < 0; a(r - bc) < 0$, (c)$ac > 0; a(r - bc) < 0$, (d)$ac < 0; a(r - bc) > 0$.

(1) When $m = 1, n = 2, ac > 0, a(r - bc) > 0$, (see figure 1(a)), we have the smooth solitary wave solution given by

$$u = \frac{3c}{2a} sech^2(\frac{1}{2}\sqrt{\frac{c}{r - bc}}\xi). \tag{9}$$

(2) When $m = 2, n = 2, ac > 0, a(r - bc) > 0$, (see figure 2(a)), we have the smooth compacton given by

$$u = \frac{2c}{3a}(1 \pm sin(\frac{1}{2}\sqrt{\frac{a}{r - bc}}\xi)), \xi \in (0, \pi\sqrt{\frac{r - bc}{a}}). \tag{10}$$

(3)When $m = 3, n = 2, ac > 0, a(r - bc) > 0$, (see figure 3(a)), we have the periodic cusp traveling wave solution given by

$$u = \frac{5c}{4a} - \frac{a}{30(r - bc)}\xi^2. \tag{11}$$

(4) When $m = 1, n = 3, ac > 0, a(r - bc) > 0$, (see figure 4(a)), we have a couple of solitary wave solutions of peak form and valley form given by

$$u = \sqrt{\frac{2c}{a}} sech(\sqrt{\frac{c}{r - bc}}\xi). \tag{12}$$

(5) When $m = 2, n = 3, ac > 0, a(r - bc) > 0$, (see figure 5(a)), we have the

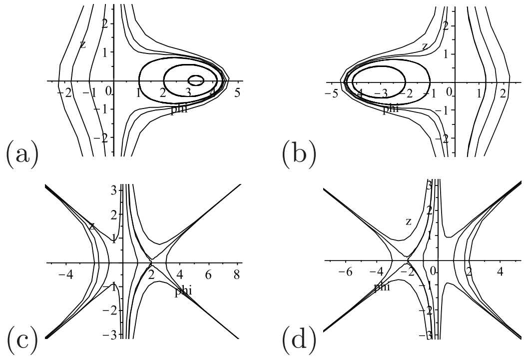

Figure 2: The phase portraits of equation (5) for $m = 2, n = 2$, (a)$ac > 0; a(r - bc) > 0$, (b)$ac < 0; a(r - bc) > 0$, (c)$ac > 0; a(r - bc) < 0$, (d)$ac < 0; a(r - bc) < 0$.

smooth compacton given by

$$u = \frac{v sn^2(\sqrt{\frac{av}{10(r-bc)}}\xi, \frac{\sqrt{2}}{2})}{2 - sn^2(\sqrt{\frac{av}{10(r-bc)}}\xi, \frac{\sqrt{2}}{2})}, \text{ where } v = \sqrt{\frac{5c}{3a}}. \tag{13}$$

(6) When $m = 2, n = 3, ac > 0, a(r - bc) < 0$, (see figure 5(b)), we have the smooth compacton given by

$$u = \frac{v sn^2(\sqrt{-\frac{av}{10(r-bc)}}\xi, \frac{\sqrt{2}}{2})}{sn^2(\sqrt{-\frac{av}{10(r-bc)}}\xi, \frac{\sqrt{2}}{2}) - 2}, \text{ where } v = \sqrt{\frac{5c}{3a}}. \tag{14}$$

(7) When $m = 3, n = 3, ac > 0, a(r - bc) > 0$, (see figure 6(a)), we have the couple of periodic cusp wave solutions given by

$$u = \pm\sqrt{\frac{3c}{2a}} sin(\sqrt{\frac{c}{6(r-bc)}}\xi), \text{where} \xi \in (0, \frac{\pi}{2}\sqrt{\frac{6(r-bc)}{c}}). \tag{15}$$

Among these exact solutions, the solutions given by equations (10), (11), (13), (14) and (15) are obtained for the first time for the generalized ZK-MEW equation. The smooth compactons given by equations (13) and (14) involve elliptic functions obtained by using elliptic integrals [2].

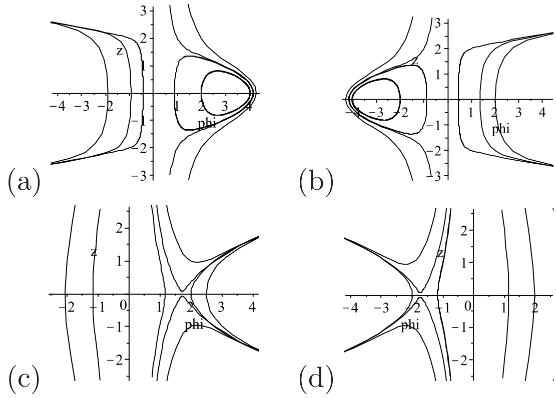

Figure 3: The phase portraits of equation (5) for $m = 3, n = 2$, (a)$ac > 0; a(r - bc) > 0$, (b)$ac < 0; a(r - bc) < 0$, (c)$ac > 0; a(r - bc) < 0$, (d)$ac < 0; a(r - bc) > 0$.

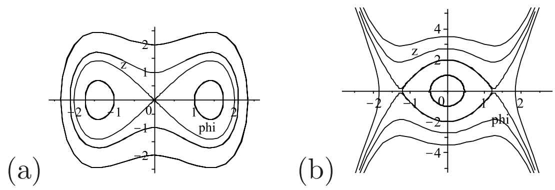

Figure 4: The phase portraits of equation (5) for $m = 1, n = 3$, (a)$ac > 0; a(r - bc) > 0$, (b)$ac > 0; a(r - bc) < 0$.

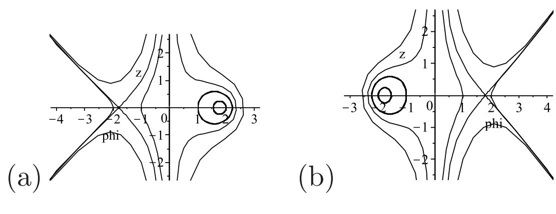

Figure 5: The phase portraits of equation (5) for $m = 2, n = 3$, (a)$ac > 0; a(r - bc) > 0$, (b)$ac > 0; a(r - bc) < 0$.

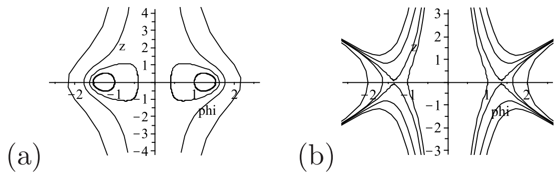

Figure 6: The phase portraits of equation (5) for $m = 3, n = 3$, (a)$ac > 0; a(r - bc) > 0$, (b)$ac > 0; a(r - bc) < 0$.

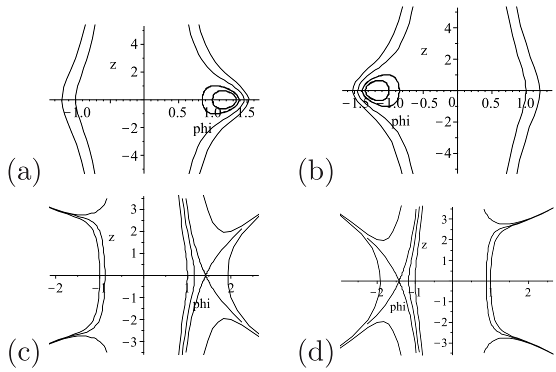

Figure 7: The phase portraits of equation (5) for $m = 2p > 2, n = 2q > 2, p, q \in N$: (a)$ac > 0; a(r - bc) > 0$, (b)$ac < 0; a(r - bc) > 0$, (c)$ac > 0; a(r - bc) < 0$, (d)$ac < 0; a(r - bc) < 0$.

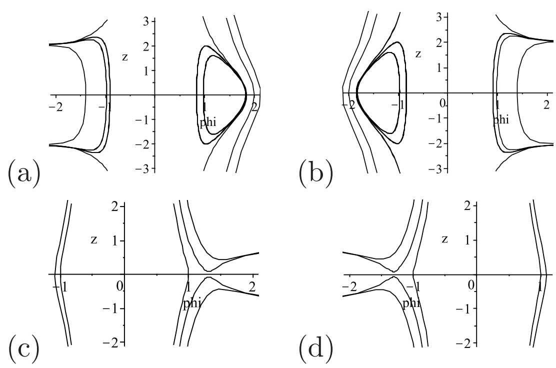

Figure 8: The phase portraits of equation (5) for $m = 2p + 1 > 3, n = 2q > 2, p, q \in N$: (a)$ac > 0; a(r - bc) > 0$, (b)$ac < 0; a(r - bc) < 0$, (c)$ac > 0; a(r - bc) < 0$, (d)$ac < 0; a(r - bc) > 0$.

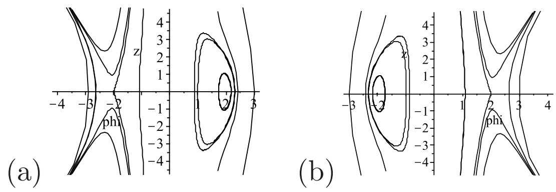

Figure 9: The phase portraits of equation (5) for $m = 2p > 2, n = 2q + 1 > 3, p, q \in N$: (a)$ac > 0; a(r - bc) > 0$, (b)$ac > 0; a(r - bc) < 0$.

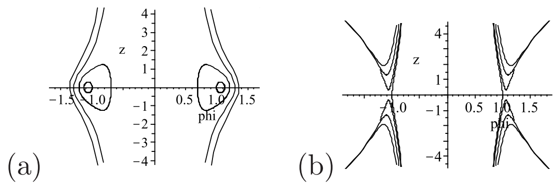

Figure 10: The phase portraits of equation (5) for $m = 2p + 1 > 3, n = 2q + 1 > 3, p, q \in N$: (a)$ac > 0; a(r - bc) > 0$, (b)$ac > 0; a(r - bc) < 0$.

4. The Existence of Traveling Wave Solutions of Equation (2)

In this section, we discuss the existence of smooth and non-smooth solitary wave solutions, kink and anti-kink wave solutions, smooth and non-smooth periodic wave solutions, and compacton solutions of equation (2). We denote that $h_i = H(\phi_i, 0)$ is the Hamiltonian at $(\phi_i, 0)$ given by equation (6). We notice that the singular system (5) has the same orbit as the regular system (2) except for the straight line $\phi = 0$. It means that in the (ϕ, z) plane the profiles defined by the orbits far from the straight line $\phi = 0$ are smooth with respect to $\xi = x + y - ct$.

Lemma 4.1. The boundary curves of a periodic annulus are the limit curves of closed orbits inside the annulus. If these boundary curves contain a segment of the singular straight line $\phi = 0$ of equation (2), then along this segment and near this segment, in very short time interval $z = \phi_\xi$ jumps rapidly.

It is clear from the lemma that if a phase point (ϕ, z) moves near the straight line $\phi = 0$ along the periodic orbit, then $z = \phi_\xi$ changes rapidly its sign which forms a profile of a cusp wave. Using Lemma 4.1. and figures 1-10, we obtain the following results.

Theorem 4.2. When $m = 1, n = 2$, equation (2) has a family of smooth periodic wave solutions and a smooth solitary wave solution under the following conditions:

i) $a > 0, c > 0, r - bc > 0 (a < 0, c < 0, r - bc < 0.)$ Then, corresponding to a branch of the curves $H(\phi, z) = h, h \in (h_3, 0)(H(\phi, z) = h, h \in (0, h_3))$ defined by equation (6), equation (2) has a family of smooth periodic wave solutions, corresponding to a branch of the curves $H(\phi, z) = 0$ defined by equation (6), equation (2) has a smooth solitary wave solution (see figure 1(a)).

ii) $a < 0, c > 0, r - bc > 0 (a > 0, c < 0, r - bc < 0.)$ Then, corresponding to a branch of the curves $H(\phi, z) = h, h \in (h_3, 0)(H(\phi, z) = h, h \in (0, h_3))$ defined by equation (6), equation (2) has a family of smooth periodic wave solutions, corresponding to a branch of the curves $H(\phi, z) = 0$ defined by equation (6), equation (2) has a smooth solitary wave solution (see figure 1(b)).

Theorem 4.3. When $m = 1, n = 2$, equation (2) has a family of smooth periodic wave solutions and a smooth solitary wave solution under the following conditions:

i) $a > 0, c > 0, r - bc < 0 (a < 0, c < 0, r - bc > 0.)$ Then, corresponding to a branch of the curves $H(\phi, z) = h, h \in (h_3, 0)(H(\phi, z) = h, h \in (0, h_3))$ defined by equation (6), equation (2) has a family of smooth periodic wave solutions, corresponding to a branch of the curves $H(\phi, z) = h_3$ defined by equation (6), equation (2) has a smooth solitary wave solution (see figure 1(c)).

ii) $a < 0, c > 0, r - bc < 0 (a > 0, c < 0, r - bc > 0.)$ Then, corresponding to a branch of the curves $H(\phi, z) = h, h \in (h_3, 0)(H(\phi, z) = h, h \in (0, h_3))$ defined by equation (6), equation (2) has a family of smooth periodic wave solutions, corresponding to a branch of the curves $H(\phi, z) = h_3$ defined by equation (6), equation (2) has a smooth solitary wave solution (see figure 1(d)).

Theorem 4.4. When $m = 2, n = 2$, equation (2) has a family of smooth periodic wave solutions and a compacton under the following conditions:

i) $a > 0, c > 0, r - bc > 0 (a < 0, c < 0, r - bc < 0.)$ Then, corresponding to a branch of the curves $H(\phi, z) = h, h \in (h_3, 0)(H(\phi, z) = h, h \in (0, h_3))$ defined by equation (6), equation (2) has a family of smooth periodic wave solutions, corresponding to a branch of the curves $H(\phi, z) = 0$ defined by equation (6), equation (2) has a compacton (see figure 2(a)).

ii) $a > 0, c < 0, r - bc > 0 (a < 0, > 0, r - bc < 0.)$ Then, corresponding to a branch of the curves $H(\phi, z) = h, h \in (h_3, 0)(H(\phi, z) = h, h \in (0, h_3))$ defined by equation (6), equation (2) has a family of smooth periodic wave solutions, corresponding to a branch of the curves $H(\phi, z) = 0$ defined by equation (6), equation (2) has a compacton (see figure 2(b)).

Theorem 4.5. When $m = 1, n = 3$, equation (2) has three families of smooth periodic wave solutions and two solitary wave solutions of peak form and valley form under the following conditions:

i) $a > 0, c > 0, r - bc > 0 (a < 0, c < 0, r - bc < 0.)$ Then, corresponding to a branch of the curves $H(\phi, z) = h, h \in (h_1 = h_2, 0)$or $h \in (0, \infty)(H(\phi, z) = h, h \in (0, h_1 = h_2)$or $h \in (-\infty, 0))$ defined by equation (6), equation (2) has three families of smooth periodic wave solutions, corresponding to a branch of the curves $H(\phi, z) = 0$ defined by equation (6), equation (2) has two solitary wave solutions of peak form and valley form (see figure 4(a)).

Theorem 4.6. When $m = 1, n = 3$, equation (2) has a family of smooth periodic wave solutions and a kink and an anti-kink wave solutions under the following conditions:

i) $a > 0, c > 0, r - bc < 0(a < 0, c < 0, r - bc > 0.)$ Then, corresponding to a branch of the curves $H(\phi, z) = h, h \in (h_1 = h_2, 0)(H(\phi, z) = h, h \in (0, h_1 = h_2))$ defined by equation (6), equation (2) has a family of smooth periodic wave solutions, corresponding to a branch of the curves $H(\phi, z) = h_1$ defined by equation (6), equation (2) has a kink and an anti-kink wave solutions (see figure 4(b)).

Theorem 4.7. When $m = 2, n = 3$, equation (2) has a family of smooth periodic wave solutions, a solitary cusp wave solution and a compacton under the following conditions:

i) $a > 0, c > 0, r - bc > 0(a < 0, c < 0, r - bc < 0.)$ Then, corresponding to a branch of the curves $H(\phi, z) = h, h \in (h_1, 0)(H(\phi, z) = h, h \in (0, h_1))$ defined by equation (6), equation (2) has a family of smooth periodic wave solutions, corresponding to a branch of the curves $H(\phi, z) = h_2$ defined by equation (6), equation (2) has a solitary cusp wave solution, corresponding to a branch of the curves $H(\phi, z) = 0$ defined by equation (6), equation (2) has a compacton (see figure 5(a)).

ii) $a > 0, c > 0, r - bc < 0(a < 0, c < 0, r - bc > 0.)$ Then, corresponding to a branch of the curves $H(\phi, z) = h, h \in (0, h_2)(H(\phi, z) = h, h \in (h_2, 0))$ defined by equation (6), equation (2) has a family of smooth periodic wave solutions, corresponding to a branch of the curves $H(\phi, z) = h_1$ defined by equation (6), equation (2) has a solitary cusp wave solution, corresponding to a branch of the curves $H(\phi, z) = 0$ defined by equation (6), equation (2) has a compacton (see figure 5(b)).

Theorem 4.8. When $m = 3, n = 3$, equation (2) has two families of smooth periodic wave solutions and two periodic cusp wave solutions of peak form and valley form under the following conditions:

i) $a > 0, c > 0, r - bc > 0(a < 0, c < 0, r - bc < 0.)$ Then, corresponding to a branch of the curves $H(\phi, z) = h, h \in (h_1 = h_2, 0)(H(\phi, z) = h, h \in (0, h_1 = h_2))$ defined by equation (6), equation (2) has two families of smooth periodic wave solutions, corresponding to a branch of the curves $H(\phi, z) = 0$ defined by equation (6), equation (2) has two periodic cusp wave solutions of peak form and valley form (see figure 6(a)).

Theorem 4.9. When $n = 2q, q \in N$, equation (2) has a solitary cusp wave solution under the following conditions:

i) $ac > 0, a(r - bc) < 0$ and $m = 2p; p \in N$ (see figures 2(c) and 7(c)).

ii) $ac > 0, a(r - bc) < 0$ and $m = 2p + 1; p \in N$ (see figures 3(c) and 8(c)).

iii) $ac < 0, a(r - bc) < 0$ and $m = 2p; p \in N$ (see figures 2(d) and 7(d)).

iv) $ac < 0, a(r - bc) > 0$ and $m = 2p + 1; p \in N$ (see figures 3(d) and 8(d)).

Theorem 4.10. When $m = 2p > 2, n = 2q > 2, p, q \in N$, equation (2) has a smooth family of uncountable infinity many periodic wave solutions under the following conditions:

i) $ac > 0, a(r - bc) > 0$ (see figure 7(a)).

ii) $ac < 0, a(r - bc) > 0$ (see figure 7(b)).

Theorem 4.11. When $m = 2p + 1, n = 2q, p, q \in N$, equation (2) has the periodic cusp traveling wave solutions under the following conditions:

i) $ac > 0, a(r - bc) > 0$ (see figures 3(a) and 8(a)).

ii) $ac > 0, a(r - bc) < 0$ (see figures 3(b) and 8(b)).

Theorem 4.12. When $m = 2p > 2, n = 2q + 1 > 3, p, q \in N$, equation (2) has a family of smooth periodic wave solutions and a solitary cusp wave solution under the following conditions:

i) $a > 0, c > 0, r - bc > 0 (a < 0, c < 0, r - bc < 0.)$ Then, corresponding to a branch of the curves $H(\phi, z) = h, h \in (h_1, 0)(H(\phi, z) = h, h \in (0, h_1))$ defined by equation (6), equation (2) has a family of smooth periodic wave solutions, corresponding to a branch of the curves $H(\phi, z) = h_2$ defined by equation (6), equation (2) has a solitary cusp wave solution (see figure 9(a)).

ii) $a > 0, c > 0, r - bc < 0 (a < 0, c < 0, r - bc > 0.)$ Then, corresponding to a branch of the curves $H(\phi, z) = h, h \in (0, h_2)(H(\phi, z) = h, h \in (h_2, 0))$ defined by equation (6), equation (2) has a family of smooth periodic wave solutions, corresponding to a branch of the curves $H(\phi, z) = h_1$ defined by equation (6), equation (2) has a solitary cusp wave solution (see Figure 9(b)).

Theorem 4.13. When $m = 2p + 1 > 3, n = 2q + 1 > 3, p, q \in N$, and $ac > 0, a(r - bc) > 0$, equation (2) has two families of smooth periodic wave solutions (see figure 10(a)).

Theorem 4.14. When $m = 2p + 1, n = 2q + 1, p, q \in N$, and $ac > 0, a(r - bc) < 0$, equation (2) has a couple of solitary cusp wave solutions of peak form and valley form (see figures 6(b) and 10(b)).

5. Conclusion

In this paper, we have considered the bifurcation behavior of traveling wave solutions of equation (2). We have proved that equation (2) has smooth and non-smooth solitary wave solutions, kink and anti-kink wave solutions, smooth and non-smooth periodic wave solutions, and compacton depending on different regions of parametric spaces. We have obtained various sufficient conditions to guarantee the existence of the above wave solutions. Moreover, some exact traveling wave solutions of equation (2) are determined. To the best of our knowledge, the study of the bifurcation behavior of traveling wave solutions of equation (2) has not been performed. Therefore it might be helpful to the study of the Zakharov-Kuznetsov modified equal width equation. We would like to study the generalized Zakharov-Kuznetsov modified equal width equation further.

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References

[1] A. Biswas, 1-soliton solution of the generalized Zakharov-Kuznetsov modified equal width equation, Applied Mathematics Letters, **22**, No. 11 (2009), 1775-1777.

[2] P.F. Byrd, M. D. Friedman, Handbook of elliptic integrals for engineer and scientists, Springer, New York (1971).

[3] S. N. Chow, J. K Hale, Methods of bifurcation theory, Springer, New York (1981).

[4] A. Esen, S. Kutluay, Travelling wave Solutions for the generalized (2+1)-dimensional ZK-MEW Equation, International Journal of Nonlinear Science, **8** (2009), 428-434.

[5] J. Guckenheimer, P. J. Holmes, Nonlinear oscillations, dynamical Systems and bifurcations of vector fields, Springer, New York (1983).

[6] C. M. Khalique, K. R. Adem, Exact solutions of the (2 + 1)-dimensional Zakharov-Kuznetsov modified equal width equation using Lie group analysis, Mathematical and Computer Modelling, **54** (2011), 184-189.

[7] J. Li, Z. Liu, Smooth and non-smooth travelling waves in a nonlinearly dispersive equation,Applied Mathematical Modelling, **25**, No. 1 (2000), 41-56.

[8] L. Perko, Differential equations and dynamical Systems, Springer, New York (1991).

[9] A. Saha, Bifurcation of travelling wave solutions for the generalized KP-MEW equations, Communications in Nonlinear Science and Numerical Simulation, **17**, No. 9 (2012), 3539-3551.

[10] J. Shen, J. Li, W. Xu, Bifurcations of travelling wave solutions in a model of the hydrogen-bonded systems, Applied Mathematics and Computation, **171**, No. 1 (2005), 242-271.

[11] A. M. Wazwaz, Exact solutions for the ZK-MEW equation by using the tanh and sine-cosine methods, International Journal of Computer Mathematics, **82** (2005), 699-708.

[12] Z. Wang, S. Tang, Bifurcation of travelling wave solutions for the generalized ZK equations, Communications in Nonlinear Science and Numerical Simulation, **14**, No. 5 (2009), 2018-2024.

[13] Z. Wang, S. Tang, Bifurcation of travelling wave solutions for the generalized ZK-BBM equations, Communications in Nonlinear Science and Numerical Simulation, **14**, No. 7 (2009), 2948-2955.